IMA Journal of Mathematical Control and Information (2010) **27**, 145−164
doi:10.1093/imamci/dnq005
Advance Access publication on April 9, 2010

Robust invariance in uncertain discrete event systems with applications to transportation networks

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[Received on 17 February 2009; revised on 1 March 2010; accepted on 6 March 2010]

This paper studies a class of uncertain discrete event systems over the max–plus algebra, where system matrices are unknown but are convex combinations of known matrices. These systems model a wide range of applications, e.g. transportation systems with varying vehicle travel time and queueing networks with uncertain arrival and queuing time. This paper presents the computational methods for different robust invariant sets of such systems. A recursive algorithm is given to compute the supremal robust invariant sub-semimodule in a given sub-semimodule. The algorithm converges to a fixed point in a finite number of iterations under proper assumptions on the state semi-module. This paper also presents computational methods for positively robust invariant polyhedral sets. A search algorithm is presented for the positively robust invariant polyhedral sets. The main results are applied to the time table design of a public transportation network.

Keywords: robust controlled invariance; discrete event systems; max–plus algebra.

1. Introduction

Semirings can be understood as a set of objects without inverses with respect to the corresponding operators. A semimodule over a semiring can be analogized to a linear space over a field. Systems over a semiring are systems evolving with variables taking values in semimodules over the semiring. Intuitively, such systems are not equipped with 'additive inverses'. The max–plus algebra (Golan, 1999) is a special semiring consisting of real numbers embedded with max and plus operations. Systems over the max–plus algebra are often used to model discrete event systems, such as queueing systems (Cassandras, 1993), transportation systems (Baccelli *et al.*, 1992) and communication networks (Le Boudec & Thiran, 2002). In the study of linear systems over a field or a ring, controlled invariant subspaces (or sets) play a key role in the fundamental problems of the geometric control theory (Conte & Perdon, 1995, 1998; Hautus, 1982; Inaba *et al.*, 1988; Wonham, 1979), e.g. the disturbance decoupling problem, the model matching problem and the block decoupling problem. The geometric approach for systems over a semiring, however, is still a new research direction (Cohen *et al.*, 1996, 1999) compared to the geometric theory for traditional linear systems over a field (Wonham, 1979). The goal of this paper is to study different controlled invariant sets for a class of discrete event systems with parameter uncertainties, which are modelled as uncertain linear systems over the max–plus algebra. The ultimate impact of these results is to establish a foundation for the geometric control theory of uncertain linear systems over semirings.

Researchers have been studying different computational methods for controlled invariant spaces (or sets) for discrete event systems over the max–plus algebra (Katz, 2007; Bitsoris, 1988; Truffet, 2004).

However, there are many uncertain factors in the discrete event system modelling. This paper focuses on a class of discrete event systems over the max–plus algebra in which system matrices are uncertain but can be written as linear combinations of known matrices. These systems model a class of discrete event systems, such as transportation systems with varying travel time and queuing networks with uncertain arrival time and queueing time. Most of these uncertainties can be characterized by max–plus convex sets in Nitica & Singer (2007). For such uncertain discrete event systems, there are some research results on the model predictive control (Necoara *et al.*, 2006a; Van den Boom & De Schutter, 2002) and worst-case optimal control (Necoara *et al.*, 2006b). This paper, on the other hand, makes contributions in the analysis of robust invariant sets, which can be used in the geometric control problems.

The main results in this paper are the computational methods for different robust controlled invariant sub-semimodules (and sets). A recursive algorithm is given to compute the supremal robust invariant sub-semimodule in a given sub-semimodule of the state semimodule. The algorithm converges to a fixed point in a finite number of iterations under proper assumptions on the state semimodule. The fixed point is the supremal robust controlled invariant sub-semimodule of the given sub-semimodule. The main challenge for systems over a semiring (or a ring) is that the (A, B)-invariant sub-semimodule (or submodule) does not coincide with (A, B)-invariant sub-semimodule (or submodule) of feedback type (Conte & Perdon, 1995, 1998; Hautus, 1982). A computational method is introduced for robust controlled invariant sub-semimodule of feedback type in a given sub-semimodule. Moreover, this paper presents computational methods for positively robust invariant polyhedral sets in two cases, time-invariant and time-varying polyhedral sets. The reason to focus on time-varying polyhedral invariant sets is that there are many quantities in discrete event systems varying with time. For instance, in transportation networks, the goal is to control the departure time of each vehicle such that it is contained in a desired time constraint. These constraints usually vary over time and can be used to design proper time tables for transportation systems. A search algorithm for time-varying positively robust invariant sets is presented and is illustrated using a public transportation network with varying vehicle travelling time.

This paper is organized as follows. Section 2 introduces some mathematical preliminaries and discrete event systems over the max–plus algebra. Section 3 presents computational methods for robust controlled invariant sub-semimodules. Section 4 presents the computational methods for positively robust invariant sets. A public transportation network is used to illustrate the main results. Section 5 concludes this paper with future research.

2. Mathematical preliminaries

2.1 *Semiring and semimodule*

A monoid R is a semigroup $(R, \oplus)$ with an identity element e_R with respect to the binary operation $\oplus$. The term ‘semiring’ means a set, $R = (R, \oplus, e_R, \otimes, 1_R)$ with two binary associative operations, $\oplus$ and $\otimes$, such that $(R, \oplus, e_R)$ is a commutative monoid and $(R, \otimes, 1_R)$ is a monoid, which are connected by a two-sided distributive law of $\otimes$ over $\oplus$. Moreover, $e_R \otimes r = r \otimes e_R = e_R$, for all r in R. $R = (R, \oplus, e_R, \otimes, 1_R)$ is a semifield if and only if $(R \backslash \{e_R\}, \otimes, 1_R)$ is a group, i.e. all of its elements have inverse elements with respect to the $\otimes$ operator. A semifield $R = (R, \oplus, e_R, \otimes, 1_R)$ is called an ‘idempotent’ semifield if $a \oplus a = a$ for all $a \in R$. The max–plus algebra is an idempotent semifield, where the traditional addition and multiplication are replaced by the max operation and the plus operation, i.e.

$$\text{Addition}: \ a \oplus b \equiv \max\{a, b\},$$
$$\text{Multiplication}: \ a \otimes b \equiv a + b.$$

The max–plus algebra is denoted as $\mathbb{R}_{\text{Max}}=(\mathbb{R}\cup\{\epsilon\},\oplus,\epsilon,\otimes,e)$, or simply $\mathbb{R}_{\text{Max}}$, where $\mathbb{R}$ denotes the set of real numbers, $\epsilon=-\infty$ and $e=0$. Similarly, $\mathbb{Z}_{\text{Max}}=(\mathbb{Z}\cup\{\epsilon\},\oplus,\epsilon,\otimes,e)$ denotes the integer max–plus algebra, where $\mathbb{Z}$ is the set of integers.

Let $(R,\oplus,e_R,\otimes,1_R)$ be a semiring and $(M,\oplus_M,e_M)$ be a commutative monoid, where the subscript denotes the corresponding monoid for the operator $\oplus$. M is called a 'left R-semimodule' if there exists a map $\mu\colon R\times M\to M$, denoted by $\mu(r,m)=rm$, for all $r\in R$ and $m\in M$, such that the following conditions are satisfied:

1. $r(m_1\oplus_M m_2)=rm_1\oplus_M rm_2$,
2. $(r_1\oplus r_2)m=r_1m\oplus_M r_2m$,
3. $r_1(r_2m)=(r_1\otimes r_2)m$,
4. $1_Rm=m$,
5. $re_M=e_M=e_Rm$,

for any $r, r_1, r_2\in R$ and $m, m_1, m_2\in M$. In this paper, e denotes the unit semimodule. A 'sub-semimodule' K of M is a submonoid of M with $rk\in K$, for all $r\in R$ with $k\in K$. An R-morphism between two semimodules $(M,\oplus_M,e_M)$ and $(N,\oplus_N,e_N)$ is a map $f\colon M\to N$ satisfying

1. $f(m_1\oplus_M m_2)=f(m_1)\oplus_N f(m_2)$;
2. $f(rm)=rf(m)$,

for all $m, m_1, m_2\in M$ and $r\in R$.

Let N be a subset of an R-semimodule $(M,\oplus_M,e_M)$. We denote N_0 as the set of all elements of the form $\bigoplus_{M\, i\in I}\lambda_i n_i$, where $n_i\in N$, $\lambda_i\in R$ and I is the index set. The sub-semimodule N_0 is said to be generated by N, and N is called a system of generators of N_0. The subset N of an R-semimodule M is called 'linearly independent' if $\bigoplus_{M\, i}\lambda_i n_i=\bigoplus_{M\, i}\beta_i n_i$ implies $\lambda_i=\beta_i$ for all $i\in I$. An R-semimodule M is called a free R-semimodule if it has a linearly independent subset N of M which generates M and then N is called a basis of M. If N has a finite number of elements, M is called a finitely generated R-semimodule, denoted as Span N. Katz (2007) defined the concept of volume for a semimodule. Let $\mathcal{K}\subset\mathbb{Z}^n_{\text{Max}}$ be a semimodule, the volume of $\mathcal{K}$, denoted as $\text{vol}(\mathcal{K})$, is the cardinality of the set $\widetilde{\mathcal{K}}\triangleq\{x\in\mathcal{K}|x_1\oplus x_2\oplus\cdots\oplus x_n=0\}$. Also, if $K\in\mathbb{Z}^{n\times p}_{\text{Max}}$, the volume of the semimodule $\mathcal{K}=\text{Im}K$ is denoted as $\text{vol}(K)=\text{vol}(\text{Im}K)=\text{vol}(\mathcal{K})$.

