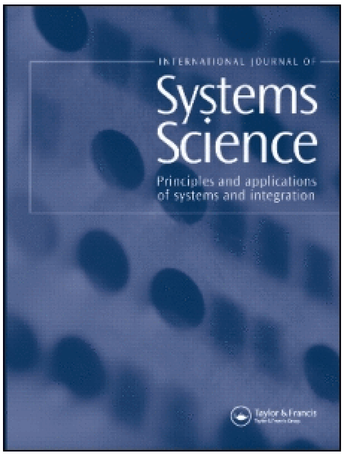

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Towards geometric control of max-plus linear systems with applications to queueing networks

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Towards geometric control of max-plus linear systems with applications to queueing networks

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The max-plus linear systems have been studied for almost three decades, however, a well-established system theory on such specific systems is still an on-going research. The geometric control theory in particular was proposed as the future direction for max-plus linear systems by Cohen et al. [Cohen, G., Gaubert, S. and Quadrat, J.P. (1999), 'Max-plus Algebra and System Theory: Where we are and Where to Go Now', *Annual Reviews in Control*, 23, 207–219]. This article generalises R.E. Kalman's abstract realisation theory for traditional linear systems over fields to max-plus linear systems. The new generalised version of Kalman's abstract realisation theory not only provides a more concrete state space representation other than just a 'set-theoretic' representation for the canonical realisation of a transfer function, but also leads to the computational methods for the controlled invariant semimodules in the kernel and the equivalence kernel of the output map. These controlled invariant semimodules play key roles in the standard geometric control problems, such as disturbance decoupling problem and block decoupling problem. A queueing network is used to illustrate the main results in this article.

Keywords: max-plus algebra; (A, B)-controlled invariance; disturbance decoupling

1. Introduction

Traditional system theory focuses on linear time-invariant systems whose coefficients belong to a field. Recent applications in communication networks (Le Boudec and Thiran 2002), genetic regulatory networks (De Jong 2002) and queueing systems (Baccelli, Cohen, Olsder, and Quadrat 1992) require a new system theory for linear time-invariant systems with coefficients in a semiring. A semiring is understood as a set of objects without inverses with respect to the corresponding operations, for example, the max-plus algebra (Baccelli et al. 1992), the min-plus algebra (Le Boudec and Thiran 2002) and the Boolean semiring (Golan 1999). Especially, max-plus linear systems have been studied by researchers for the past three decades, for example, controllability (Prou and Wagneur 1999), observability (Hardouin, Maia, Cottenceau, and Lhommeau 2010) and the model reference control problem (Maia, Hardouin, Santos-Mendes, and Cottenceau 2005). Another new research area for max-plus linear systems is to establish the geometric control theory (Wonham 1979) for systems over semirings as predicted in Cohen, Gaubert, and Quadrat (1999). There are some existing research results on generalising fundamental concepts and problems in geometric control to max-plus linear systems, such as computation of different controlled invariant sets (Katz 2007; Di Loreto, Gaubert, Katz, and Loiseau 2010) and the solvability of the disturbance decoupling problem (Lhommeau, Hardouin, and Cottenceau 2002; Shang and Sain 2008). However, even nowadays, researchers still have only a basic understanding of geometric control theory for max-plus linear systems (Baccelli et al. 1992).

This article revisits R.E. Kalman's abstract realisation theory for traditional linear systems over fields. The advantages of Kalman's theory include that the frequency-domain method and the state-variable method are merged into a single framework, linear systems over fields are special cases of this theory, and it provides new computational methods for important controlled invariant sets. These controlled invariant sets play key roles in the standard geometric control problems, such as the disturbance decoupling problem and the block decoupling problem (Wonham 1979). Because Kalman's realisation theory does not demand operation inverses, it overcomes the difficulty of lack of the subtraction operation for max-plus linear systems.

The researchers in Cohen et al. (1999) mentioned, Kalman's set-theoretic realisation for a canonical or

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minimal realisation, i.e. both reachable and observable, has the form of $\mathrm{im}\,\mathcal{R}/\mathrm{Ker}\,\mathcal{O}$, where $\mathcal{R}$ is the controllability matrix and $\mathcal{O}$ is the observability matrix. There are a few un-answered questions regarding this abstract structure, such as whether it can lead to a more concrete representation of a state space realisation and when the minimality from the 'set-theoretic' point of view will imply the minimal number of coordinates in the vector space. This article shows that the generalised Kalman's abstract realisation identifies a state space matrix A such that, for some proper B, C and D matrices, this realisation is canonical or minimal when it is both reachable and observable. Because there are different reachability and observability definitions in max-plus literature, this article adopts the definition from Gaubert and Katz (2004).

Moreover, this article discovers that the new generalised realisation theory displays distinctly different characters when proceeding from the observability point of view than when proceeding from the reachability point of view. This article also presents different computational methods of the supremal controlled invariant sub-semimodules in the kernel and equivalence kernel of the output map C. For max-plus linear systems, due to lack of additive inverses, the two kernels are not the same any more. Therefore, special studies need to focus on different kernels and images for max-plus linear systems. These controlled invariant sub-semimodules are crucial in a well-established geometric control theory for max-plus linear systems, such as the disturbance decoupling problem, the block decoupling problem and the model matching problem. A queueing system is modelled as a max-plus linear system to illustrate the main results in this article.

The remainder of this article is organised as follows. Section 2 presents mathematical preliminaries and defines max-plus linear systems. Section 3 presents the generalisation of Kalman's abstract system theory to max-plus linear systems. Section 4 introduces different controlled invariant sub-semimodules and presents different computational methods for the supremal controlled invariant sub-semimodules in the kernel and equivalence kernel of the output map. Section 5 connects these controlled invariant sub-semimoudles to the standard geometric control problem, the disturbance decoupling problem, with an application in a queueing network. Section 6 concludes this article.

2. Mathematical preliminaries

2.1. *Semirings and semimodules*

A monoid R is a semigroup $(R, \boxplus)$ with an identity element e_R with respect to the binary operation $\boxplus$. The term *semiring* means a set, $R = (R, \boxplus, e_R, \boxtimes, 1_R)$ with two binary associative operations, $\boxplus$ and $\boxtimes$, such that $(R, \boxplus, e_R)$ is a commutative monoid and $(R, \boxtimes, 1_R)$ is a monoid, which are connected by a two-sided distributive law of $\boxtimes$ over $\boxplus$. Moreover, $e_R \boxtimes r = r \boxplus e_R = e_R$, for all r in R. $R = (R, \boxplus, e_R, \boxtimes, 1_R)$ is a semifield if and only if $(R\backslash\{e_R\}, \boxtimes, 1_R)$ is a group, i.e. all its elements have inverse elements with respect to the $\boxtimes$ operator. An idempotent semifield R is a semifield satisfying $a \boxplus a = a$ for all $a \in R$. The common example for idempotent semifields is the max-plus algebra, which replaces the traditional addition and multiplication into the max operation and the plus operation,

$$\text{Addition}: a \oplus b \equiv \max\{a, b\},$$
$$\text{Multiplication}: a \otimes b \equiv a + b.$$

In max-plus algebra literature, we usually denote it as $\mathbb{R}_{\text{Max}} = (\mathbb{R} \cup \{\epsilon\}, \oplus, \epsilon, \otimes, e)$, where $\mathbb{R}$ is the set of real numbers, $\epsilon = -\infty$ and $e = 0$.

Let $(R, \boxplus, e_R, \boxtimes, 1_R)$ be a semiring, and $(M, \Box_M, e_M)$ be a commutative monoid. The operator $\Box_M$ is adopted from the module definition in Sain (1981), where the subscript denotes the corresponding R-semimodule. M is called a *left R-semimodule* if there exists a map $\mu : R \times M \to M$, denoted by $\mu(r, m) = rm$, for all $r \in R$ and $m \in M$, such that the following conditions are satisfied:

(1) $r(m_1 \Box_M m_2) = rm_1 \Box_M rm_2$,
(2) $(r_1 \boxplus r_2)m = r_1 m \Box_M r_2 m$,
(3) $r_1(r_2 m) = (r_1 \boxtimes r_2)m$,
(4) $1_R m = m$,
(5) $re_M = e_M = e_R m$

for any $r, r_1, r_2 \in R$ and $m, m_1, m_2 \in M$. In this article, e denotes the unit semimodule. For the max-plus algebra, the mapping μ is an isotone mapping, that is $\mu(r_1, m_1) \leq \mu(r_2, m_2)$ if $r_1 \leq r_2$ and $m_1 \leq m_2$. A *sub-semimodule* K of M is a submonoid of M with $rk \in K$ for all $r \in R$ with $k \in K$. A sub-semimodule K of M is called *subtractive* if $k \in K$ and $k \Box_M m \in K$ imply $m \in K$ for $m \in M$. An R-morphism between two semimodules $(M, \Box_M, e_M)$ and $(N, \Box_N, e_N)$ is a map $f: M \to N$ satisfying

(1) $f(m_1 \Box_M m_2) = f(m_1) \Box_N f(m_2)$,
(2) $f(rm) = rf(m)$

for all $m, m_1, m_2 \in M$ and $r \in R$.

2.2. *Kernels, images and Bourne equivalence relation*

There are two different kernels for R-semimodule morphisms. The kernel of an R-semimodule morphism $f: M \to N$ is defined as $\mathrm{Ker} f = \{x \in M | f(x) = e_N\}$.

The equivalence kernel is an equivalence relation defined by

$$\mathrm{Ker}_{\mathrm{eq}} f = \{(x_1, x_2) \in M \times M | f(x_1) = f(x_2)\}.$$

For the module case, any element (x_1, x_2) in the equivalence kernel, $\mathrm{Ker}_{\mathrm{eq}} f$, then there exists an element $k \in \mathrm{Ker} f$ such that $x_1 = x_2 + k$. This conclusion does not hold for the semimodule case without subtraction.

There are two different images for R-semimodule morphisms (Takahashi 1981). Given an R-semimodule morphism $f: M \to N$, one image is defined to be the set of all values $f(m)$, $m \in M$, i.e.

$$f(M) = \{n \in N | n = f(m), \text{ for some } m \in M\}. \quad (1)$$

It is called the *proper image* of an R-semimodule morphism f. The other image of f is defined as

$$\mathrm{Im} f = \{n \in N | n \square_N f(m) = f(m') \text{ for some } m, m' \in M\}. \quad (2)$$

It is called the *image* of f to distinguish from the proper image. In the max-plus algebra, if $f(m) \leq f(m')$, then $\mathrm{Im} f$ is not empty. The two images coincide for the module case. For the semimodule case, if the two images are the same, the R-morphism of semimodules $f: M \to N$ is called *i-regular* or *image-regular*.

The Bourne relation, introduced in Golan (1999, pp. 164), is an equivalence relation for an R-semimodule. If K is a sub-semimodule of an R-semimodule M, then the *Bourne relation* is defined by setting $m \equiv_K m'$ if and only if there exist two elements k and k' of K such that $m \square_M k = m' \square_M k'$. In the module case, the Bourne relation is usually defined as $m \equiv_K m'$ if and only if there exists one element k in K such that $m = m' + k'$ because of the existence of the subtraction operation. The factor semimodule $M/\equiv_K$ induced by $\equiv_K$ is also written as M/K. If K is equal to the kernel of an R-semimodule morphism $f: M \to N$, then $m \equiv_{\mathrm{Ker} f} m'$ if and only if there exist two elements k, k' of $\mathrm{Ker} f$, such that $m \square_M k = m' \square_M k'$. Applying f on both sides, we obtain that $f(m) = f(m')$. In other words, m and m' are equivalent with respect to the morphism f, i.e. $m \equiv_f m'$. Hence, this special Bourne relation and the equivalence relation induced by the morphism f satisfy the partial order $\leq$, i.e. $\equiv_{\mathrm{Ker} f} \leq \equiv_f$. In general, we do not have $\equiv_f \leq \equiv_{\mathrm{Ker} f}$ for an R-semimodule morphism f. If an R-semimodule morphism $f: M \to N$ satisfies $\equiv_f \leq \equiv_{\mathrm{Ker} f}$, then f is called a *steady* or *k-regular* R-semimodule morphism. In the module case, the image equivalence is the same as the kernel equivalence because $f(m) = f(m')$ implies $f(m - m') = 0$, that is $m - m' \in \mathrm{Ker} f$, i.e. $m = m' + k$, where $k \in \mathrm{Ker} f$.

2.3. *Factor theorem*

Factor theorem (Al-Thani 1996, p. 50) is used often in this article, so it is included here.

Theorem 2.1: *Let M, M', N and N' be left R-semimodules and let $f: M \to N$ be an R-semimodule morphism.*

(1) *If $g: M \to M'$ is a surjective k-regular R-semimodule morphism with $\mathrm{Ker}\, g \subset \mathrm{Ker} f$, then there exists a unique R-semimodule morphism $h: M' \to N$ such that $f = h \circ g$. Moreover, if f is injective then h is also injective. Also, $\mathrm{Ker}\, h = g(\mathrm{Ker} f)$, $\mathrm{Im}\, h = \mathrm{Im} f$ and $f(M) = h(M')$, so that h has a unit kernel, i.e. $\mathrm{Ker}\, h = e_{M'}$, if and only if $\mathrm{Ker}\, g = \mathrm{Ker} f$. h is surjective if and only if f is surjective* (*See* (1) in *Figure* 1).

(2) *If $g: N' \to N$ is an i-regular injective R-semimodule morphism with $\mathrm{Im} f \subset \mathrm{Im}\, g$, then there exists a unique R-semimodule morphism $h: M \to N'$ such that $f = g \circ h$. Moreover, $\mathrm{Ker}\, h = \mathrm{Ker} f$ and $\mathrm{Im}\, h = g^{-1}(\mathrm{Im} f)$, h is injective if and only if f is injective, h is surjective if and only if $g(N') = f(M)$* (*See* (2) *in Figure* 1).

2.4. *Max-plus linear systems*

Max-plus linear systems over the max-plus algebra $\mathbb{R}_{\mathrm{Max}}$ are described by the following equations:

$$\begin{aligned} x(k+1) &= A \otimes x(k) \oplus B \otimes u(k), \\ y(k) &= C \otimes x(k) \oplus D \otimes u(k), \end{aligned} \quad (3)$$

where x is in the state semimodule $X \cong \mathbb{R}^n_{\mathrm{Max}}$, y is in the output semimodule $Y \cong \mathbb{R}^m_{\mathrm{Max}}$ and u is in the input semimodule $U \cong \mathbb{R}^r_{\mathrm{Max}}$. $A: X \to X$, $B: U \to X$, $C: X \to Y$ and $D: U \to Y$ are four R-semimodule morphisms. $R[z]$ denotes the polynomial semiring with coefficients in R, and ΩX, ΩU and ΩY denote the polynomial $R[z]$-semimodules of states, inputs and outputs, respectively. The operator ΩX is an alternative notation for the polynomial $R[z]$-semimodules $X[z]$ of states. For instance, given a sequence of states,

$$\{\ldots, x(-2), x(-1), x(0), x(1), x(2), \ldots\},$$

$\Omega X = x(0) \oplus x(-1)z \oplus x(-2)z^2 \oplus \cdots \oplus x(-k)z^k$, for a finite $k \in \mathbb{N}$, is isomorphic to a sequence of states

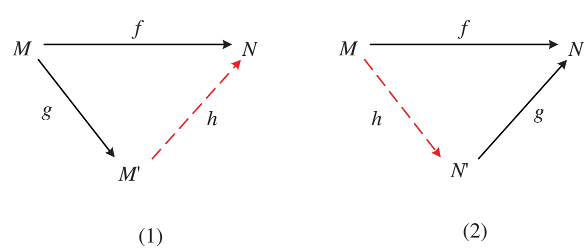

Figure 1. Triangle diagrams in Theorem 2.1.

starting from the time instant $-k$ and ending at the time instant 0. The z-transform is analogical to γ-transform in the max-plus linear systems literature. This article adopts z-transform from Kalman's abstract system theory (Kalman, Falb, and Arbib 1969), where the indeterminant z may be viewed as a time-marker instead of a complex variable. Let $R(z)$ denote the set of formal Laurent series in z^{-1}, with coefficients in R. In like manner, let $X(z)$ denote the set of formal Laurent series in z^{-1}, with coefficients in X. We define $U(z)$ and $Y(z)$ similarly to $X(z)$. The transfer function $G(z): U(z) \to Y(z)$ of the system in Equaion (3) is

$$\begin{aligned} G(z) &= C(z^{-1}A)^* Bz^{-1} \oplus D \\ &= CBz^{-1} \oplus CABz^{-2} \oplus CA^2Bz^{-3} \oplus \cdots \oplus D. \end{aligned} \quad (4)$$

The star operator A^* for an $n \times n$ matrix mapping $A: X \to X$ is defined as

$$A^* = I_{n\times n} \oplus A \oplus \cdots \oplus A^n \oplus \cdots, \quad (5)$$

where $I_{n\times n}$ denotes the identify matrix mapping from X to X. Without loss of generality, we assume that $D=0$ for a given transfer function $G(z): U(z) \to Y(z)$ in this article.

3. Generalisation of Kalman's algebraic system theory to max-plus linear systems

This section generalises the Kalman's algebraic theory from systems over fields in Kalman et al. (1969) to max-plus linear systems. Kalman defined the linear, zero-state, input/output map over a field as a map from the input sequence, which starts from the past, ends at zero time instant and remains zero for future time instants, to an output sequence, which has zero values in the past and starts to have non-zero values at zero time instant and future time instants. The input sequence is denoted as the set of polynomial inputs, $\Omega U = U[z]$. The output semimodule is defined as $\Gamma Y = Y(z)/\Omega Y$, which is induced from the Bourne relation. For any $y_1(z)$, $y_2(z) \in Y(z)$, they can be written as the polynomial component y_p^i combines with the strictly proper part y_{sp}^i, for $i=1$, 2, i.e. $y_1(z) = y_p^1 \oplus y_{sp}^1$ and $y_2(z) = y_p^2 \oplus y_{sp}^2$. $y_1(z)$ is equivalent as $y_1(z)$ by the Bourne relation means that there exists a pair of polynomial outputs $(\bar{y}_p^1, \bar{y}_p^2)$ such that $y_p^1 \oplus y_{sp}^1 \oplus \bar{y}_p^1 = y_p^2 \oplus y_{sp}^2 \oplus \bar{y}_p^2$. By selecting $\bar{y}_p^1 = y_p^2$ and $\bar{y}_p^2 = y_p^1$, the equality is satisfied if and only if the two strictly proper components y_{sp}^1 and y_{sp}^2 are the same. Therefore, the factor semimodule ΓY consists of equivalence classes having the same strictly proper component.

Kalman's input/output map automatically build causality in the formulation because the input and output functions are defined on two disjoint subsets of $(-\infty, 0]$ and $[1, \infty)$ of the integers. The Kalman input/output map, denoted as $G^{\#}(z)$, can be constructed from the commutative diagram shown in Figure 2, where i is the insertion and p is the natural projection. In addition to the input/output map, Kalman defined the state space as a module by the following proposition.

Proposition 3.1 (Proposition (5.1) in Kalman et al. (1969)**):** *Let $A: V \to V$ be an arbitrary K-endomorphism of K-vector space V. Then V admits the structure of a K[z]-module with scalar multiplication*

$$(\pi, v) \mapsto \pi \cdot v = \pi(A)v, \quad \pi \in K[z], \quad v \in V,$$

where $\pi(A)$ is 'A substituted into π' that is, $\pi(A)$ is the endomorphism $\pi_0 I + \cdots + \pi_n A^n$ and $\pi(A)v$ is the usual action of the endomorphism $\pi(A)$ on v.

Conversely, given any K[z]-module V, the K[z]-endomorphism $v \mapsto z \cdot v$ induces the K-endomorphism $v \mapsto Av = z \cdot v$.

Kalman's algebraic system theory characterises a canonical or minimal realisation in Kalman et al. (1969) by the property that is reachable from the unit element and observable with respect to the unit input. A realisation (A, B, C) of a transfer function $G(z)$ is *canonical* or *minimal* if and only if it is both reachable from the unit element and observable with respect to the unit input. In other words, if a transfer function can be factored through X by an onto map g and a one-to-one map h, then X is a canonical realisation for the given transfer function. If g is an onto map, but h is not a one-to-one map, then X is called a reachable realisation. On the other hand, if h is a one-to-one map, but g is an onto map, then X is called an observable realisation. The advantages of Kalman's theory include that the frequency-domain method and the state-variable method are merged into a single framework, linear systems over fields are special cases of this theory, and it provides new computational methods for important controlled invariant sets. Because Kalman's realisation theory does not demand operation inverses, it overcomes the difficulty of lack of the subtraction operation for systems over semirings. Moreover, Kalman's realisation characterises the canonical realisation using reachability and observability without the concept of dimension.

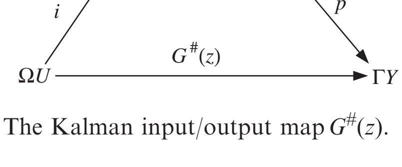

Figure 2. The Kalman input/output map $G^{\#}(z)$.

3.1. *Pole semimodules*

Researchers, Conte and Perdon, Wyman and Sain, continued the study of the Kalman's abstract system theory and defined two isomorphic pole modules to the state module. This article defines pole semimodules similarly as the module case, for the transfer function $G(z)$: $U(z) \to Y(z)$: the pole semimodule of output type as

$$X_O(G(z)) = \frac{G(\Omega U) \oplus \Omega Y}{\Omega Y}, \qquad (6)$$

and the pole semimodule of input type as

$$X_I(G(z)) = \frac{\Omega U}{G^{-1}(\Omega Y) \cap \Omega U}. \qquad (7)$$

$X_I(G(z))$ is actually equal to $\Omega U/\mathrm{Ker}\, G^{\#}$, because $\mathrm{Ker}\, G^{\#} = G^{-1}(\Omega Y) \cap \Omega U$. The quotient structures are induced by the Bourne relation, therefore, any equivalence classes in the pole semimodule of output type have the same strictly proper part in the output sequences. Hence, the pole semimodule of output type can be understood as output signals with strictly proper components produced by polynomial outputs, therefore, the strictly proper components are the pole elements in the plant transfer function. Similarly, the pole semimodule of input type can be understood as polynomial inputs which produce the outputs with strictly proper parts by removing the polynomial outputs produced by the polynomial inputs.