REMARK. In Wagneur (1991, 2005), independence and weakly independence are defined differently. It can be shown that if the semimodule is linearly independent according to the definition in this paper, then it is also independent and weakly independent according to the definitions in Wagneur (1991, 2005).

2.2 *Uncertain discrete event systems over the max–plus algebra*

A class of uncertain discrete event system over the max–plus algebra $\mathbb{R}_{\text{Max}}$ is described by the following equation:

$$x(k)=\widetilde{A}x(k-1)\oplus Bu(k), \tag{2.1}$$

where the state semimodule $X\cong\mathbb{R}^n_{\text{Max}}$ and the input semimodule $U\cong\mathbb{R}^r_{\text{Max}}$ are free, $A\colon X\to X$ and $B\colon U\to X$ are R-semimodule morphisms. Although we assume the state semimodule, the input semimodule and the output semimodule are free for convenience, the main results in this paper did not require the free assumption. The system's state matrix morphism $\widetilde{A}: X \to X$ is unknown, but it can be represented as the linear combination of m known matrix morphisms $A_1, A_2, \ldots, A_m$, i.e.

$$\widetilde{A} = \bigoplus_{i=1}^{m} (\lambda_i \otimes A_i) \quad \text{with} \quad \bigoplus_{i=1}^{m} \lambda_i = e,$$

where the parameters λ_i are unknown. A set $\mathcal{P}$ is convex if $(a \otimes x) \oplus (b \otimes y) \in \mathcal{P}$ for any $x, y \in \mathcal{P}$ and $a \oplus b = 1_R$, where $a, b \in R$. Therefore, $\widetilde{A}$ is in the convex hull of $A_1, \ldots, A_m$, denoted as $\widetilde{A} \in \mathrm{co}\{A_1, \ldots, A_m\}$. The geometric concepts of different invariant subspaces for systems over a field can be generalized to systems over a semiring. Given a system of the form (2.1) over the max–plus algebra $\mathbb{R}_{\mathrm{Max}}$, a sub-semimodule $\mathcal{V}$ of the state semimodule X is

- called '$(\widetilde{A}, B)$-invariant' or 'robust controlled invariant' if and only if, for all $x_0 \in \mathcal{V}$ and an arbitrary matrix morphism $\widetilde{A} \in \mathrm{co}\{A_1, \ldots, A_m\}$, there exists a sequence of control inputs, $\underline{u} = \{u_1, u_2, \ldots\}$, such that every component in the state trajectory produced by this input, $\underline{x}(x_0; \underline{u}) = \{x_0, x_1, \ldots\}$, remains inside of $\mathcal{V}$.
- called '$(\widetilde{A}, B)$-invariant of feedback type' or '$(\widetilde{A} \oplus BF)$-invariant', if and only if there exists a state feedback $F: X \to U$ such that $(\widetilde{A} \oplus BF)\mathcal{V} \subset \mathcal{V}$, for an arbitrary matrix morphism $\widetilde{A} \in \mathrm{co}\{A_1, \ldots, A_m\}$.

Unlike systems over a field, (A, B)-invariant sub-semimodules of feedback type are not same as (A, B)-invariant sub-semimodules for systems over a semiring or even a ring. Conte & Perdon (1998) proved that, for systems over a ring, if an (A, B)-invariant submodule is a direct summand of the free state semimodule X, then it is (A, B)-invariant of feedback type. However, direct summand in semimodules is far more complicated than modules, the same result in Conte & Perdon (1998) is not true any more.

This paper also generalizes positively invariant sets for deterministic systems to uncertain systems of the form (2.1) over the max–plus algebra $\mathbb{R}_{\mathrm{Max}}$. A non-empty subset $\mathcal{P}$ of $\mathbb{R}^n_{\mathrm{Max}}$ is said to be 'positively invariant' or 'A-positively invariant' for the systems of the form $x(k+1) = Ax(k)$, if, for every initial condition $x(0) \in \mathcal{P}$, the trajectory $x(k)$ remains in $\mathcal{P}$ for all $k \geqslant 0$. If a non-empty subset $\mathcal{P}$ of $\mathbb{R}^n_{\mathrm{Max}}$ is said to be 'robust positively invariant' or $\widetilde{A}$-positively invariant for systems of the form $x(k+1) = \widetilde{A}x(k)$, if, for every initial condition $x(0) \in \mathcal{P}$, the state trajectory $x(k)$ remains in $\mathcal{P}$ with respect to any uncertain matrix morphism $\widetilde{A} \in \mathrm{co}\{A_1, \ldots, A_n\}$.

A non-empty subset $\mathcal{P}$ of $\mathbb{R}^n_{\mathrm{Max}}$ is said to be a 'controlled positively invariant set' for systems of the form $x(k+1) = Ax(k) \oplus Bu(k)$ if, for every initial condition $x(0) \in \mathcal{P}$, there exists a control input u such that the trajectory $x(k)$ remains in $\mathcal{P}$ with respect to any uncertain matrix morphism $\widetilde{A}$. A non-empty subset $\mathcal{P}$ of $\mathbb{R}^n_{\mathrm{Max}}$ is said to be a 'controlled robust positively invariant set' for systems of the form (2.1) if, for every initial condition $x(0) \in \mathcal{P}$, there exists a control input u such that the trajectory $x(k)$ remains in $\mathcal{P}$ with respect to any uncertain matrix morphism $\widetilde{A} \in \mathrm{co}\{A_1, \ldots, A_n\}$.

3. Robust invariant sub-semimodules

This section presents the computational methods for $(\widetilde{A}, B)$-invariant sub-semimodules and $(\widetilde{A} \oplus BF)$-invariant sub-semimodules in a given sub-semimodule. These computational methods are generalizations of deterministic discrete event systems in Katz (2007) to uncertain discrete event systems of the form (2.1) with parameter uncertainties.

3.1 $(\widetilde{A}, B)$-*invariant sub-semimodules*

A sub-semimodule $\mathcal{V}$ of the state semimodule is $(\widetilde{A}, B)$-invariant if and only if

$$\mathcal{V} = \mathcal{V} \cap \widetilde{A}^{-1}(\mathcal{V} \ominus \mathcal{B}), \tag{3.1}$$

where $\mathcal{B} = \mathrm{Im} B$ and

$$\widetilde{A}^{-1}(\mathcal{V} \ominus \mathcal{B}) = \{x \in \mathbb{R}^n_{\mathrm{Max}} | \exists u \in U, \text{s.t. } \widetilde{A}x \oplus Bu \in \mathcal{V}, \forall\, \widetilde{A} \in \mathrm{co}\{A_1, \ldots, A_m\}\}. \tag{3.2}$$

REMARK. Equation (3.1) is not a definition of $(\widetilde{A}, B)$-invariant sub-semimodules, but it is a direct generalization from the definition of the $(\widetilde{A}, B)$-invariant sub-semimodules in the last section. Moreover, it is often used in the calculations of the robust controlled invariant sub-semimodules.

LEMMA 3.1 For an uncertain discrete event system of the form (2.1) over the max–plus algebra, a sub-semimodule $\mathcal{V}$ in the state semimodule is $(\widetilde{A}, B)$-invariant if and only if $\mathcal{V}$ is (A_i, B)-invariant for any $i \in \{1, 2, \ldots, m\}$.

Proof of Lemma 3.1. $\Longrightarrow$. $\mathcal{V}$ is $(\widetilde{A}, B)$-invariant, then, for all $x \in \mathcal{V}$, $\widetilde{A}x \oplus Bu \in \mathcal{V}$, i.e. $(\bigoplus_{i=1}^m \lambda_i A_i)x \oplus Bu$ is also in $\mathcal{V}$. If $\lambda_i = e$ and $\lambda_j = \epsilon$ for $j \neq i$ and $i, j \in \{1, \ldots, m\}$, then $\bigoplus_{i=1}^m \lambda_i A_i$ will just produce the matrix morphism A_i for any $i \in \{1, 2, \ldots, m\}$. Therefore, $A_i x \oplus Bu \in \mathcal{V}$ is true. Therefore, $\mathcal{V}$ is (A_i, B)-invariant for any $i \in \{1, 2, \ldots, m\}$.

$\Longleftarrow$. If $\mathcal{V}$ is (A_i, B)-invariant for any $i \in \{1, 2, \ldots, m\}$, then, for all $x \in \mathcal{V}$, there exists a control input $u_i \in U$ such that $A_i x \oplus Bu_i \in \mathcal{V}$. Because the sub-semimodule $\mathcal{V}$ is closed under the addition operation, $\bigoplus_{i=1}^m \lambda_i A_i x \oplus B \bigoplus_{i=1}^m \lambda_i u_i = \widetilde{A}x \oplus Bu \in \mathcal{V}$, where $u = \bigoplus_{i=1}^m \lambda_i u_i$. Therefore, $\mathcal{V}$ is $(\widetilde{A}, B)$-invariant.