In the module case, each pole module has been used by the preference of the researchers, because $X_I(G(z))$ is isomorphic to $X_O(G(z))$. However, for the semimodule case, there exists an $R[z]$-semiisomorphism between $X_I(G(z))$ and $X_O(G(z))$ instead, i.e. an unit kernel $R[z]$-semimodule epimorphism. Moreover, with the steady assumption on the Kalman input/output map, the two pole semimodules become isomorphic to each other in the semimodule case.

Lemma 3.2: *Given a transfer function* $G(z)$: $U(z) \to Y(z)$ *and the pole semimodules of input and output type as shown in Equations* (6) *and* (7), *there exists an* $R[z]$-*semimodule semiisomorphism* $\overline{G}(z)$ *from* $X_I(G(z))$ *to* $X_O(G(z))$. *The semiisomorphism* $\overline{G}(z)$ *becomes an isomorphism if and only if* $G^{\#}(z)$ *is steady or k-regular*.

Proof: We need to prove that there exists an $R[z]$-semimodule morphism $\overline{G}(z) : X_I(G(z)) \to X_O(G(z))$ such that the following diagram is commutative.

$$\begin{array}{ccccccc}
 & & \Omega U & \xrightarrow{G(z)} & G(\Omega U) & & \\
 & & \downarrow p_1 & & \downarrow p_2 & & \\
e & \longrightarrow & \frac{\Omega U}{G^{-1}(\Omega Y)\cap\Omega U} & \xrightarrow{\overline{G}(z)} & \frac{G(\Omega U)\oplus\Omega Y}{\Omega Y} & \longrightarrow & e
\end{array}$$

The projection p_1 is k-regular by Lemma 1 in Shang and Sain (2005). Using the Factor theorem (Al-Thani 1996, p. 50), if $\mathrm{Ker}\, p_1 \subset \mathrm{Ker}(p_2 \circ G)$, then there exists a unique $R[z]$-semimodule morphism $\overline{G}(z)$ such that the diagram is commutative. Proving $\mathrm{Ker}\, p_1 \subset \mathrm{Ker}(p_2 \circ G)$ only needs the definition of kernel, i.e.

$$\begin{aligned}
\mathrm{Ker}\, p_1 &= \{u \in \Omega U | u \equiv_{G^{-1}(\Omega Y)\cap\Omega U} e_{U(z)}\} \\
&= \{u \in \Omega U | u \oplus u_1 = u_2, \\
&\qquad \exists u_1, u_2 \in G^{-1}(\Omega Y) \cap \Omega U\} \\
&= G^{-1}(\Omega Y) \cap \Omega U, \\
\mathrm{Ker}(p_2 \circ G) &= \{u \in \Omega U | G(u) \equiv_{G(\Omega U)\cap\Omega Y} e_Y\} \\
&= \{u \in \Omega U | G(u) \oplus y_1 = y_2, \exists y_1, y_2 \in \Omega Y\}, \\
&= G^{-1}(\Omega Y) \cap \Omega U,
\end{aligned}$$

where the last equalities of both kernels hold because $G^{-1}(\Omega Y) \cap \Omega U$ is subtractive. Not only $\mathrm{Ker}\, p_1 \subset \mathrm{Ker}(p_2 \circ G)$ is satisfied, they are actually equal to each other. Therefore, there exists a unique $R[z]$-semimodule morphism $\overline{G}(z)$ such that the diagram is commutative. The morphism $\overline{G}(z)$ is surjective because $p_2 \circ G(z)$ is surjective. The reason for the morphism $\overline{G}(z)$ having the unit kernel is the following:

$$\begin{aligned}
\mathrm{Ker}\, \overline{G}(z) &= \left\{ \frac{u}{G^{-1}(\Omega Y) \cap \Omega U}, u \in \Omega U \middle| \frac{G(u)}{\Omega Y} = e_{X_O(G(z))} \right\} \\
&= \left\{ \frac{u}{G^{-1}(\Omega Y) \cap \Omega U}, u \in \Omega U \middle| G(u) \oplus y_1 = y_2, \right. \\
&\qquad \left. y_1, y_2 \in \Omega Y \right\} \\
&= G^{-1}(\Omega Y) \cap \Omega U. \qquad (8)
\end{aligned}$$

When $G^{\#}(z)$ is steady or k-regular, for any $u_1, u_2 \in \Omega U$, we have

$$\begin{aligned}
&\overline{G}\left(\frac{u_1}{G^{-1}(\Omega Y)\cap\Omega U}\right) = \overline{G}\left(\frac{u_2}{G^{-1}(\Omega Y)\cap\Omega U}\right) \\
&\qquad \Longrightarrow G^{\#}u_1 = G^{\#}u_2 \\
&\qquad \Longrightarrow u_1 \oplus u_p^1 = u_2 \oplus u_p^2, \\
&\qquad u_p^1, u_p^2 \in G^{-1}(\Omega Y)\cap\Omega U \\
&\qquad \Longrightarrow \frac{u_1}{G^{-1}(\Omega Y)\cap\Omega U} = \frac{u_2}{G^{-1}(\Omega Y)\cap\Omega U}.
\end{aligned}$$

Therefore, if $G^{\#}(z)$ is steady, the pole semimodule of input and output types become isomorphic to each other. In order to prove the 'only if' part of the steady assumption, when $\bar{G}(z)$ is a one-on-one map, then we have

$$\begin{aligned}
&G^{\#}u_1 = G^{\#}u_2 \\
&\quad \Longrightarrow \frac{G(u_1)}{\Omega Y} = \frac{G(u_1)}{\Omega Y} \\
&\quad \Longrightarrow \overline{G}\left(\frac{u_1}{G^{-1}(\Omega Y) \cap \Omega U}\right) = \overline{G}\left(\frac{u_2}{G^{-1}(\Omega Y) \cap \Omega U}\right) \\
&\quad \Longrightarrow \frac{u_1}{G^{-1}(\Omega Y) \cap \Omega U} = \frac{u_2}{G^{-1}(\Omega Y) \cap \Omega U} \\
&\quad \Longrightarrow u_1 \oplus u_p^1 = u_2 \oplus u_p^2, u_p^1, u_p^2 \in G^{-1}(\Omega Y) \cap \Omega U.
\end{aligned}$$

Therefore, $G^{\#}$ is a steady or k-regular by definition. □

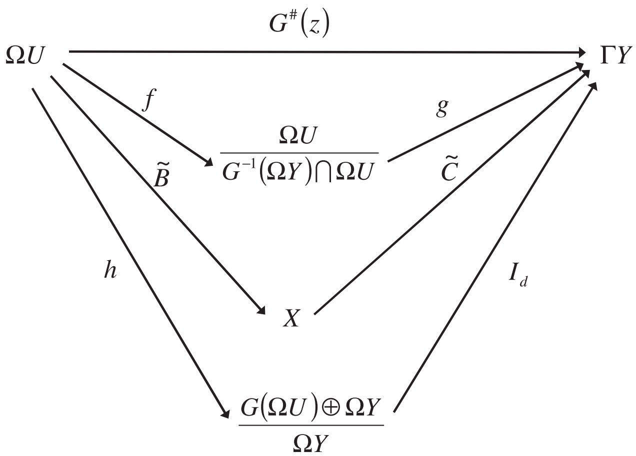

Figure 3. The generalised Kalman realisation diagram.

3.2. *Generalisation of Kalman's abstract realisation theory*

Kalman's realisation theory can be extended to the semimodule case with some modifications. If considering the Kalman input/output map $G^{\#}(z): \Omega U \to \Gamma Y$, then we can obtain the commutative diagram shown in Figure 2. The mappings $\widetilde{B}: \Omega U \to X$ and $\widetilde{C}: X \to \Gamma Y$ are defined using the Bourne equivalence relation as

$$\widetilde{B}(z \cdot u) = ABu; \tag{9}$$

$$\widetilde{C}(x) = \frac{Cxz^{-1} \oplus C(Ax)z^{-2} \oplus C(A^2x)z^{-3} \oplus \cdots}{\Omega Y}. \tag{10}$$

The Kalman realisation diagram for the max-plus linear systems is illustrated in Figure 3. For the pole semimodule of output type $X_O(G(z))$, the mapping h is defined by the action $h: u \mapsto \frac{G(u)\oplus\Omega Y}{\Omega Y}$, where $u \in \Omega U$. The identity mapping I_d is defined by the action $I_d: \frac{G(u)\oplus\Omega Y}{\Omega Y} \mapsto \frac{G(u)}{\Omega Y}$. For the pole semimodule of input type $X_I(G(z))$, the mapping f is a natural projection, i.e. $f: u \mapsto \frac{u}{G^{-1}(\Omega Y)\cap\Omega U}$ for $u \in \Omega U$. The mapping g is defined by the action $g: \frac{u}{G^{-1}(\Omega Y)\cap\Omega U} \mapsto \frac{G(u)}{\Omega Y}$.

In the module case, the pole modules are isomorphic to the state space X (Wyman and Sain 1981) in a canonical realisation. The differences between the new abstract realisation diagram to the traditional Kalman's realisation diagram are: the quotient structure is the Bourne relation and the pole semimodule of input type $X_I(G(z))$ is a reachable but not observable realisation of $G^{\#}(z)$, because f is onto and g is not one-to-one. Using Lemma 3.2, the pole semimodule of input type becomes a controllable and observable realisation if and only if the Kalman input/output map $G^{\#}(z)$ is steady.

The pole semimodule of output type $X_O(G(z))$ is a canonical realisation of the Kalman input/output map, because $G^{\#}(z)$ is an onto map from ΩU to $X_O(G(z))$. The map I_d is one-to-one because it is the induced map from the identity map, i.e. for any element $\frac{G(u_1)\oplus\Omega Y}{\Omega Y}$, there is only one image of I_d that is $\frac{G(u_1)}{\Omega Y}$. Therefore, the pole semimodule of output type is isomorphic to the canonical state module X, when $\widetilde{B}$ is epic and $\widetilde{C}$ is one-to-one, i.e. the realisation matrices (A, B, C) are controllable and observable. At least, it is isomorphic to the canonical component of X, represented by $\mathrm{Im}_{\mathrm{proper}}\,\mathcal{R}/\mathrm{Ker}_{\mathrm{eq}}\,\mathcal{O}$, where $\mathcal{R}$ is the controllability matrix

$$[B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B \quad \cdots],$$

and $\mathcal{O}$ is the observability matrix

$$[C^T \quad A^TC^T \quad (A^T)^2C^T \quad \cdots \quad (A^T)^{n-1}C^T \quad \cdots]^T.$$

The structure can be interpreted as the quotient of the reachable space by the equivalence kernel of the observability matrix because the two kernels and the two images are different in the semimodule case. Specifically, two elements x_1 and x_2 in $\mathrm{Im}_{\mathrm{proper}}\,\mathcal{R}$, $x_1 \equiv_{\mathrm{Ker}_{\mathrm{eq}}\mathcal{O}} x_2$ implies that $\mathcal{O}x_1 = \mathcal{O}x_2$. Usually, x_1 cannot be represented as $x_2 + k$ where $k \in \mathrm{Ker}\,\mathcal{O}$ like in the module case. Unless $\mathcal{O}$ is steady or k-regular, then $x_1 \oplus k_1 = x_2 \oplus k_2$ where k_1 and k_2 are in the kernel of $\mathcal{O}$.