□

LEMMA 3.2 For an uncertain discrete event system of the form (2.1) over the max–plus algebra, the following equality holds

$$\widetilde{A}^{-1}(\mathcal{V} \ominus \mathcal{B}) = \bigcap_{i=1}^m A_i^{-1}(\mathcal{V} \ominus \mathcal{B})$$

for any sub-semimodule $\mathcal{V}$ of the state semimodule X.

Proof of Lemma 3.2. $\subseteq$. For any $x \in \widetilde{A}^{-1}(\mathcal{V} \ominus \mathcal{B})$, there exists a control input $u \in U$ such that $\widetilde{A}x \oplus Bu = \bigoplus_{i=1}^m \lambda_i A_i x \oplus Bu \in \mathcal{V}$. If $\lambda_i = e$ and $\lambda_j = \epsilon$ for $j \neq i$ and $i, j \in \{1, \ldots, m\}$, then $A_i x \oplus Bu \in \mathcal{V}$ holds. Therefore, $x \in \bigcap_{i=1}^m A_i^{-1}(\mathcal{V} \ominus \mathcal{B})$.

$\supseteq$. For any $x \in \bigcap_{i=1}^m A_i^{-1}(\mathcal{V} \ominus \mathcal{B})$, there exists a control input $u_i \in U$ such that $A_i x \oplus Bu_i \in \mathcal{V}$ for all $i \in \{1, \ldots, m\}$. Therefore, $\bigoplus_{i=1}^m \lambda_i A_i x \oplus B \bigoplus_{i=1}^m \lambda_i u_i = \widetilde{A}x \oplus Bu \in \mathcal{V}$, where $u = \bigoplus_{i=1}^m \lambda_i u_i$. Hence, $x \in \widetilde{A}^{-1}(\mathcal{V} \ominus \mathcal{B})$.

□

For a linear system over a field, the supremal (A, B)-invariant subspace $\mathcal{V}^*$ in a given subspace $\mathcal{K}$ of the state space is computed using the following algorithm (Wonham, 1979),

$$\begin{aligned} \mathcal{V}_1 &= \mathcal{K}, \\ \mathcal{V}_{k+1} &= \mathcal{V}_k \cap A^{-1}(\mathcal{V}_k + \mathcal{B}), k \in \mathbb{N}, \end{aligned} \tag{3.3}$$

where $A^{-1}(\mathcal{V}_i + \mathcal{B}) = \{x \in \mathbb{R}^n | Ax \in \mathcal{V}_i + \mathcal{B}\}$. This recursion converges to a fixed point after a finite number of iterations. The fixed point is the supremal controlled invariant subspace in $\mathcal{K}$. However, for systems over a semiring or even a ring, this algorithm does not guarantee to terminate in a finite number of steps. For uncertain linear systems over the max–plus algebra, to calculate the supremal controlled invariant sub-semimodule of a given sub-semimodule $\mathcal{K}$, the algorithm in (3.3) is modified as follows by using Lemma 3.2:

$$\begin{aligned} \mathcal{V}_1 &= \mathcal{K}, \\ \mathcal{V}_{k+1} &= \mathcal{V}_k \cap \widetilde{A}^{-1}(\mathcal{V}_k \ominus \mathcal{B}), \\ &= \mathcal{V}_k \cap \bigcap_{i=1}^{m} A_i^{-1}(\mathcal{V}_k \ominus \mathcal{B}), \quad k \in \mathbb{N}. \end{aligned} \tag{3.4}$$

LEMMA 3.3 Let $\{\mathcal{V}_k\}_{k\geqslant 0}$ be the family of sub-semimodules defined by the algorithm in (3.4). If there exists a well-defined non-empty semimodule $\cap_{k\in\mathbb{N}}\mathcal{V}_k$, then any $(\widetilde{A}, B)$-invariant sub-semimodule of $\mathcal{K}$ is contained in $\cap_{k\in\mathbb{N}}\mathcal{V}_k$, namely the supremal controlled invariant sub-semimodule $\mathcal{V}^*$ is also contained in $\cap_{k\in\mathbb{N}}\mathcal{V}_k$. Moreover, if the algorithm in (3.4) terminates in r steps, then $\mathcal{V}^* = \mathcal{V}_r$.

Proof of Lemma 3.3. The sequence of sub-semimodules $\{\mathcal{V}_k\}_{k\in\mathbb{N}}$ is decreasing: clearly $\mathcal{V}_2 \subset \mathcal{V}_1$, if $\mathcal{V}_k \subset \mathcal{V}_{k-1}$, then

$$\begin{aligned} \mathcal{V}_{k+1} &= \mathcal{V}_k \cap \widetilde{A}^{-1}(\mathcal{V}_k \ominus \mathcal{B}) \subset \mathcal{V}_{k-1} \cap \widetilde{A}^{-1}(\mathcal{V}_{k-1} \ominus \mathcal{B}) \\ &= \mathcal{V}_{k-1} \cap \bigcap_{i=1}^{m} A^{-1}(\mathcal{V}_{k-1} \ominus \mathcal{B}) = \mathcal{V}_k. \end{aligned}$$

The next step is to show that any (A, B)-invariant sub-semimodule $\mathcal{V}$ in $\mathcal{K}$ is contained in $\cap_{k\in\mathbb{N}}\mathcal{V}_k$. This conclusion will imply that the supremal controlled invariant sub-semimodule $\mathcal{V}^*$ is contained in $\cap_{k\in\mathbb{N}}\mathcal{V}_k$. Clearly, $\mathcal{V}$ is contained in $\mathcal{V}_0 = \mathcal{K}$, suppose $\mathcal{V}$ is also contained in $\mathcal{V}_k$, then,

$$\begin{aligned} \mathcal{V} &= \mathcal{V} \cap \widetilde{A}^{-1}(\mathcal{V} \ominus \mathcal{B}) \subset \mathcal{V}_k \cap \widetilde{A}^{-1}(\mathcal{V}_k \ominus \mathcal{B}) \\ &= \mathcal{V}_k \cap \bigcap_{i=1}^{m} A_i^{-1}(\mathcal{V}_k \ominus \mathcal{B}) = \mathcal{V}_{k+1}. \end{aligned}$$

Hence, $\mathcal{V} \subset \mathcal{V}_k$ for all $k \in \mathbb{N}$, namely $\mathcal{V} \subset \cap_{k\in\mathbb{N}}\mathcal{V}_k$. Therefore, the supremal controlled invariant sub-semimodule $\mathcal{V}^*$ is contained in $\cap_{k\in\mathbb{N}}\mathcal{V}_k$.

If the algorithm in (3.4) terminates in r steps, then $\cap_{k\in\mathbb{N}}\mathcal{V}_k = \mathcal{V}_r$. Because $\mathcal{V}_r = \mathcal{V}_r \cap \widetilde{A}^{-1}(\mathcal{V}_r \ominus \mathcal{B})$, $\mathcal{V}_r$ is $(\widetilde{A}, B)$-invariant, namely $\mathcal{V}_r \subset \mathcal{V}^*$. On the other hand, $\mathcal{V}^* \subset \cap_{k\in\mathbb{N}}\mathcal{V}_k = \mathcal{V}_r$ from the previous step. Therefore, the equality holds for $\mathcal{V}^* = \mathcal{V}_r$. □

The preceding lemma states and proves that the algorithm in (3.4) can be used to calculate the supremal controlled invariant sub-semimodule in $\mathcal{K}$. Note that the algorithm in (3.4) does not always terminate in a finite number of steps. However, if restricting to a finite-volume semimodule in the integer max–plus algebra $\mathbb{Z}_{\mathrm{Max}}$, the algorithm in (3.4) guarantees to terminate in finite steps.

PROPOSITION 3.1 Given a sub-semimodule $\mathcal{K}$ in $\mathbb{Z}^n_{\mathrm{Max}}$ with a finite volume. The supremal $(\widetilde{A}, B)$-invariant sub-semimodule of $\mathcal{K}$ under the dynamics of the system (2.1) is finitely generated. Moreover, the sequence $\{\mathcal{V}_k\}_{k\in\mathbb{N}}$ by the algorithm in (3.4) terminates in a finite number r of steps, i.e. $\mathcal{V}^* = \mathcal{V}_r$ for some $r \leqslant \mathrm{vol}\,(\mathcal{K}) + 1$.

Proof of Proposition 3.1. The proof is a direct generalization of Theorem 2 for the deterministic discrete event systems in Katz (2007). First of all, let us recall that the set of robust controlled invariant sub-semimodules in a sub-semimodule $\mathcal{K}$ of X is a upper semilattice relative to sub-semimodule inclusion $\subset$ and operation $\oplus$. Therefore, there exists the 'supremal' element $\mathcal{V}^*$ in the family of robust controlled invariant sub-semimodules, a sub-semimodule $\mathcal{K}$ in X. Because every semimodule in $\mathbb{Z}^n_{\mathrm{Max}}$ with a finite volume is finitely generated and the supremal $(\widetilde{A}, B)$-invariant sub-semimodule $\mathcal{V}^*$ is contained in $\mathcal{K}$, we have $\mathrm{vol}(\mathcal{V}^*) \leqslant \mathrm{vol}(\mathcal{K}) < \infty$. Therefore, $\mathcal{V}^*$ is also finitely generated. Moreover, the sequence $\{\mathcal{V}_k\}_{k\in\mathbb{N}}$ is a decreasing sequence, so the sequence of the volumes of semimodules $\{\mathcal{V}_k\}_{k\in\mathbb{N}}$ is a decreasing sequence of non-negative integers. There exists the smallest non-negative integer $r \leqslant \mathrm{vol}(\mathcal{K}) + 1$, such that $\mathrm{vol}(\mathcal{V}_r) = \mathrm{vol}(\mathcal{V}_{r+1}) < \infty$. Hence, $\mathcal{V}_r = \mathcal{V}_{r+1}$ for some $r \leqslant \mathrm{vol}(\mathcal{K}) + 1$. □

The following example illustrates the termination of the algorithm in (3.4) under the assumptions in Proposition 3.1.

EXAMPLE 3.1 Consider an uncertain discrete event system over $\mathbb{Z}_{\mathrm{Max}}$ with system matrix morphism $\widetilde{A} \in \mathrm{co}\{A_1, A_2\}$, where the two matrix morphisms $A_1 \colon X \to X$ and $A_2 \colon X \to X$ are given as

$$A_1 = \begin{bmatrix} 1 & -\infty \\ -\infty & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & -\infty \\ -\infty & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We need to calculate the supremal $(\widetilde{A}, B)$-invariant sub-semimodule in $\mathcal{K} = \{(x, y)^\top \in \mathbb{Z}^2_{\mathrm{Max}} | x + 1 \leqslant y \leqslant x + 4\}$. The set $\widetilde{\mathcal{K}}$ is

$$\widetilde{\mathcal{K}} = \{(x, y)^\top \in \mathcal{K} | x \oplus y = 0\} = \{(-1, 0)^\top, (-2, 0)^\top, (-3, 0)^\top, (-4, 0)^\top\}.$$

The volume of $\mathcal{K}$ is 4. Then the algorithm in (3.4) to calculate the supremal controlled invariant sub-semimodule will terminate at most $\mathrm{vol}(\mathcal{K}) + 1 = 5$ steps. We verify it by computing the algorithm in (3.4):

$$\begin{aligned}
\mathcal{V}_1 &= \mathcal{K}, \\
\mathcal{V}_2 &= \mathcal{V}_1 \cap \bigcap_{i=1}^{2} A_i^{-1}(\mathcal{V}_1 \ominus \mathcal{B}) \\
&= \{(x, y)^\top \in \mathbb{Z}^2_{\mathrm{Max}} | x + 1 \leqslant y \leqslant x + 4\} \\
&\quad \cap \{(x, y)^\top \in \mathbb{Z}^2_{\mathrm{Max}} | x + 2 \leqslant y \leqslant x + 5\} \\
&\quad \cap \{(x, y)^\top \in \mathbb{Z}^2_{\mathrm{Max}} | x + 3 \leqslant y \leqslant x + 6\} \\
&= \{(x, y)^\top \in \mathbb{Z}^2_{\mathrm{Max}} | x + 3 \leqslant y \leqslant x + 4\}, \\
\mathcal{V}_3 &= \mathcal{V}_2 \cap \bigcap_{i=1}^{2} A_i^{-1}(\mathcal{V}_2 \ominus \mathcal{B})
\end{aligned}$$

$$
\begin{aligned}
&= \{(x, y)^\top \in \mathbb{Z}^2_{\mathrm{Max}} | x + 3 \leqslant y \leqslant x + 4\} \\
&\quad \cap \{(x, y)^\top \in \mathbb{Z}^2_{\mathrm{Max}} | x + 4 \leqslant y \leqslant x + 5\} \\
&\quad \cap \{(x, y)^\top \in \mathbb{Z}^2_{\mathrm{Max}} | x + 5 \leqslant y \leqslant x + 6\} \\
&= \{(-\infty, -\infty)^\top\}, \\
\mathcal{V}_4 &= \mathcal{V}_3, \\
&\vdots \\
\mathcal{V}_{k+1} &= \mathcal{V}_3, \\
&\vdots
\end{aligned}
$$

The algorithm terminates in three steps and the supremal $(\widetilde{A}, B)$-invariant sub-semimodule in $\mathcal{K}$ is $\mathcal{V}^* = \mathcal{V}_3 = \{(-\infty, -\infty)^\top\}$.

3.2 $(\widetilde{A}, B)$-*invariant sub-semimodules of feedback type*

A sub-semimodule $\mathcal{V}$ is $(\widetilde{A} \oplus BF)$-invariant if and only if there exists a state feedback $F\colon X \to U$ for any $\widetilde{A} \in \mathrm{co}\{A_1, \ldots, A_m\}$ such that $(\widetilde{A} \oplus BF)\mathcal{V} \subset \mathcal{V}$. For systems over a semiring or even a ring, a controlled invariant sub-semimodule is not identical with a control invariant sub-semimodule of feedback type, therefore, the computational method in the algorithm in (3.4) cannot be used to calculate $(\widetilde{A}, B)$-invariant sub-semimodules of feedback type.

LEMMA 3.4 A sub-semimodule $\mathcal{V}$ is $(\widetilde{A}, B)$-invariant of feedback type if and only if the sub-semimodule $\mathcal{V}$ is (A_i, B)-invariant of feedback type, for any $A_i \in \{A_1, \ldots, A_m\}$ and any $\widetilde{A} \in \mathrm{co}\{A_1, \ldots, A_m\}$.

Proof of Lemma 3.4. $\Longrightarrow$. If a sub-semimodule $\mathcal{V}$ is $(\widetilde{A}, B)$-invariant of feedback type for all $\widetilde{A} \in \mathrm{co}\{A_1, \ldots, A_m\}$, then there exists a state feedback $F\colon X \to U$ such that $(\widetilde{A} \oplus BF)\mathcal{V} \subset \mathcal{V}$, which implies $(A_i \oplus BF)\mathcal{V} \subset \mathcal{V}$ for any $i \in \{1, \ldots, m\}$. Therefore, the sub-semimodule $\mathcal{V}$ is (A_i, B)-invariant of feedback type.

$\Longleftarrow$. If the sub-semimodule $\mathcal{V}$ is (A_i, B)-invariant of feedback type, for any $A_i \in \{A_1, \ldots, A_m\}$, there there exists an $F_i\colon X \to U$ such that $(A_i \oplus BF_i)\mathcal{V} \subset \mathcal{V}$. Therefore,

$$\left(\bigoplus_{i=1}^{m} \lambda_i (A_i \oplus BF_i)\right)\mathcal{V} \subset \mathcal{V} \Longrightarrow (\widetilde{A} \oplus B\widetilde{F})\mathcal{V} \subset \mathcal{V},$$

where $\widetilde{F} \in \mathrm{co}\{F_1, \ldots, F_m\}$. Hence, the sub-semimodule $\mathcal{V}$ is $(\widetilde{A}, B)$-invariant of feedback type for all $\widetilde{A} \in \mathrm{co}\{A_1, \ldots, A_m\}$.

□

For linear systems over the integer max–plus algebra $\mathbb{Z}_{\mathrm{Max}}$, if a given sub-semimodule $\mathcal{V} = \mathrm{Im}Q$, where $Q \in \mathbb{Z}^{n\times r}_{\mathrm{Max}}$, then $\mathcal{V}$ is $(\widetilde{A} \oplus BF)$-invariant if and only if there exists matrices $F \in \mathbb{Z}^{q\times n}_{\mathrm{Max}}$ and $G \in \mathbb{Z}^{r\times r}_{\mathrm{Max}}$ such that

$$(\widetilde{A} \oplus BF)Q = QG \tag{3.5}$$

holds for all $\widetilde{A} \in \mathrm{co}\{A_1, \ldots, A_m\}$.

LEMMA 3.5 Given a sub-semimodule $\mathcal{V} = \mathrm{Im}Q$, where $Q \in \mathbb{Z}_{\mathrm{Max}}^{n\times r}$, then $\mathcal{V}$ is $(\widetilde{A} \oplus BF)$-invariant if and only if there exists matrices $F_i \in \mathbb{Z}_{\mathrm{Max}}^{q\times n}$ and $G_i \in \mathbb{Z}_{\mathrm{Max}}^{r\times r}$ such that the equality

$$(A_i \oplus BF_i)Q = QG_i, \forall\, i \in \{1, \ldots, m\} \tag{3.6}$$

is true.

Proof of Lemma 3.5. $\Longleftarrow$. If $(A_i \oplus BF_i)Q = QG_i$ is true for any $i \in \{1, \ldots, m\}$, then

$$\bigoplus_{i=1}^{m} \lambda_i (A_i \oplus BF_i)Q = \bigoplus_{i=1}^{m} \lambda_i QG_i \,\text{implies}$$

$$\left(\bigoplus_{i=1}^{m} \lambda_i A_i \oplus B \bigoplus_{i=1}^{m} \lambda_i F_i\right) Q = Q \bigoplus_{i=1}^{m} \lambda_i G_i \,\text{implies}\ (\widetilde{A} \oplus B\widetilde{F})Q = Q\widetilde{G},$$

where $\widetilde{F} \in \mathrm{co}\{F_1, \ldots, F_m\}$ and $\widetilde{G} \in \mathrm{co}\{G_1, \ldots, G_m\}$. Then $\mathcal{V}$ is $(\widetilde{A} \oplus BF)$-invariant. $\Longrightarrow$. $\mathcal{V}$ is $(\widetilde{A} \oplus BF)$-invariant, then $(\widetilde{A} \oplus BF)Q = QG, \forall\, \widetilde{A} \in \mathrm{co}\{A_1, \ldots, A_m\}$. Hence, $(A_i \oplus BF_i)Q = QG_i$, for all $i \in \{1, \ldots, m\}$.