Therefore, Kalman's original realisation theory for systems over fields, specifically max-plus linear systems, is generalised to systems over semirings, even there is no subtraction operation at present. By revisiting Kalman's realisation theory, we are able to define the pole structures in terms of semimodules instead of point poles, which cannot be traditionally defined without the subtractions operation. The pole semimodule of input type can be understood as inputs without poles producing outputs with poles. The pole semimodule of output type can be understood as outputs with poles produced by inputs without poles. The poles in the output come from the poles of the given plant, therefore, it helps the discoveries of the poles in the plant transfer function. Moreover, pole semimodules can generate the state matrix A for max-plus linear systems, which is related to the eigenvalues or point poles of the given system. The pole structures provide alternative computational methods for different controlled invariant sets often used in various geometric control problems.

3.3. *Numerical example*

The following example illustrates how to relate the pole semimodule of output type, which is a canonical realisation of the transfer function $G(z)$, to a more concrete representation of the state space realisation.

Example 3.3: Consider a transfer function $G(z): U(z) \to Y(z)$ over the max-plus algebra $\mathbb{R}_{\text{Max}}$ as $G(z) = \begin{bmatrix} z^{-2} & \epsilon \\ \epsilon & \epsilon \end{bmatrix}$. The construction for the pole semimodule of output type $X_O(G(z)) = \frac{G(\Omega U) \oplus \Omega Y}{\Omega Y}$ can be intuitively understood as the set of $G(\Omega U) \oplus \Omega Y$ by removing the kernel of ΩY. The reason is that the Bourne equivalence class induced by ΩY is a set of elements in $G(\Omega U)$ having the same strictly proper parts. For arbitrary control input $u = [u_1 \ \ u_2]^T \in \Omega U^2$, where $u_1 = v_0 \oplus v_1 z \oplus v_2 z \cdots$ and $u_2 = w_0 \oplus w_1 z \oplus w_2 z \cdots$, we can represent $G(\Omega U)$ as

$$G(\Omega U) = \left\{ \begin{bmatrix} z^{-2} & \epsilon \\ \epsilon & \epsilon \end{bmatrix} \begin{bmatrix} v_0 \oplus v_1 z \oplus v_2 z \cdots \\ w_0 \oplus w_1 z \oplus w_2 z \cdots \end{bmatrix} \right\} = \left\{ \begin{bmatrix} v_0 z^{-2} \oplus v_1 z^{-1} \oplus v_2 \oplus \cdots \\ \epsilon \end{bmatrix} \right\}. \tag{11}$$

The pole semimodule of output type is a quotient structure induced from the polynomial output by the Bourne relation, it can be understood as removing the polynomial output ΩY from $G(\Omega U)$. The two basis elements are obtained from the strictly proper part in the set of $G(\Omega U)$ in the previous equation:

$$e_1 = \begin{bmatrix} z^{-2} \\ \epsilon \end{bmatrix}, \quad e_2 = \begin{bmatrix} z^{-1} \\ \epsilon \end{bmatrix}.$$

We can understand the equivalence classes in the pole semimodule of output type are generated from the first two elements of Equaion (11). Consider an operation by z upon the two basis vectors:

$$\begin{bmatrix} z^{-2} \\ \epsilon \end{bmatrix} \cdot z = \begin{bmatrix} z^{-1} \\ \epsilon \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} z^{-1} \\ \epsilon \end{bmatrix} \cdot z = \begin{bmatrix} e \\ \epsilon \end{bmatrix}.$$

When applying z to the basis vectors, whenever polynomial elements appear, they will be dropped because polynomial components are zero elements in the pole semimodule. For a representative in $X_O(G(z))$, the scalar multiplication of z will be mapped as matrix multiplication A in the state semimodule X.

$$[e_1 e_2] \cdot z = [e_1 e_2] A \Longrightarrow A = \begin{bmatrix} \epsilon & \epsilon \\ e & \epsilon \end{bmatrix}.$$

The B and C matrices can also be obtained as

$$B = \begin{bmatrix} e & \epsilon \\ \epsilon & \epsilon \end{bmatrix}, \quad C = \begin{bmatrix} \epsilon & e \\ \epsilon & \epsilon \end{bmatrix}.$$

Moreover, the realisation (A, B, C) is a canonical realisation, i.e. both reachable and observable. In summary, this example illustrates how to construct the pole semimodule of output type and demonstrates the connection between the pole semimodule of the output type and the matrix A in the state semimodule which can result in a canonical realisation for the transfer function.

4. Controlled invariance

This section presents different computational methods for the controlled invariant semimodules for max-plus linear systems. These controlled invariant semimodules, generalised from the traditional geometric control theory (Wonham 1979; Conte and Perdon 1998), are often used in the solvability conditions of the disturbance decoupling problem, the block decoupling problem, and the model matching problem. The main results in this section are motivated from Wyman and Sain (1983) for systems over fields.

4.1. *(A, B)-invariant sub-semimodules*

Given the max-plus linear system (3), a sub-semimodule V of the state semimodule X is called

- (A, B)-*invariant*, or *controlled invariant*, if and only if, for all $x_0 \in V$, there exists a sequence of control inputs, $\underline{u} = \{u_1, u_2, \ldots\}$, such that every component in the state trajectory produced by this input, $\underline{x}(x_0; \underline{u}) = \{x_0, x_1, \ldots\}$, remains inside of V.
- called (A, B)-*invariant of feedback type* if and only if there exists a state feedback $F: X \to U$ such that $(A \oplus BF)V \subset V$.

Unlike for systems over fields or even rings, (A, B)-invariant sub-semimodules of feedback type are not identical as (A, B)-invariant sub-semimodules for systems over semirings. Recall that, in the module case, a submodule V of the state module X is (A, B)-invariant if and only if $AV \subset V + \mathcal{B}$, where $\mathcal{B} \triangleq B(U)$. However, due to lack of the subtraction operation, this condition needs modifications for systems over semirings. For max-plus linear systems, computational methods for (A, B)-controlled invariant sub-semimodules are established in Katz (2007). The following results summarise that the computational methods for (A, B)-controlled invariant sub-semimodules in Katz (2007).

Lemma 4.1 (Katz 2007): *A sub-semimodule V of the state semimodule X is an (A, B)-invariant sub-semimodule if and only if $AV \subset V \widehat{\oplus} \mathcal{B}$, where*

$$V \widehat{\oplus} \mathcal{B} \triangleq \{x \in X | \exists b \in \mathcal{B}, \ s.t. \ x \oplus b \in V\},$$

and $\mathcal{B} \triangleq B(U)$.

Lemma 4.2 (Katz 2007): *V is an (A,B)-invariant sub-semimodule of a sub-semimodule $\mathcal{K}$ of X if and only if*

$$V = V \cap A^{-1}(V \widehat{\oplus} \mathcal{B}), \tag{12}$$

where A^{-1} is the set inverse map of $A: X \to X$.

The family of the controlled invariant sub-semimodules in a sub-semimodule $\mathcal{K}$ of X is closed under the operation $\oplus$. It is a upper semilattice relative to sub-semimodule inclusion $\subset$. Therefore, there exists the *supremal* element $\mathcal{V}^*$ in the family of controlled invariant sub-semimodules in a sub-semimodule $\mathcal{K}$ of the state semimodule X and it can be computed by the following algorithm.

Theorem 4.3 (Katz 2007): *Let $\{V_k\}_{k\geq 0}$ be the family of sub-semimodules defined recursively by*

$$\begin{aligned} V_0 &= \mathcal{K} \\ V_{k+1} &= V_k \cap A^{-1}(V_k \widehat{\oplus} \mathcal{B}). \end{aligned} \tag{13}$$

If there exists a nonempty $\cap_{k\in\mathbb{N}} V_k$, then any (A,B)-invariant sub-semimodule of $\mathcal{K}$ is contained in $\cap_{k\in\mathbb{N}} V_k$, namely the supremal controlled invariant sub-semimodule $\mathcal{V}^*$ is also contained in $\cap_{k\in\mathbb{N}} V_k$. Moreover, if the algorithm in Equation (13) *terminates in r steps, then $\mathcal{V}^* = V_r$.*

Because the null kernel for max-plus linear systems is not as crucial as the equivalence kernel in geometric control theory, this article introduces another (A,B)-invariant sub-semimodule in the equivalence kernel of an R_{Max}-morphism $P: X \to X$, which is a sub-semimodule of $X \times X$. Two elements x_1 and x_2 are in the equivalence kernel of the morphism P if and only if $P(x_1) = P(x_2)$. A sub-semimodule in the equivalence kernel of the morphism P is (A,B)-*invariant* or *controlled invariant* if and only if there exists a pair of control inputs such that $P(Ax_1 \oplus Bu_1) = P(Ax_2 \oplus Bu_2)$. In other words, there exists a pair of control inputs such that the states produced by the two inputs are still equivalent based on the morphism P. The computational methods of different invariant sub-semimodules will be studied in the next section.

4.2. *Supremal controlled invariant sub-semimodules in* $\ker C$ *and* $\ker_{\text{eq}} C$

Define an R-morphism $p': G^{-1}(\Omega Y) \to \Omega U$ as $p'(u(z)) = u_{\text{poly}}$, where $u(z) = u_{\text{poly}} \oplus u_{\text{sp}}$ is the sum of a polynomial component and a strictly proper component. Therefore, this mapping induces an R-morphism, $p_1: G^{-1}(\Omega Y) \to \text{Im}_{\text{proper}} \mathcal{R}$ as $p_1 = \widetilde{B} \circ p'$, where $\mathcal{R}$ is the controllability matrix for max-plus linear systems.

Proposition 4.4: *Given a max-plus linear system, the nonempty proper image of $p_1(G^{-1}(\Omega Y))$ is the supremal (A,B)-invariant sub-semimodule $\mathcal{V}^*$ in* $\text{Ker}\, C \cap \text{Im}_{\text{proper}} \mathcal{R}$. *If the max-plus linear system is reachable, i.e. $X = \text{Im}_{\text{proper}} \mathcal{R}$, then the nonempty proper image of $p_1(G^{-1}(\Omega Y))$ is the supremal controlled invariant sub-semimodule $\mathcal{V}^*$ in* $\text{Ker}\, C$.

Proof: In order to prove that $p_1(G^{-1}(\Omega Y))$ is the supremal controlled invariant sub-semimodule $\mathcal{V}^*$ in $\text{Ker}\, C \cap \text{Im}_{\text{proper}} \mathcal{R}$, we need three steps: $p_1(G^{-1}(\Omega Y))$ is contained in $\text{Ker}\, C \cap \text{Im}_{\text{proper}} \mathcal{R}$ and $p_1(G^{-1}(\Omega Y))$ is (A,B)-invariant, any (A,B)-invariant subsemimodule in $\text{Ker}\, C \cap \text{Im}_{\text{proper}} \mathcal{R}$ is contained in $p_1(G^{-1}(\Omega Y))$.