□

In order to find each pair of (F_i, G_i) satisfying (3.6), the elimination method in Katz (2007) and the residuation theory in Baccelli *et al.* (1992) can be used by adding a variable t_i to form the homogenous linear equation over the integer max–plus algebra $\mathbb{R}_{\mathrm{Max}}$:

$$(A_i t_i \oplus BF_i)Q = QG_i, \quad \text{for } i \in \{1, \ldots, m\}. \tag{3.7}$$

Equation (3.6) has at least one set of solutions (F_i, G_i) if and only if (3.7) has at least one set of solutions (t_i, F_i, G_i) with $t_i \neq -\infty$. This implies that the semimodule $X = \mathrm{Im}Q$ is (A_i, B)-invariant of feedback type with $F_i = (-t_i)F_i$. The necessary and sufficient condition for (t_i, F_i, G_i) to be a solution of (3.7) is

$$t_i \leqslant (A_i Q)\backslash(QG_i),$$
$$F_i \leqslant B\backslash(QG_i)/Q,$$
$$G_i \leqslant Q\backslash((A_i t_i \oplus BF)Q),$$

for $i \in \{1, \ldots, m\}$. The operations $\cdot\backslash\cdot$ and $\cdot/\cdot$ are defined

$$M\backslash N = \bigvee\{X \in \mathbb{Z}_{\mathrm{Max}}^{p\times r} | M \otimes X \leqslant N\},\ D/C = \bigvee\{X \in \mathbb{Z}_{\mathrm{Max}}^{p\times r} | X \otimes C \leqslant D\},$$

for $M \in \mathbb{Z}_{\mathrm{Max}}^{n\times p}$ and $N \in \mathbb{Z}_{\mathrm{Max}}^{n\times r}$, and for $C \in \mathbb{Z}_{\mathrm{Max}}^{r\times n}$ and $D \in \mathbb{Z}_{\mathrm{Max}}^{p\times n}$.

4. Positively robust invariant polyhedral Sets

This section presents the necessary and sufficient conditions for the polyhedral sets to be positively robust invariant under the dynamics of an uncertain linear system of the form (2.1) over an idempotent semiring R. Because the max–plus algebra is a special idempotent semiring, the results in this section are applicable to systems of the form (2.1) over the max–plus algebra.

4.1 *Time-invariant polyhedral sets*

Considering the following time-invariant polyhedral set

$$\mathcal{P}(F, \phi, \psi) = \{x \in R^n | \phi \leqslant F \otimes x \leqslant \psi\},$$
$$\phi, \psi \in R^p \text{ and } F \in R^{p \times n}.$$

For deterministic linear systems over an idempotent semiring, the necessary and sufficient condition for positively invariance of the set $\mathcal{P}(I_n, \phi, \psi)$ was established in Truffet (2004).

LEMMA 4.1 (Truffet, 2004) Assume $n = p$ and $F = I_n$, where I_n denotes the $n \times n$ identity matrix. $\mathcal{P}(I_n, \phi, \psi)$ is positively invariant under the dynamics of the system over an idempotent semiring R,

$$x(k) = Ax(k-1), \quad A \in R^{n \times n}, \tag{4.1}$$

if and only if the condition

$$(A \otimes \psi \leqslant \psi) \wedge (\phi \leqslant A \otimes \phi) \tag{4.2}$$

is true, where $\wedge$ denotes the logic AND.

For uncertain discrete event systems of the form (2.1) over an idempotent semiring R, similar results can obtained if the input signals are zero.

PROPOSITION 4.1 Assume $n = p$ and $F = I_n$, where I_n denotes the $n \times n$ identity matrix. $\mathcal{P}(I_n, \phi, \psi)$ is positively robust invariant under the dynamics of the system (2.1) over an idempotent semiring R and the inputs are zero, i.e. $u = e_U$, if and only if

$$(A_i \otimes \psi \leqslant \psi) \wedge (\phi \leqslant A_i \otimes \phi), \tag{4.3}$$

for all $i \in \{1, \dots, m\}$.

Proof of Proposition 4.1. $\Longrightarrow$. If $\mathcal{P}(I_n, \phi, \psi)$ is positively invariant under the dynamics of the system (2.1) over an idempotent and with zero inputs, then for any $x \in R^n$ such that $\phi \leqslant x \leqslant \psi$, the condition $\phi \leqslant \widetilde{A} \otimes x \leqslant \psi$ holds for any possible choice of $\widetilde{A} \in \mathrm{co}\{A_1, \dots, A_m\}$. Therefore, the inequality $\phi \leqslant A_i \otimes x \leqslant \psi$ is true, for any A_i, where $i \in \{1, \dots, m\}$. Using Lemma 4.1, $(A_i \otimes \psi \leqslant \psi) \wedge (\phi \leqslant A_i \otimes \phi)$ holds for all $i \in \{1, \dots, m\}$.

$\Longleftarrow$. Because (4.3) holds for all $i \in \{1, \dots, m\}$, we have

$$\left(\bigoplus_{i=1}^{m} \lambda_i \otimes A_i \otimes \psi \leqslant \bigoplus_{i=1}^{m} \lambda_i \otimes \psi \right) \wedge \left(\bigoplus_{i=1}^{m} \lambda_i \otimes \phi \leqslant \bigoplus_{i=1}^{m} \lambda_i \otimes A_i \otimes \phi \right).$$

Since $\bigoplus_{i=1}^{m} \lambda_i = 1_R$ and $\bigoplus_{i=1}^{m} \lambda_i \otimes A_i = \widetilde{A}$, the above condition becomes

$$(\widetilde{A} \otimes \psi \leqslant \psi) \wedge (\phi \leqslant \widetilde{A} \otimes \phi).$$

Using Lemma 4.1, $\mathcal{P}(I_n, \phi, \psi)$ is positively robust invariant for the uncertain discrete event system (2.1) over an idempotent semiring and with zero inputs. □

THEOREM 4.1 Given $E \in R^{n \times p}$ satisfying

$$\begin{cases} E \otimes F \leqslant I_n, \\ \phi \leqslant F \otimes E \otimes \phi. \end{cases}$$

The set $\mathcal{P}(F, \phi, \psi)$ is positively robust invariant under the dynamics of the system (2.1) over an idempotent semiring R by $x \mapsto \widetilde{A} \otimes x$ with zero inputs if and only if the condition

$$(A_i \otimes (F \backslash \psi) \leqslant (F \backslash \psi)) \wedge (E \otimes \phi \leqslant A_i \otimes E \otimes \phi) \tag{4.4}$$

is true for all $i \in \{1, \ldots, m\}$. The symbol $F \backslash \psi$ denotes $\bigvee \{x \in X | F \otimes x \leqslant \psi\}$.

Proof of Theorem 4.1. $\Longrightarrow$. If $\mathcal{P}(F, \phi, \psi)$ is positively invariant under the dynamics of the system (2.1) over an idempotent semiring R by $x \mapsto \widetilde{A} \otimes x$ with zero inputs, then, for any $x \in R^n$ such that $\phi \leqslant x \leqslant \psi$, we have $\phi \leqslant \widetilde{A} \otimes x \leqslant \psi$, for any possible choice of $\widetilde{A} \in \mathrm{co}\{A_1, \ldots, A_m\}$. Therefore, using Theorem 3.1 in Truffet (2004), the following condition holds:

$$[\widetilde{A} \otimes (F \backslash \psi) \leqslant (F \backslash \psi)] \wedge [E \otimes \phi \leqslant \widetilde{A} \otimes E \otimes \phi].$$

Therefore, for all $i \in \{1, \ldots, m\}$, the equality

$$[A_i \otimes (F \backslash \psi) \leqslant (F \backslash \psi)] \wedge [E \otimes \phi \leqslant A_i \otimes E \otimes \phi]$$

is true.

$\Longleftarrow$. Because (4.4) holds for all $i \in \{1, \ldots, m\}$, we have

$$\bigoplus_{i=1}^{m} \lambda_i \otimes A_i \otimes (F \backslash \psi) \leqslant \bigoplus_{i=1}^{m} \lambda_i \otimes (F \backslash \psi),$$

$$\bigoplus_{i=1}^{m} \lambda_i \otimes E \otimes \phi \leqslant \bigoplus_{i=1}^{m} \lambda_i \otimes A_i \otimes E \otimes \phi.$$

Since $\bigoplus_{i=1}^{m} \lambda_i = 1_R$ and $\bigoplus_{i=1}^{m} \lambda_i \otimes A_i = \widetilde{A}$, the above condition becomes

$$[\widetilde{A} \otimes (F \backslash \psi) \leqslant (F \backslash \psi)] \wedge [E \otimes \phi \leqslant \widetilde{A} \otimes E \otimes \phi]. \tag{4.5}$$

Using Theorem 3.1 in Truffet (2004), $\mathcal{P}(F, \phi, \psi)$ is positively robust invariant for the discrete event system (2.1) over an idempotent semiring R by $x \mapsto \widetilde{A}x$ with zero inputs. □

PROPOSITION 4.2 Assume $n = p$, $\mathcal{P}(F, \phi, \psi)$ is positively invariant under the dynamics $x \mapsto Ax$ of the system (4.1) if there exists a matrix $H \in R^{p \times p}$ such that

$$F \otimes A = H \otimes F \text{ and}$$

$$(H \otimes \psi \leqslant \psi) \wedge (\phi \leqslant H \otimes \phi). \tag{4.6}$$

The second condition means that $\mathcal{P}(I_n, \phi, \psi)$ is H-positively invariant.

Proof of Proposition 4.2. We need to prove for any $\phi \leqslant F \otimes x \leqslant \psi$, the condition $\phi \leqslant F \otimes A \otimes x \leqslant \psi$ holds. Since $F \otimes A = H \otimes F$, we only need to show that $\phi \leqslant H \otimes F \otimes x \leqslant \psi$. Since $\mathcal{P}(I_n, \phi, \psi)$ is H-positively invariant, for all $x \in R^n$, $\phi \leqslant x \leqslant \psi$, we have $\phi \leqslant H \otimes x \leqslant \psi$. Therefore, $F \otimes x \in \mathcal{P}(I_n, \phi, \psi)$. We have $\phi \leqslant H \otimes F \otimes x \leqslant \psi$. Thus, $\mathcal{P}(F, \phi, \psi)$ is positively invariant under the dynamics of the system (4.1).

□

The preceding proposition is a sufficient condition for a polyhedral set, $\mathcal{P}(F, \phi, \psi)$, to be positively invariant under the dynamics of the system (4.1). The following proposition states a sufficient condition for a polyhedral set $\mathcal{P}(F, \phi, \psi)$ to be positively robust invariant under the dynamic of the system (2.1) over an idempotent semiring R and with zero inputs.

PROPOSITION 4.3 Assume $n = p$, $\mathcal{P}(F, \phi, \psi)$ is positively robust invariant under the dynamics of the system (2.1) over an idempotent semiring R and with zero inputs, i.e. $x \mapsto \widetilde{A}x$, if there exists a matrix $H_i \in R^{p \times p}$ such that

$$F \otimes A_i = H_i \otimes F \text{ and}$$

$$(H_i \otimes \psi \leqslant \psi) \wedge (\phi \leqslant H_i \otimes \phi). \tag{4.7}$$

The second condition means that $\mathcal{P}(I_n, \phi, \psi)$ is H_i-positively invariant, for all $i \in \{1, \ldots, m\}$.

Proof of Proposition 4.3. We need to prove that, for any $\phi \leqslant F \otimes x \leqslant \psi$, the condition $\phi \leqslant F \otimes \widetilde{A} \otimes x \leqslant \psi$ holds for any arbitrary $\widetilde{A} \in \mathrm{co}\{A_1, \ldots, A_m\}$. Since $F \otimes A_i = H_i \otimes F$ and $\mathcal{P}(I_n, \phi, \psi)$ is H_i-positively invariant, for all $x \in R^n$, $\phi \leqslant x \leqslant \psi$, we have $\phi \leqslant H_i \otimes x \leqslant \psi$. Therefore, $F \otimes x \in \mathcal{P}(I_n, \phi, \psi)$. We have $\phi \leqslant H_i \otimes F \otimes x = F \otimes A_i \otimes x \leqslant \psi$, for all $i \in \{1, \ldots, m\}$. Combining them together to obtain

$$\phi = \bigoplus_{i=1}^{m} \lambda_i \otimes \phi \leqslant \bigoplus_{i=1}^{m} (\lambda_i \otimes F \otimes A_i \otimes x) \leqslant \bigoplus_{i=1}^{m} \lambda_i \otimes \psi = \psi,$$

where $\bigoplus_{i=1}^{m} (\lambda_i \otimes F \otimes A_i \otimes x) = F \otimes (\bigoplus_{i=1}^{m} \lambda_i \otimes A_i) \otimes x = F \otimes \widetilde{A} \otimes x$. Therefore, we have $\phi \leqslant F \otimes \widetilde{A} \otimes x \leqslant \psi$ for all $\phi \leqslant F \otimes x \leqslant \psi$. Thus, $\mathcal{P}(F, \phi, \psi)$ is positively invariant under the dynamics of the system (2.1) over an idempotent semiring R and with zero inputs. □

4.2 *Time-variant polyhedral sets*

Consider the following time-variant polyhedral sets:

$$\begin{aligned} \widetilde{\mathcal{P}}(F, \phi(k), \psi(k)) &= \{x(k) \in R^n | \phi(k) \leqslant F \otimes x(k) \leqslant \psi(k)\}, \\ \phi(k) &= K_\phi \otimes \phi(k-1), \\ \psi(k) &= K_\psi \otimes \psi(k-1), \end{aligned}$$

where $\phi, \psi \in R^p$, $k \in \mathbb{Z}^+$ and $F \in R^{p \times n}$ and $x(k)$ is governed by system (2.1) or system (4.1). Time-variant polyhedral sets are polyhedrons with time-variant boundaries, conditions that are very common in applications such as public transportation networks with time-varying time tables and queuing networks with time-varying arrival or service time.

LEMMA 4.2 Assume $n = p$ and $F = I_n$, where I_n denotes the $n \times n$ identity matrix. $\widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$ is positively robust invariant under the dynamics of system (4.1) by $x \mapsto Ax$ if and only if

$$(A \otimes \psi(k) \leqslant K_\psi \otimes \psi(k)) \wedge (K_\phi \otimes \phi(k) \leqslant A \otimes \phi(k)), \tag{4.8}$$

for all $k \in \mathbb{Z}^+$.

Proof of Lemma 4.2. $\Longrightarrow$. If $\widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$ is positively invariant under the dynamics of the system (4.1), then for any $x \in R^n$ such that $\phi(k) \leqslant x(k) \leqslant \psi(k)$, we have $\phi(k+1) = K_\phi \otimes \phi(k) \leqslant A \otimes x(k) \leqslant K_\psi \otimes \psi(k) = \psi(k+1)$. Since $\phi(k)$ and $\psi(k) \in \widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$, we have

$$(A \otimes \psi(k) \leqslant K_\psi \otimes \psi(k)) \wedge (K_\phi \otimes \phi(k) \leqslant A \otimes \phi(k)).$$

$\Longleftarrow$. If the condition $(A \otimes \psi(k) \leqslant K_\psi \otimes \psi(k)) \wedge (K_\phi \otimes \phi(k) \leqslant A \otimes \phi(k))$ holds, then for any $x \in \widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$, i.e. $\phi(k) \leqslant x(k) \leqslant \psi(k)$, hence the condition

$$K_\phi \otimes \phi(k) \leqslant A \otimes \phi(k) \leqslant A \otimes x(k) \leqslant A \otimes \psi(k) \leqslant K_\psi \otimes \psi(k)$$

implies $\phi(k+1) \leqslant x(k+1) \leqslant \psi(k+1)$. Therefore, $\widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$ is positively invariant under the dynamics of the system (4.1).

□

Lemma 4.2 presents the necessary and sufficient conditions for a time-variant polyhedral set $\widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$ to be positively invariant for the system (4.1). If we are given K_ϕ and K_ψ, the searching method for $\phi(k) \leqslant \psi(k)$, such that $\widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$, $k \geqslant 1$, is positively invariant for the system (4.1), can be performed as follows.

1. Solve the following equations for $\phi^* \leqslant \psi^*$,

$$A \otimes \psi^* = K_\psi \otimes \psi^*,$$

$$A \otimes \phi^* = K_\phi \otimes \phi^*.$$

2. For any $x(0) \in R^n$ such that let $\phi(0) = \phi^*$ and $\psi(0) = \psi^*$ and $\phi(0) \leqslant x(0) \leqslant \psi(0)$, we have

$$A \otimes \phi(0) \leqslant A \otimes x(0) \leqslant A \otimes \psi(0) \Longrightarrow$$

$$K_\phi \otimes \phi(0) \leqslant A \otimes x(0) \leqslant K_\psi \otimes \psi(0) \Longrightarrow$$

$$\phi(1) \leqslant x(1) \leqslant \psi(1).$$

Continuing this process, we are able to show that $\widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$ is positively invariant with respect to system (4.1). Therefore, ϕ^* and ψ^* generate the possible boundaries for a positively invariant polyhedral set.

PROPOSITION 4.4 Assume $n = p$ and $F = I_n$, where I_n denotes the $n \times n$ identity matrix. $\widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$ is positively robust invariant under the dynamics of the system (2.1) over an idempotent semiring R and with zero inputs by $x \mapsto \widetilde{A}x$ if and only if

$$(A_i \otimes \psi(k) \leqslant K_\psi \otimes \psi(k)) \wedge (K_\phi \otimes \phi(k) \leqslant A_i \otimes \phi(k)), \tag{4.9}$$

for all $i \in \{1, \ldots, m\}$ and $k \in \mathbb{Z}^+$.

Proof of Proposition 4.4. $\Longrightarrow$. If $\widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$ is positively robust invariant under the dynamics of the system (2.1) over an idempotent semiring R and with zero inputs, then, for any $x \in R^n$ such that $\phi(k) \leqslant x(k) \leqslant \psi(k)$, we have $\phi(k+1) = K_\phi \otimes \phi(k) \leqslant \widetilde{A} \otimes x(k) \leqslant K_\psi \otimes \psi(k) = \psi(k+1)$, for an arbitrary $\widetilde{A} \in \mathrm{co}\{A_1, \ldots, A_m\}$. Since $\phi(k)$ and $\psi(k) \in \widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$, we have

$$(\widetilde{A} \otimes \psi(k) \leqslant K_\psi \otimes \psi(k)) \wedge (K_\phi \otimes \phi(k) \leqslant \widetilde{A} \otimes \phi(k)) \Longrightarrow$$

$$(A_i \otimes \psi(k) \leqslant K_\psi \otimes \psi(k)) \wedge (K_\phi \otimes \phi(k) \leqslant A_i \otimes \phi(k))$$

for all $i \in \{1, \ldots, m\}$.