Step 1: $p_1(G^{-1}(\Omega Y)) \subset \text{Ker}\, C \cap \text{Im}_{\text{proper}} \mathcal{R}$.

In fact, because $p_1: G^{-1}(\Omega Y) \to \text{Im}_{\text{proper}} \mathcal{R}$, we only need to prove $p_1(G^{-1}(\Omega Y)) \subset \text{Ker}\, C$. Suppose that $u(z) \in G^{-1}(\Omega Y)$ and $u(z) = u_{\text{poly}} \oplus u_{\text{sp}}$, that is a combination of a polynomial component and a strictly proper component, then $G(z)u(z) = G(z)u_{\text{poly}} \oplus G(z)u_{\text{sp}}$ is an element in the polynomial output. Using the Kalman's realisation diagram for max-plus linear systems, we can construct $G(z)u_{\text{poly}}$ as

$$G(z)u_{\text{poly}} = y_{\text{poly}} \oplus (C \circ \widetilde{B}u_{\text{poly}}) \cdot z^{-1} \oplus (C \circ \widetilde{B}zu_{\text{poly}}) \cdot z^{-2} \ldots,$$

where $y_{\text{poly}} \in \Omega Y$. Because $G(z)u(z) \in \Omega Y$ and $G(z)u_{\text{sp}}$ has no z^{-1} terms, it implies that $G(z)u_{\text{poly}}$ also has no z^{-1} terms, that is $C \circ \widetilde{B}u_{\text{poly}} = C(p_1(u(z))) = \epsilon$. So $\widetilde{B}u_{\text{poly}}$ is in $\text{Ker}\, C$.

Step 2: $p_1(G^{-1}(\Omega Y))$ is (A,B)-invariant, that is $p_1(G^{-1}(\Omega Y)) \subset \mathcal{V}^*$.

Suppose x in $p_1(G^{-1}(\Omega Y))$, namely $x = p_1(u(z)) = \widetilde{B}u_{\text{poly}}$ and there exists a strictly proper element in $U(z)$ such that

$$u(z) = u_{\text{poly}} \oplus u_1 z^{-1} \oplus u_2 z^{-2} \oplus u_3 z^{-3} \cdots \in G^{-1}(\Omega Y).$$

If we pick u as u_1, which is the coefficient for z^{-1}, then $Ax \oplus Bu = \widetilde{B}zu_{\text{poly}} \oplus \widetilde{B}u = \widetilde{B}(zu_{\text{poly}} \oplus u_1)$, where $\widetilde{B}(zu_{\text{poly}} \oplus u_1)$ is still an element in $p_1(G^{-1}(\Omega Y))$, because there exists a strictly proper element such that

$$(zu_{\text{poly}} \oplus u_1) \oplus u_2 z^{-1} \oplus u_3 z^{-2} \cdots \in G^{-1}(\Omega Y).$$

Therefore, $p_1(G^{-1}(\Omega Y))$ is (A,B)-invariant.

Step 3: Finally, we need to show that the supremal controlled invariant sub-semimodule $\mathcal{V}^*$ in $\text{Ker}\, C \cap \text{Im}_{\text{proper}} \mathcal{R}$ is contained in $p_1(G^{-1}(\Omega Y))$, i.e. $\mathcal{V}^* \subset p_1(G^{-1}(\Omega Y))$.

If any (A,B)-invariant sub-semimodule V in $\text{Ker}\, C \cap \text{Im}_{\text{proper}} \mathcal{R}$ is contained in $p_1(G^{-1}(\Omega Y))$; then the supremal invariant sub-semimodule $\mathcal{V}^*$ must be inside of $p_1(G^{-1}(\Omega Y))$. For an arbitrary element $x \in V$, because $V \subset \text{Range}[B\ AB\ A^2\ B\ \cdots]$, x can be represented as $\widetilde{B}u_{\text{poly}}$. We need to find a strictly proper element

$$u_{sp} = u_0 z^{-1} \oplus u_1 z^{-2} \oplus u_2 z^{-3} \ldots,$$

such that $G(z)(u_{\text{poly}} \oplus u_{\text{sp}}) \in \Omega Y$. The following equality holds directly from Kalman's realisation diagram,

$$\begin{aligned} G(z)u_{\text{poly}} &= y_{\text{poly}} \oplus (C \circ \widetilde{B}u_{\text{poly}}) \\ &\quad \cdot z^{-1} \oplus (C \circ \widetilde{B}zu_{\text{poly}}) \cdot z^{-2} \cdots \\ &= y_{\text{poly}} \oplus (Cx) \cdot z^{-1} \oplus (CAx) \\ &\quad \cdot z^{-2} \oplus (CA^2x) \cdot z^{-3} \cdots. \end{aligned}$$

Using the given strictly proper transfer function representation $G(z)$, we can obtain $G(z)u_{\text{sp}}$ as

$$\begin{aligned} G(z)u_{sp} &= (CBz^{-1} \oplus CABz^{-2} \oplus CA^2Bz^{-3} \cdots) \\ &\quad \cdot (u_0 z^{-1} \oplus u_1 z^{-2} \oplus u_2 z^{-3} \cdots) \\ &= (CBu_0) \cdot z^{-2} \oplus (CABu_0 \oplus CBu_1) \cdot z^{-3} \oplus \cdots. \end{aligned}$$

The last step is with the assumption of the distributive law of the operation $\oplus$ over $\oplus$. Sum up the last two equations to obtain

$$\begin{aligned} G(z)u(z) = G(z)(u_{\text{poly}} \oplus u_{sp}) &= y_{\text{poly}} \oplus (Cx) \cdot z^{-1} \\ &\oplus (C(Ax \oplus Bu_0)) \cdot z^{-2} \\ &\oplus (C(A^2x \oplus ABu_0 \oplus Bu_1)) \cdot z^{-3} \cdots. \end{aligned}$$

$G(z)u(z)$ produces a polynomial output if and only if each strictly proper coefficients is the unit element, namely

$$\begin{aligned} Cx &= e_Y, \\ C(Ax \oplus Bu_0) &= e_Y, \\ C(A^2x \oplus ABu_0 \oplus Bu_1) &= e_Y, \\ &\vdots \end{aligned}$$

$Cx = e_Y$ because $x \in V \subset \text{Ker}\, C$. Since V is an (A, B)-invariant sub-semimodule in $\text{Ker}\, C$, for any $x \in V$, there exists an input u_0 such that $x_1 = Ax \oplus Bu_0$ is still inside of $V \subset \text{Ker}\, C$, which means that $Cx_1 = e_Y$. To obtain this, we only need to pick the coefficient of z^{-1} of u_{sp} as such a u_0. For this x_1, there exists a new control u_1, such that $x_2 = Ax_1 \oplus Bu_1$ remains inside of V, which means that $C(x_2) = C(Ax_1 \oplus Bu_1) = C(A^2x \oplus ABu_0 \oplus Bu_1) = e$. To obtain this, we only need to pick the coefficient of z^{-2} of u_{sp} as such a u_1. So on and so forth, we can pick the coefficients of the strictly proper input u_{sp} such that $G(z)(u_{\text{poly}} \oplus u_{\text{sp}})$ is a polynomial output. This means that, for any $x = \widetilde{B}u_{\text{poly}}$, there exists a pre-image $u(z) \in G^{-1}(\Omega Y)$ of p_1, that is $x \in p_1(G^{-1}(\Omega Y))$. Hence, any (A, B)-invariant sub-semimodule V in is contained in $\mathcal{V}^*$ in $\text{Ker}\, C \cap \text{Im}_{\text{proper}}\, \mathcal{R}$, which implies $\mathcal{V}^* \subset p_1(G(z))$.

Combining these three steps, we proved that $\mathcal{V}^* = p_1(G^{-1}(\Omega Y))$ in $\text{Ker}\, C \cap \text{Im}_{\text{proper}}\, \mathcal{R}$. Specifically, if the max-plus linear system is reachable, therefore, we obtain $\mathcal{V}^* = p_1(G^{-1}(\Omega Y))$ in $\text{Ker}\, C$. □

Next, we will discuss the relationship between the factor semimodule of $\text{Im}_{\text{proper}}\, \mathcal{R}$ induced by the proper image of p_1, $\frac{\text{Im}_{\text{proper}}\mathcal{R}}{p_1(G^{-1}(\Omega Y))}$, and the factor semimodule induced by the equivalence kernel of C, $\frac{\text{Im}_{\text{proper}}\mathcal{R}}{\text{Ker}_{\text{eq}}C}$.

Proposition 4.5: *$\frac{\text{Im}_{\text{proper}}\mathcal{R}}{p_1(G^{-1}(\Omega Y))}$ is an (A, B)-invariant sub-semimodule of $\frac{\text{Im}_{\text{proper}}\mathcal{R}}{\text{Ker}_{\text{eq}}C}$.*

Proof: Considering a projection $\pi_1 : \text{Im}_{\text{proper}}\mathcal{R} \to \frac{\text{Im}_{\text{proper}}\mathcal{R}}{p_1(G^{-1}(\Omega Y))}$ based on the Bourne relation, then two elements x_1 and x_2 are equivalent to each other if $x_1 \oplus k_1 = x_1 \oplus k_2$, where k_1 and k_2 are in $p_1(G^{-1}(\Omega Y))$, i.e. $k_1 = \widetilde{B}u_p^1$ and $k_2 = \widetilde{B}u_p^2$, $G(u_p^1 \oplus u_{sp}^1) \in \Omega Y$, and $G(u_p^2 \oplus u_{sp}^2) \in \Omega Y$. Therefore, $Cx_1 \oplus Ck_1 = Cx_1 \oplus Ck_1$ implies that $Cx_1 = Cx_1$ because $p_1(G^{-1}(\Omega Y))$ is in kernel of C. In other words, $x_1 \equiv_{\text{Ker}_{\text{eq}}C} x_2$, so $\frac{\text{Im}_{\text{proper}}\mathcal{R}}{p_1(G^{-1}(\Omega Y))}$ is a sub-semimodule of $\frac{\text{Im}_{\text{proper}}\mathcal{R}}{\text{Ker}_{\text{eq}}C}$.