$\Longleftarrow$. If we have $(A_i \otimes \psi(k) \leqslant K_\psi \otimes \psi(k)) \wedge (K_\phi \otimes \phi(k) \leqslant A_i \otimes \phi(k))$ for all $i \in \{1, \ldots, m\}$, then $\widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$ is A_i-positively invariant. Therefore, for any $x \in \widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$, i.e. $\phi(k) \leqslant x(k) \leqslant \psi(k)$, we have

$$K_\phi \otimes \phi(k) \leqslant A_i \otimes x(k) \leqslant K_\psi \otimes \psi(k) \Longrightarrow$$

$$\lambda_i \otimes K_\phi \otimes \phi(k) \leqslant \lambda_i \otimes A_i \otimes x(k) \leqslant \lambda_i \otimes K_\psi \otimes \psi(k).$$

Combining them together for all $i \in \{1, \ldots, m\}$ to obtain

$$\bigoplus_{i=1}^{m} \lambda_i \otimes K_\phi \otimes \phi(k) \leqslant \bigoplus_{i=1}^{m} \lambda_i \otimes A_i \otimes x(k) \leqslant \bigoplus_{i=1}^{m} \lambda_i \otimes K_\psi \otimes \psi(k).$$

Since $\bigoplus_{i=1}^{m} \lambda_i = 1_R$, the condition $K_\phi \otimes \phi(k) \leqslant \widetilde{A} \otimes x(k) \leqslant K_\psi \otimes \psi(k)$ holds for arbitrary $\widetilde{A} \in \mathrm{co}\{A_1, \ldots, A_m\}$. Therefore, $\widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$ is positively robust invariant under the dynamics of the system (2.1) over an idempotent semiring R and with zero inputs. □

The preceding proposition presents a necessary and sufficient condition for $\widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$ to be positively robust invariant under the dynamics of the system (2.1) with zero inputs. If we are given K_ϕ and K_ψ, the searching method for $\phi(k) \leqslant \psi(k)$, such that $\widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$, $k \geqslant 1$, is positively robust invariant for the system (2.1) with zero inputs, is similar to the positively invariant case.

1. Solve the following equations for $\phi^* \leqslant \psi^*$,

$$A_i \otimes \psi^* = K_\psi \otimes \psi^*,$$

$$A_i \otimes \phi^* = K_\phi \otimes \phi^*,$$

   for all possible A_i, $i \in \{1, \ldots, m\}$.
2. For any $x(0) \in R^n$ such that let $\phi(0) = \phi^*$ and $\psi(0) = \psi^*$ and $\phi(0) \leqslant x(0) \leqslant \psi(0)$, we have

$$A_i \otimes \phi(0) \leqslant A_i \otimes x(0) \leqslant A_i \otimes \psi(0) \Longrightarrow$$

$$K_\phi \otimes \phi(0) \leqslant A_i \otimes x(0) \leqslant K_\psi \otimes \psi(0) \Longrightarrow$$

$$\phi(1) \leqslant x(1) \leqslant \psi(1),$$

for all $i \in \{1, \dots, m\}$. Continuing this process, we can show that $\widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$ is positively robust invariant with respect to system (2.1) with zero inputs. Therefore, ϕ^* and ψ^* generate the possible boundaries for a positively robust invariant polyhedral set.

The following proposition is a sufficient condition for a polyhedral set $\widetilde{\mathcal{P}}(F, \phi(k), \psi(k))$ to be positively invariant under the dynamic of the system (4.1).

PROPOSITION 4.5 Assume $n = p$, $\widetilde{\mathcal{P}}(F, \phi(k), \psi(k))$ is positively invariant under the dynamics of the system (4.1) if there exists a matrix $H \in R^{p\times p}$ such that

$$F \otimes A = H \otimes F \text{ and}$$

$$(H \otimes \psi(k) \leqslant K_\psi \otimes \psi(k)) \wedge (K_\psi \otimes \phi(k) \leqslant H \otimes \phi(k)).$$

The second condition means that $\widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$ is H-positively invariant.

Proof of Proposition 4.5. We need to prove that, for any $\phi(k) \leqslant F \otimes x(k) \leqslant \psi(k)$, the condition $K_\phi \otimes \phi(k) = \phi(k+1) \leqslant F \otimes A \otimes x(k) \leqslant \psi(k+1) = K_\psi \otimes \psi(k)$ is true. Since $F \otimes A = H \otimes F$, we only need to show that $K_\phi \otimes \phi(k) \leqslant H \otimes F \otimes x(k) \leqslant K_\psi \otimes \psi(k)$. Since $\widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$ is H-positively invariant, for all $x \in R^n$, $\phi(k) \leqslant x(k) \leqslant \psi(k)$, we have $\phi(k+1) \leqslant H \otimes x(k) \leqslant \psi(k+1)$. Therefore, $F \otimes x(k) \in \widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$. We have $\phi(k+1) \leqslant H \otimes F \otimes x(k) \leqslant \psi(k+1)$. Thus, $\widetilde{\mathcal{P}}(F, \phi(k), \psi(k))$ is positively invariant under the dynamics of the system (4.1). □

The following proposition states a sufficient condition for a polyhedral set $\widetilde{\mathcal{P}}(F, \phi(k), \psi(k))$ to be positively robust invariant.

PROPOSITION 4.6 Assume $n = p$, $\widetilde{\mathcal{P}}(F, \phi(k), \psi(k))$ is positively robust invariant under the dynamics of the system (2.1) over an idempotent semiring R and with zero inputs if there exists a matrix $H \in R^{p\times p}$ such that

$$F \otimes A_i = H_i \otimes F \text{ and}$$

$$(H_i \otimes \psi(k) \leqslant K_\psi \otimes \psi(k)) \wedge (K_\phi \otimes \phi(k) \leqslant H_i \otimes \phi(k)).$$

The second condition means that $\widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$ is H_i-positively invariant, where $i \in \{1, \dots, m\}$ and $k \in \mathbb{Z}^+$.

Proof of Proposition 4.6. We need to prove that, for any $\phi(k) \leqslant F \otimes x(k) \leqslant \psi(k)$, the condition $\phi(k+1) \leqslant F \otimes \widetilde{A} \otimes x(k) \leqslant \psi(k+1)$ holds for an arbitrary $\widetilde{A} \in \mathrm{co}\{A_1, \dots, A_m\}$. Since $F \otimes A_i = H_i \otimes F$ and $\widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$ is H_i-positively invariant, for all $x \in R^n$, $\phi(k) \leqslant x(k) \leqslant \psi(k)$, we have $\phi(k+1) \leqslant H_i \otimes x(k) \leqslant \psi(k+1)$. Therefore, $F \otimes x(k) \in \mathcal{P}(I_n, \phi, \psi)$. We have

$$\phi(k+1) \leqslant H_i \otimes F \otimes x(k) = F \otimes A_i \otimes x(k) \leqslant \psi(k+1),$$

for all $i \in \{1, \dots, m\}$. Adding them together to obtain

$$\phi(k+1) \leqslant \bigoplus_{i=1}^{m} (\lambda_i \otimes F \otimes A_i \otimes x(k)) \leqslant \psi(k+1),$$

where $\bigoplus_{i=1}^{m} (\lambda_i \otimes F \otimes A_i \otimes x(k)) = F \otimes (\bigoplus_{i=1}^{m} \lambda_i \otimes A_i) \otimes x(k) = F \otimes \widetilde{A} \otimes x(k)$. Therefore, we have $\phi(k+1) \leqslant F \otimes \widetilde{A} \otimes x(k) \in \psi(k+1)$ for all $\phi(k) \leqslant F \otimes x(k) \leqslant \psi(k)$. Thus, $\widetilde{\mathcal{P}}(F, \phi(k), \psi(k))$

is positively invariant under the dynamics of the system (2.1) over an idempotent semiring R and with zero inputs. □

4.3 *Example: a public transportation network*

In this section, a public transportation network (De Vries *et al.*, 1998) is modelled by an uncertain linear system over the max–plus algebra, whose system matrices are uncertain. In a small public transportation network shown in Fig. 1, there are train services from P via Q to S and back and from Q to R and back. Trains from P to S have to stop at Q for the connection to trains with destination R and vice versa. If $x_i(\cdot)$ denotes the departure time of the train in the direction i, $i \in \{1, \ldots, 4\}$. The train which is about to leave in the direction i for the kth time cannot leave if the train has not arrived yet. This condition can be represented as

$$x_i(k) \geqslant a_{ij} \otimes x_j(k-1), \tag{4.10}$$

where $x_i(k)$ denotes the kth departure time in direction i and a_{ij} is the travelling time from direction j to i, including the loading time of passengers. Another condition is that the train needs to wait for the possible connecting trains, i.e.