In order to prove that $\frac{\text{Im}_{\text{proper}}\mathcal{R}}{p_1(G^{-1}(\Omega Y))}$ is an (A, B)-invariant, we need to show that for a pair $x_1 \equiv_{p_1(G^{-1}(\Omega Y))} x_2$, we need to find a pair of control inputs v_1 and v_2 such that $Ax_1 \oplus Bv_1 \equiv_{p_1(G^{-1}(\Omega Y))} Ax_2 \oplus Bv_2$. Because $x_1 \equiv_{p_1(G^{-1}(\Omega Y))} x_2$ implies that $x_1 \oplus k_1 = x_1 \oplus k_2$, where k_1 and k_2 are in $p_1(G^{-1}(\Omega Y))$, then $Ax_1 \oplus Ak_1 = Ax_2 \oplus Ak_2$. Because $p_1(G^{-1}(\Omega Y))$ is (A, B)-invariant by Proposition 4.4, then there exists a pair of control inputs m_1 and m_2 such that $Ak_1 \oplus Bm_1$ and $Ak_2 \oplus Bm_2$ are still in $p_1(G^{-1}(\Omega Y))$. We can obtain the following equation

$$\begin{aligned} &Ax_1 \oplus Ak_1 \oplus Bm_1 \oplus Bm_2 \\ &\quad = Ax_2 \oplus Ak_2 \oplus Bm_1 \oplus Bm_2 \Longrightarrow \\ &(Ax_1 \oplus Bm_2) \oplus (Ak_1 \oplus Bm_1) \\ &\quad = (Ax_2 \oplus Bm_1) \oplus (Ak_2 \oplus Bm_2). \end{aligned}$$

Therefore, $Ax_1 \oplus Bv_1 \equiv_{p_1(G^{-1}(\Omega Y))} Ax_2 \oplus Bv_2$, where $v_1 = m_2$ and $v_2 = m_1$. In summary, $\frac{\text{Im}_{\text{proper}}\mathcal{R}}{p_1(G^{-1}(\Omega Y))}$ is an (A, B)-invariant sub-semimodule of $\frac{\text{Im}_{\text{proper}}\mathcal{R}}{\text{Ker}_{\text{eq}}C}$. □

Proposition 4.6: *$\frac{\text{Im}_{\text{proper}}\mathcal{R}}{p_1(G^{-1}(\Omega Y))}$ is the supremal (A, B)-invariant sub-semimodule of $\frac{\text{Im}_{\text{proper}}\mathcal{R}}{\text{Ker}_{\text{eq}}C}$ if the mapping $p \circ G(z) : U(z) \to \Gamma Y = \frac{Y(z)}{\Omega Y}$ on the Kalman's realisation diagram in Figure 3 is steady or k-regular.*

Proof: In order to prove that $\frac{\text{Im}_{\text{proper}}\mathcal{R}}{p_1(G^{-1}(\Omega Y))}$ is the supremal (A, B)-invariant sub-semimodule of $\frac{\text{Im}_{\text{proper}}\mathcal{R}}{\text{Ker}_{\text{eq}}C}$, we only need to show that for any (A, B)-invariant sub-semimodule in $\frac{\text{Im}_{\text{proper}}\mathcal{R}}{\text{Ker}_{\text{eq}}C}$, it is contained in $\frac{\text{Im}_{\text{proper}}\mathcal{R}}{p_1(G^{-1}(\Omega Y))}$.

For any pair $(x_1 = \widetilde{B}u^1_{\text{poly}}, x_2 = \widetilde{B}u^2_{\text{poly}})$ in $\frac{\text{Im}_{\text{proper}}\mathcal{R}}{\text{Ker}_{\text{eq}}C}$, $x_1 \equiv_{\text{Ker}_{\text{eq}}C} x_2$ implies that $Cx_1 = Cx_2$. If the pair (x_1, x_2) is also contained in an (A, B)-invariant sub-semimodule in $\frac{\text{Im}_{\text{proper}}\mathcal{R}}{\text{Ker}_{\text{eq}}C}$, it means that there exists sequences of inputs such that the following equations are satisfied:

$$\begin{aligned} Cx_1 &= Cx_2,\\ C(Ax_1 \oplus Bu^1_0) &= C(Ax_2 \oplus Bu^2_0),\\ C(A^2x_1 \oplus ABu^1_0 \oplus Bu^1_1) &= C(A^2x_2 \oplus ABu^2_0 \oplus Bu^2_1),\\ &\vdots \end{aligned}$$

Based on the Kalman's realisation diagram, we have

$$\begin{aligned} G(z)u_{\text{poly}} &= y_{\text{poly}} \oplus (C \circ \widetilde{B}u_{\text{poly}})\\ &\quad \cdot z^{-1} \oplus (C \circ \widetilde{B}zu_{\text{poly}}) \cdot z^{-2} \cdots\\ &= y_{\text{poly}} \oplus (Cx) \cdot z^{-1} \oplus (CAx)\\ &\quad \cdot z^{-2} \oplus (CA^2x) \cdot z^{-3} \cdots, \end{aligned}$$

and $G(z)u_{\text{sp}}$ as

$$\begin{aligned} G(z)u_{sp} &= (CBz^{-1} \oplus CABz^{-2} \oplus CA^2Bz^{-3} \cdots)\\ &\quad \cdot (u_0z^{-1} \oplus u_1z^{-2} \oplus u_2z^{-3} \cdots)\\ &= (CBu_0) \cdot z^{-2} \oplus (CABu_0 \oplus CBu_1) \cdot z^{-3} \oplus \cdots. \end{aligned}$$

Sum up the last two equations to obtain

$$\begin{aligned} G(z)u(z) = G(z)(u_{\text{poly}} \oplus u_{sp}) &= y_{\text{poly}} \oplus (Cx) \cdot z^{-1}\\ &\oplus (C(Ax \oplus Bu_0)) \cdot z^{-2}\\ &\oplus (C(A^2x \oplus ABu_0 \oplus Bu_1)) \cdot z^{-3} \cdots. \end{aligned}$$

Therefore, an (A, B)-invariant sub-semimodule in $\frac{\text{Im}_{\text{proper}}\mathcal{R}}{\text{Ker}_{\text{eq}}C}$ implies for u^1_{poly} and u^2_{poly}, that there exists a pair of u^1_{sp} and u^2_{sp} such that

$$\begin{aligned} &\frac{G(z)(u^1_{\text{poly}} \oplus u^1_{sp})}{\Omega Y} = \frac{G(z)(u^2_{\text{poly}} \oplus u^2_{sp})}{\Omega Y}\\ &\Longrightarrow p \circ G(z)(u^1_{\text{poly}} \oplus u^1_{sp}) = p \circ G(z)(u^2_{\text{poly}} \oplus u^2_{sp}). \end{aligned}$$

If the mapping $p \circ G(z)$ is steady, then there exists $k_1 = k^1_{\text{poly}} \oplus k^1_{sp}$ and $k_2 = k^2_{\text{poly}} \oplus k^2_{sp} \in \text{Ker}\, p \circ G(z)$, i.e. $k_1, k_2 \in G^{-1}(\Omega Y)$, such that

$$\begin{aligned} u^1_{\text{poly}} \oplus u^1_{sp} \oplus k^1_{\text{poly}} \oplus k^1_{sp} &= u^2_{\text{poly}} \oplus u^2_{sp} \oplus k^2_{\text{poly}} \oplus k^2_{sp}\\ &\Longrightarrow u^1_{\text{poly}} \oplus k^1_{\text{poly}} = u^2_{\text{poly}} \oplus k^2_{\text{poly}}\\ &\Longrightarrow \widetilde{B}u^1_{\text{poly}} \oplus \widetilde{B}k^1_{\text{poly}} = \widetilde{B}u^2_{\text{poly}} \oplus \widetilde{B}k^2_{\text{poly}}\\ &\Longrightarrow x_1 \oplus \widetilde{B}k^1_{\text{poly}} = x_2 \oplus \widetilde{B}k^2_{\text{poly}}, \end{aligned}$$

where $\widetilde{B}k^1_{\text{poly}}$ and $\widetilde{B}k^2_{\text{poly}}$ are contained in $p_1(G^{-1}(\Omega Y))$ because there exists strictly proper components such as the two inputs producing the polynomial outputs. Therefore, $x_1 \equiv_{p_1(G^{-1}(\Omega Y))} x_2$ and it implies that any pair (x_1, x_2) in an (A, B)-invariant subsemimodule in of $\frac{\text{Im}_{\text{proper}}\mathcal{R}}{\text{Ker}_{\text{eq}}C}$ is contained in $\frac{\text{Im}_{\text{proper}}\mathcal{R}}{p_1(G^{-1}(\Omega Y))}$. In this case, $\frac{\text{Im}_{\text{proper}}\mathcal{R}}{p_1(G^{-1}(\Omega Y))}$ becomes the supremal controlled invariant sub-semimodule in $\frac{\text{Im}_{\text{proper}}\mathcal{R}}{\text{Ker}_{\text{eq}}C}$. □

4.3. *Numerical example*

This example illustrates that the computational methods for the supremal (A, B)-invariant sub-semimodule in the kernel of C and the equivalence kernel of C using both the recursive algorithm and the proper image of p_1. Given a strictly proper transfer function $G(z)$ for a max-plus linear system,

$$G(z) = \begin{bmatrix} z^{-2} & \epsilon \\ \epsilon & \epsilon \end{bmatrix}$$

and its state space representation (A, B, C) as

$$A = \begin{bmatrix} \epsilon & \epsilon & \epsilon & \epsilon \\ e & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & e & \epsilon \end{bmatrix}, \quad B = \begin{bmatrix} e & \epsilon \\ \epsilon & \epsilon \\ e & e \\ \epsilon & \epsilon \end{bmatrix}, \quad C = \begin{bmatrix} \epsilon & e & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon \end{bmatrix}.$$

The controllability matrix of this system is

$$\begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix} = \begin{bmatrix} e & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \cdots \\ \epsilon & \epsilon & e & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \cdots \\ e & e & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \cdots \\ \epsilon & \epsilon & e & e & \epsilon & \epsilon & \epsilon & \epsilon & \cdots \end{bmatrix},$$

Therefore the proper image of the reachability space is the same as $\widetilde{B}(\Omega U)$. If

$$\begin{aligned} u_{\text{poly}} &= \begin{bmatrix} u^1_0 \oplus u^1_1 z \oplus \cdots \oplus u^1_n z^n \\ u^2_0 \oplus u^2_1 z \oplus \cdots \oplus u^2_n z^n \end{bmatrix} \Longrightarrow\\ \widetilde{B}u_{\text{poly}} &= \widetilde{B}\begin{bmatrix} u^1_0 \oplus u^1_1 z \oplus \cdots \oplus u^1_n z^n \\ u^2_0 \oplus u^2_1 z \oplus \cdots \oplus u^2_n z^n \end{bmatrix}\\ &= B\begin{bmatrix} u^1_0 \\ u^2_0 \end{bmatrix} \oplus AB\begin{bmatrix} u^1_1 \\ u^2_1 \end{bmatrix} \oplus A^2B\begin{bmatrix} u^1_2 \\ u^2_2 \end{bmatrix} \cdots \oplus A^nB\begin{bmatrix} u^1_n \\ u^2_n \end{bmatrix}\\ &= \begin{bmatrix} e & \epsilon \\ \epsilon & \epsilon \\ e & \epsilon \\ \epsilon & \epsilon \end{bmatrix}\begin{bmatrix} u^1_0 \\ u^2_0 \end{bmatrix} \oplus \begin{bmatrix} \epsilon & \epsilon \\ e & \epsilon \\ \epsilon & \epsilon \\ e & e \end{bmatrix}\begin{bmatrix} u^1_1 \\ u^2_1 \end{bmatrix}\\ &= \begin{bmatrix} u^1_0 \\ u^1_1 \\ u^1_0 \oplus u^2_0 \\ u^1_1 \oplus u^1_2 \end{bmatrix}. \end{aligned}$$

Hence, the reachability space $\mathrm{Im}_{\mathrm{proper}}\mathcal{R}$ is the semi-module consisting of states in $\mathbb{R}^4_{\mathrm{Max}}$ in this form:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},$$

where x_1, x_2 are arbitrary elements in $\mathbb{R}_{\mathrm{Max}}$, and $x_4 \geq x_2$, $x_3 \geq x_1$ for x_3, x_4 in $\mathbb{R}_{\mathrm{Max}}$. Therefore, $\mathrm{Ker}\, C \cap \mathrm{Im}_{\mathrm{proper}}\mathcal{R}$ is a semimodule consisting the states in this form:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 = \epsilon \\ x_3 \\ x_4 \end{bmatrix},$$

where x_1 and x_4 are arbitrary elements in $\mathbb{R}_{\mathrm{Max}}$ and $x_3 \geq x_1$.