$$x_i(k) \geqslant a_{il} \otimes x_l(k-1), \tag{4.11}$$

where l is the possible connecting direction and a_{il} denotes the travelling time from direction l to i, also including the loading time of passengers. For the simple transportation network in Fig. 1, the system equation is the following:

$$\begin{aligned}
x_1(k) &= a_2 \otimes x_2(k-1),\\
x_2(k) &= a_3 \otimes x_3(k-1) \oplus a_4 \otimes x_4(k-1),\\
x_3(k) &= a_1 \otimes x_1(k-1) \oplus a_3 \otimes x_3(k-1),\\
&\quad \oplus a_4 \otimes x_4(k-1),\\
x_4(k) &= a_1 \otimes x_1(k-1) \oplus a_3 \otimes x_3(k-1),
\end{aligned}$$

where a_i denotes the travelling time on direction $i \in \{1, \ldots, 4\}$. The state equation can be rewritten in the matrix form

$$x(k) = Ax(k-1), \quad \text{where } A = \begin{bmatrix} \epsilon & a_2 & \epsilon & \epsilon \\ \epsilon & \epsilon & a_3 & a_4 \\ a_1 & \epsilon & a_3 & a_4 \\ a_1 & \epsilon & a_3 & \epsilon \end{bmatrix}.$$

If the travelling time a_i is deterministic, then the linear system is a deterministic discrete event system. In reality, however, the travelling time usually varies due to traffic and other emergency situations. For instance, assume $a_1 \in [10, 20]$ minutes, $a_2 \in [15, 25]$ minutes, $a_3 \in [10, 20]$ minutes and $a_4 \in [15, 25]$ minutes, then the system becomes an uncertain linear system $x(k) = \widetilde{A}x(k-1)$, where $\widetilde{A} \in \mathrm{co}\{A_1, A_2\}$

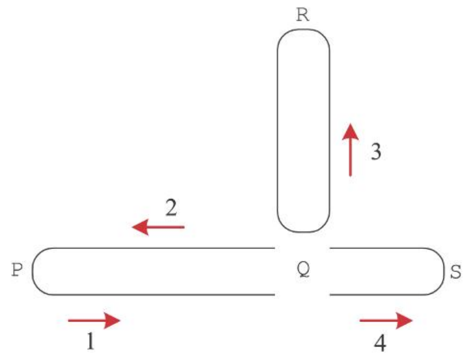

FIG. 1. A small public transportation network De Vries *et al.* (1998).

and

$$A_1 = \begin{bmatrix} \epsilon & 15 & \epsilon & \epsilon \\ \epsilon & \epsilon & 10 & 15 \\ 10 & \epsilon & 10 & 15 \\ 10 & \epsilon & 10 & \epsilon \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} \epsilon & 25 & \epsilon & \epsilon \\ \epsilon & \epsilon & 20 & 25 \\ 20 & \epsilon & 20 & 25 \\ 20 & \epsilon & 20 & \epsilon \end{bmatrix}.$$

Considering a time-varying polyhedral set

$$\widetilde{\mathcal{P}}(I_4, \phi(k), \psi(k)) = \{x \in \mathbb{R} | \phi(k) \leqslant x(k) \leqslant \psi(k)\},$$

where $K_\phi = 13.3333$, $K_\psi = 23.3333$ and

$$\phi(k) = K_\phi \otimes \phi(k-1) \quad \text{and} \quad \psi(k) = K_\psi \otimes \psi(k-1).$$

Firstly, we solve the following equations for $\phi^* \leqslant \psi^*$,

$$A_i \otimes \psi^* = K_\psi \otimes \psi^*,$$
$$A_i \otimes \phi^* = K_\phi \otimes \phi^*,$$

for all possible A_i, $i \in \{1, 2\}$. Because K_ϕ and K_ψ are eigenvalues for A_1 and A_2, respectively, we obtain that

$$\phi(0) = \phi^* = [30, 28.3333, 28.3333, 26.6667]^\top \text{ and}$$

$$\psi(0) = \psi^* = [50, 48.3333, 48.3333, 46.6667]^\top,$$

which are the eigenvectors for both A_i matrices, for $i = 1, 2$. Using Proposition 4.4, we can verify that

$$(A_i \otimes \psi(k) \leqslant K_\psi \otimes \psi(k)) \wedge (K_\phi \otimes \phi(k) \leqslant A_i \otimes \phi(k)), \tag{4.12}$$

for all $i \in \{1, \ldots, m\}$ and $k \in \mathbb{Z}^+$. Therefore, the given polyhedral set is positively robust invariant with respect to the public transportation network for arbitrary choice of matrix $\widetilde{A} \in \mathrm{co}\{A_1, A_2\}$. After computing the system trajectories, the four states, $x_i(k)$, are in the set, $\widetilde{\mathcal{P}}(I_4, \phi(k), \psi(k))$, as shown in Fig. 2. The values for $\phi(k)$ and $\psi(k)$ can be used as time-variant time tables for the train station because the departure time in each direction is constrained in this positively robust invariant set.

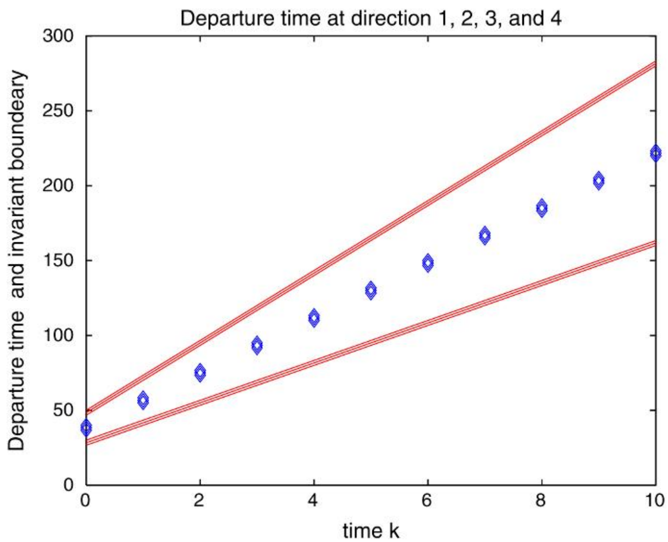

FIG. 2. The travelling time in direction 1, 2, 3, 4.

4.4 *Positively invariant under the state feedback control*

Considering uncertain discrete event systems of the form (2.1) over an idempotent semiring R and with non-zero control input signals, the necessary and sufficient condition is stated in the following theorem for the polyhedral set $\mathcal{P}(I_n, \phi, \psi)$ to be positively robust invariant under a state feedback controller $F\colon X \to U$.

THEOREM 4.2 There exists a state feedback controller $F\colon X \to U$ for the system described by (2.1) over an idempotent semiring R such that the polyhedral set $\mathcal{P}(I_n, \phi, \psi)$ is positively robust invariant if and only if

$$[\phi \leqslant (A_i \oplus B(B\backslash(\psi/\psi))) \otimes \phi] \wedge (A_i \leqslant \psi/\psi), \tag{4.13}$$

for all $A_i \in \{A_1, \ldots, A_m\}$.

Proof of Theorem 4.2. If the polyhedral set $\mathcal{P}(I_n, \phi, \psi)$ is positively robust invariant under a state feedback controller $F\colon X \to U$ for the system described by(2.1) over an idempotent semiring R, then for any $x \in \mathcal{P}(I_n, \phi, \psi)$,

$$\phi \leqslant (\widetilde{A} \oplus BF)x \leqslant \psi \Longleftrightarrow \phi \leqslant (A_i \oplus BF)x \leqslant \psi, \text{ for } i \in \{1, \ldots, m\}.$$

Therefore, $\mathcal{P}(I_n, \phi, \psi)$ is positively invariant under $x \mapsto (A_i \oplus BF)x$. By Lemma 4.1, this is equivalent to

$$\exists F | (\phi \leqslant (A_i \oplus BF) \otimes \phi) \wedge (A_i \oplus BF) \otimes \psi \leqslant \psi. \tag{4.14}$$

The condition $\exists F | (A_i \oplus BF) \otimes \psi \leqslant \psi$ is equivalent to $(A_i \leqslant \psi/\psi) \wedge (\exists F | F \leqslant B\backslash(\psi/\psi))$. Therefore, the condition in (4.14) is equivalent to (4.13), i.e.

$$[\phi \leqslant (A_i \oplus B(B\backslash(\psi/\psi))) \otimes \phi] \wedge (A_i \leqslant \psi/\psi),$$

for all $A_i \in \{A_1, \ldots, A_m\}$. □

5. Conclusion

This paper studies a class of uncertain discrete event systems over the max–plus algebra, where system matrices are unknown but are convex combinations of known matrices. This class of systems has been used to model transportation systems with varying vehicle travel time and queueing networks with uncertain arrival and queuing time. This paper presents computational methods for different robust invariant sets of such systems. These invariant sets are important in many control synthesis problems in geometric control. Because the geometric control for linear systems over the max–plus algebra has not been well established as for traditional linear systems over a field, the main results in this paper will serve as a foundation for the geometric control theory of discrete event systems with parameter uncertainty. Future research will explore different types of controlled invariant sets besides polyhedral sets, such as ellipsoidal invariant sets, in the geometric control theory of uncertain discrete event systems.

Acknowledgment

A preliminary form of this paper was presented at the 2009 American Control Conference, St. Louis, MO, USA.

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