In order to calculate the supremal (A, B)-invariant sub-semimodule $\mathcal{V}^*$, we can either use the algorithm in Equation (13) or the mapping $p_1(G^{-1}(\Omega Y))$ in Proposition 4.4. Using the algorithm in Equation (13), we obtain that

$$\begin{aligned} V_0 &= \mathcal{K} = \mathrm{Ker}\, C \cap \mathrm{Im}_{\mathrm{proper}}\mathcal{R} \\ V_1 &= V_0 \cap A^{-1}(V_0 \widehat{\oplus} \mathcal{B}) = \mathrm{Im}_{\mathrm{proper}}[\mathbf{e_3}; \mathbf{e_4}] \\ V_2 &= V_1 \cap A^{-1}(V_1 \widehat{\oplus} \mathcal{B}) = \mathrm{Im}_{\mathrm{proper}}[\mathbf{e_3}; \mathbf{e_4}] \\ &\vdots \\ V_k &= \mathrm{Im}_{\mathrm{proper}}[\mathbf{e_3}; \mathbf{e_4}] = \mathcal{V}^*, \end{aligned}$$

where the basis elements are

$$\mathbf{e_3} = \begin{bmatrix} \epsilon \\ \epsilon \\ e \\ \epsilon \end{bmatrix}, \quad \mathbf{e_4} = \begin{bmatrix} \epsilon \\ \epsilon \\ \epsilon \\ e \end{bmatrix}.$$

We can also use the mapping $p_1(G^{-1}(\Omega Y))$ to the supremal controlled invariant sub-semimodule in $\mathrm{Ker}\, C \cap \mathrm{Im}_{\mathrm{proper}}\mathcal{R}$.

$$\begin{aligned} & p_1(G^{-1}(\Omega Y)) \\ &= \widetilde{B} u_{\mathrm{poly}} \\ &= \widetilde{B} \left\{ \begin{bmatrix} \epsilon \\ \bar{v}_0 \end{bmatrix} \oplus \begin{bmatrix} \epsilon \\ \bar{v}_1 \end{bmatrix} z \oplus \begin{bmatrix} v_2 \\ \bar{v}_2 \end{bmatrix} z^2 \oplus \begin{bmatrix} v_3 \\ \bar{v}_3 \end{bmatrix} z^3 \oplus \cdots \right\} \\ &= B \begin{bmatrix} \epsilon \\ \bar{v}_0 \end{bmatrix} \oplus AB \begin{bmatrix} \epsilon \\ \bar{v}_1 \end{bmatrix} \oplus A^2 B \begin{bmatrix} v_2 \\ \bar{v}_2 \end{bmatrix} \oplus A^3 B \begin{bmatrix} v_3 \\ \bar{v}_3 \end{bmatrix} \oplus \cdots \\ &= \mathrm{Im}_{\mathrm{proper}}[\mathbf{e_3}; \mathbf{e_4}] \\ &= \mathcal{V}^*. \end{aligned}$$

Next, we will demonstrate Propositions 4.5 and 4.6 by calculating the supremal controlled invariant semimodule in the equivalence kernel of C in $\mathrm{Im}_{\mathrm{proper}}\mathcal{R}$. Given

$$\mathbf{x_1} = \begin{bmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \\ x_4^1 \end{bmatrix}, \quad \text{and} \quad \mathbf{x_2} = \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \\ x_4^2 \end{bmatrix}$$

contained in the supremal (A, B)-invariant sub-semi-module in $\frac{\mathrm{Im}_{\mathrm{proper}}\mathcal{R}}{\mathrm{Ker}_{\mathrm{eq}}C}$, then $C\mathbf{x}_1 = \mathbf{C}\mathbf{x}_2$, it implies that $x_2^1 = x_2^2$. Moreover, there exists a pair of control signals (u_1, u_2) such that $C(A\mathbf{x}_1 \oplus Bu_1) = C(A\mathbf{x}_2 \oplus Bu_2)$, i.e.

$$C\left(\begin{bmatrix} \epsilon & \epsilon & \epsilon & \epsilon \\ e & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & e & \epsilon \end{bmatrix} \begin{bmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \\ x_4^1 \end{bmatrix} \oplus \begin{bmatrix} e & \epsilon \\ \epsilon & \epsilon \\ e & e \\ \epsilon & \epsilon \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_2^1 \end{bmatrix} \right)$$

$$= C\left(\begin{bmatrix} \epsilon & \epsilon & \epsilon & \epsilon \\ e & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & e & \epsilon \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \\ x_4^2 \end{bmatrix} \oplus \begin{bmatrix} e & \epsilon \\ \epsilon & \epsilon \\ e & e \\ \epsilon & \epsilon \end{bmatrix} \begin{bmatrix} u_1^2 \\ u_2^2 \end{bmatrix} \right)$$

$$C \begin{bmatrix} u_1^1 \\ x_1^1 \\ u_1^1 \oplus u_2^1 \\ x_3^1 \end{bmatrix} = C \begin{bmatrix} u_1^2 \\ x_1^2 \\ u_1^2 \oplus u_2^2 \\ x_3^3 \end{bmatrix},$$

so it implies that $x_1^1 = x_1^2$. In summary, the first two elements on the vectors need to be the same for two equivalent states in the supremal controlled invariant sub-semimodule in $\frac{\mathrm{Im}_{\mathrm{proper}}\mathcal{R}}{\mathrm{Ker}_{\mathrm{eq}}C}$. Because $p_1(G^{-1}(\Omega Y)) = \mathrm{Im}_{\mathrm{proper}}[\mathbf{e}_3; \mathbf{e}_4]$, we can find two vectors in $p_1(G^{-1}(\Omega Y))$ such that

$$\begin{bmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \\ x_4^1 \end{bmatrix} \oplus \begin{bmatrix} \epsilon \\ \epsilon \\ x_3^2 \\ x_4^2 \end{bmatrix} = \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \\ x_4^2 \end{bmatrix} \oplus \begin{bmatrix} \epsilon \\ \epsilon \\ x_3^1 \\ x_4^1 \end{bmatrix}$$

$$\Longrightarrow \mathbf{x_1} \equiv_{p_1(G^{-1}(\Omega Y))} \mathbf{x_2}.$$

The supremal controlled invariant sub-semimodule in $\frac{\mathrm{Im}_{\mathrm{proper}}\mathcal{R}}{\mathrm{Ker}_{\mathrm{eq}}C}$ coincides with the factor semimodule $\frac{\mathrm{Im}_{\mathrm{proper}}\mathcal{R}}{p_1(G^{-1}(\Omega Y))}$ because the mapping $p \circ G(z) : U(z) \to \frac{Y(z)}{\Omega Y}$ is steady or k-regular. The reason is illustrated as follows: for two input signals $u_1(z)$ and $u_2(z)$ in $U(z)$, if they satisfy $\frac{G(z)u_1(z)}{\Omega Y} = \frac{G(z)u_2(z)}{\Omega Y}$, then we have

$$\frac{G(z) \begin{bmatrix} \cdots \oplus u_{-1}^1 z^{-1} \oplus u_0^1 \oplus u_1^1 z \oplus u_2^1 z^2 \cdots \\ \cdots \oplus u_{-1}^2 z^{-1} \oplus u_0^2 \oplus u_1^2 z \oplus u_2^2 z^2 \cdots \end{bmatrix}}{\Omega Y}$$

$$= \frac{G(z) \begin{bmatrix} \cdots \oplus v_{-1}^1 z^{-1} \oplus v_0^1 \oplus v_1^1 z \oplus v_2^1 z^2 \cdots \\ \cdots \oplus v_{-1}^2 z^{-1} \oplus v_0^2 \oplus v_1^2 z \oplus v_2^2 z^2 \cdots \end{bmatrix}}{\Omega Y}$$

$$\Longrightarrow\begin{bmatrix} \cdots \oplus u_{-1}^1 z^{-3} \oplus u_0^1 z^{-2} \oplus u_1^1 z^{-1} \\ \epsilon \end{bmatrix} = \begin{bmatrix} \cdots \oplus v_{-1}^1 z^{-3} \oplus v_0^1 z^{-2} \oplus v_1^1 z^{-1} \\ \epsilon \end{bmatrix}.$$

Therefore, the two input signals have the same coefficients on the first row for the terms z^n for $n \le 1$. The kernel of the mapping $p \circ G(z)$ is $G^{-1}(\Omega Y)$, which consists of the states with the order of the first row polynomial higher than 2 and the second row can be arbitrary, i.e.

$$\begin{bmatrix} u_2 z^2 \oplus u_3 z^3 \oplus \cdots \\ u(z) \end{bmatrix}.$$

Therefore, for two input signals $u_1(z)$ and $u_2(z)$ in $U(z)$, if they satisfy $\frac{G(z)u_1(z)}{\Omega Y} = \frac{G(z)u_2(z)}{\Omega Y}$, we can find two elements $l_1(z)$ and $l_2(z)$ in the kernel of $p \circ G(z)$ to satisfy $u_1(z) \oplus l_1(z) = u_2(z) \oplus l_2(z)$, where

$$l_1(z) = \begin{bmatrix} v_2^1 z^2 \oplus v_3^1 z^3 \oplus \cdots \\ \cdots \oplus v_{-1}^2 z^{-1} \oplus v_0^2 \oplus v_1^2 z \oplus v_2^2 z^2 \cdots \end{bmatrix},$$

$$l_2(z) = \begin{bmatrix} u_2^1 z^2 \oplus u_3^1 z^3 \oplus \cdots\cdots \\ \cdots \oplus u_{-1}^2 z^{-1} \oplus u_0^2 \oplus u_1^2 z \oplus u_2^2 z^2 \cdots \end{bmatrix}.$$

Hence, the mapping $p \circ G(z)$ is steady or k-regular. In summary, this example demonstrates the computation method for the supremal controlled invariant sub-semimodule in $\operatorname{Ker} C \cap \operatorname{Im}_{\text{proper}} \mathcal{R}$ and $\frac{\operatorname{Im}_{\text{proper}} \mathcal{R}}{\operatorname{Ker}_{\text{eq}} C}$.

5. Disturbance decoupling problem with a queueing network application

The computational methods in this article can be used in the solvability condition of the disturbance decoupling problem (DDP), that is a standard geometric control problem in traditional linear systems. A max-plus linear system with a disturbance signal $d(k) \in D$ is defined as

$$\begin{aligned} x(k+1) &= Ax(k) \oplus Bu(k) \oplus Sd(k), \\ y(k) &= Cx(k), \end{aligned} \tag{14}$$

where $x(k) \in X$, $u(k) \in U$ and $d(k) \in D$. The system (14) is called *disturbance decoupled* if there exists a state feedback controller $u(k) = Fx(k)$ such that the disturbance signals will not affect the system output $y(k)$ for all $k \in \mathbb{Z}$.

For linear systems over a field, the (A, B)-controlled invariant subspaces are equivalent to the (A, B)-controlled invariant subspaces of feedback type. The solvability of DDP can be obtained by knowing the supremal controlled invariant sub-semimodule in the kernel of C. $\langle A + BF | \operatorname{Im} S \rangle$ is defined as

$$\langle A + BF | \operatorname{Im} S \rangle = \operatorname{Im} S + (A + BF)\operatorname{Im} S + (A + BF)^2 \operatorname{Im} S + \cdots + (A + BF)^{(n-1)} \operatorname{Im} S.$$

Let $\mathcal{K} = \operatorname{Ker} C$ and $\mathcal{T} = \operatorname{Im} S$. The DDP is to find (if possible) a state feedback $F: X \to U$ such that $\langle A + BF | \mathcal{T} \rangle \subset \mathcal{K}$.

Theorem 5.1 (Wonham 1979)**:** *The DDP is solvable for linear systems over a field if and only if the supremal controlled invariant subspace $\mathcal{V}^*$ in $\mathcal{K}$ contains $\mathcal{T}$.*

This condition is not enough for the max-plus linear systems because the monotone non-decreasing property of such systems. For linear systems over semirings, or specifically max-plus linear systems in this article, the traditional null kernel of C is not as crucial as in traditional linear systems because the system trajectories are monotone nondecreasing. Therefore, the equivalence kernel of C will play a more meaningful role in the DDP. Existing work on the DDP for max-plus linear systems (Lhommeau et al. 2002) used the residuation theory to find the greatest control signal in order to satisfy the just-in-time control requirement. This result focuses on the existence of such a state feedback control signal to solve the DDP rather than the optimal control perspective.

Proposition 5.2: *The DDP is solvable for a max-plus linear system of the form* (14), *if and only if there exists a state feedback control signal $u(k) = Fx(k)$ such that, for any initial state $x(0)$ and any time instant n, $(A \oplus BF)^n x(0)$ and*

$$(A \oplus BF)^n x(0) \oplus \langle A \oplus BF | \operatorname{Im}_{\text{proper}} S \rangle_n$$

are equivalent by the equivalent kernel of C for any integer $n \ge 0$, where

$$\langle A \oplus BF | \operatorname{Im}_{\text{proper}} S \rangle_n = \operatorname{Im}_{\text{proper}} S \oplus (A \oplus BF)\operatorname{Im}_{\text{proper}} S \oplus \cdots \oplus (A \oplus BF)^{(n-1)} \operatorname{Im}_{\text{proper}} S.$$

Proof: Based on the system equation's evolution, we have

$$\begin{aligned} x(1) &= (A \oplus BF)x(0) \oplus Sd(0), \\ x(2) &= (A \oplus BF)^2 x(0) \oplus Sd(1) \oplus (A \oplus BF)Sd(0), \\ &\vdots \\ x(n) &= (A \oplus BF)^n x(0) \oplus Sd(n-1) \oplus (A \oplus BF)Sd(n-2) \\ &\quad \oplus \cdots \oplus (A \oplus BF)^{(n-1)} Sd(0) \\ \Longrightarrow y(n) &= C(A \oplus BF)^n x(0)) \\ &\quad \oplus C\big(Sd(n-1) \oplus (A \oplus BF)Sd(n-2) \\ &\quad \oplus \cdots \oplus (A \oplus BF)^{(n-1)} Sd(0)\big). \end{aligned} \tag{15}$$

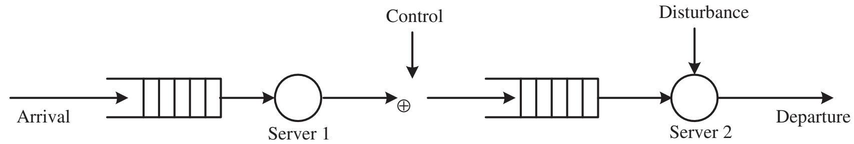

Figure 4. A queueing system with two servers.

It implies that the DDP is solvable for the max-plus linear system if and only if, for any initial state $x(0)$ and any disturbance signal $d \in D$, there exists a state feedback control signal $u(k) = Fx(k)$ such that, for any integer $n \geq 0$, $(A \oplus BF)^n x(0)$ and

$$(A \oplus BF)^n x(0) \oplus \langle A \oplus BF | \mathrm{Im}_{\mathrm{proper}} S \rangle_n$$

are equivalent by the equivalent kernel of C. □

For the max-plus linear systems, we have a new definition for $\langle A \oplus BF | \mathrm{Im}\, S \rangle$ as

$$\langle A \oplus BF | \mathrm{Im}\, S \rangle = \mathrm{Im}_{\mathrm{proper}} S \oplus (A \oplus BF) \mathrm{Im}_{\mathrm{proper}} S \oplus \cdots \oplus (A \oplus BF)^{(n-1)} \mathrm{Im}_{\mathrm{proper}} S \oplus \cdots.$$

If it is contained in the kernel of C like the traditional linear systems, the DDP is of course solvable for the max-plus linear systems. In order to avoid checking the conditions for an infinite number of integers, we can obtain a more computational friendly result.

Proposition 5.3: *The DDP is solvable for a max-plus linear system of the form* (14) *if the following equivalence relation equation is satisfied*

$$\mathcal{V}_{FB} \equiv_{\mathrm{Ker}_{\mathrm{eq}} C} \left(\mathcal{V}_{FB} \oplus \langle A \oplus BF | \mathrm{Im}_{\mathrm{proper}} S \rangle \right), \qquad (16)$$

and the pair belongs to the $(A \oplus BF)$*-invariant sub-semimodule in the equivalence kernel* $\mathrm{Ker}_{\mathrm{eq}}\, C$ *of the output map* C*, where* $\mathcal{V}_{FB}$ *is an* $(A \oplus BF)$*-invariant sub-semimodule in the state space* X.

Proof: If Equaion (16), then any x in an $(A \oplus BF)$-invariant sub-semimodule $\mathcal{V}_{FB}$ of the state space X, we have $Cx = C(x \oplus \langle A \oplus BF | \mathrm{Im}_{\mathrm{proper}} S \rangle)$, which implies that Equaion (15) holds for all n. Moreover, the pair belongs to an $(A \oplus BF)$-invariant sub-semimodule in $\mathrm{Ker}_{\mathrm{eq}}\, C$, it implies that

$$C(A \oplus BF)x = C\left\{ (A \oplus BF)x \oplus \langle A \oplus BF | \mathrm{Im}_{\mathrm{proper}} S \rangle \right\}.$$

Therefore, Equation (15) holds for all n. Hence, the DDP is solvable. □

5.1. *Queuing network application*

The DDP is studied for two queueing networks modelled by a timed Petri net (Cassandras 1993). A *Petri net* is a four-tuple (P, T, A, w), where P is a finite set of *places*, T is a finite set of *transitions*, A is a set of *arcs*, and w is a *weight function*, $w : A \to \{1, 2, 3, \ldots\}$. $I(t_j)$ represents the set of input places to the transition t_j and $O(t_j)$ represents the set of output places from the transition t_j, i.e.

$$I(t_j) = \{p_i : (p_i, t_j) \in A\} \quad \text{and} \quad O(t_j) = \{p_i : (t_j, p_i) \in A\}.$$

A marking x of a Petri net is a function $x : P \to \{0, 1, 2, \ldots\}$. The number represents the number of tokens in a place. A marked Petri net is a five-tuple (P, T, A, w, x_0) where (P, T, A, w) is a Petri net and x_0 is the initial marking. For a timed Petri net, when the transition t_j is enabled for the k-th time, it does not fire immediately, but it has a firing delay, $v_{j,k}$, during the tokens are kept in the input places of t_j. The clock structure associated with the set of timed transitions, $T_D \subseteq T$, of a marked Petri net (P, T, A, w, x) is a set $\mathbf{V} = \{\mathbf{v}_j : t_j \in T_D\}$ of lifetime sequences $\mathbf{v}_j = \{v_{j,1}, v_{j,2}, \ldots\}$, $t_j \in T_D$, $v_{j,k} \in \mathbb{R}^+$, $k = 1, 2, \ldots$. A *timed Petri net* is a six-tuple $(P, T, A, w, x, \mathbf{V})$ where (P, T, A, w, x) is a marked Petri net and $\mathbf{V} = \{\mathbf{v}_j : t_j \in T_D\}$ is a clock structure.

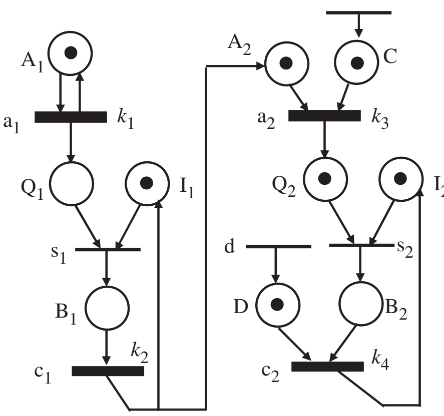

Figure 5. The timed Petri net model for this queueing system.

If we consider a queueing system with two servers as shown in Figure 4, where the second server has a controller for the customer arrival times and a service disturbance, such as service breakdown. The timed Petri net model for this queueing system is shown in Figure 5. There are four places for the server $i = 1, 2$: A_i (arrival), Q_i (queue), I_i (idle) and B_i (busy). For the server 2, there are two more places: C (server 1 completion) and D (disturbance). So $P=\{A_1,Q_1,I_1,B_1;A_2,C,Q_2,I_2,B_2,D\}$. The transitions (events) for each server are a_i (customer arrives), s_i (service starts) and c_i (service completes and customer departs). For the server 2, there are transitions u (control input) and d (disturbance). The timed transition $T_D=\{a_1, a_2, c_1, c_2\}$. The clock structure of this model has constant sequences $\mathbf{v}_{a_1}=\{k_1,k_1,\ldots\}$, $\mathbf{v}_{c_1}=\{k_2,k_2,\ldots\}$, $\mathbf{v}_{a_3}=\{k_3,k_3,\ldots\}$ and $\mathbf{v}_{c_2}=\{k_4,k_4,\ldots\}$. The rectangles present the timed transitions. The initial marking is $x_0=\{1,0,1,0,1,1,1,1,0,1\}$.

We define a_k^i as the k-th arrival time of customers for service i, s_k^i as the k-th service starting time for service i and c_k^i as the k-th service completion and the customer departure time, where $i=\{1,2\}$. We define $x_k=[a_k^1,c_k^1,a_k^2,s_k^2,c_k^2]^T$, where the service time s_k^1 is omitted because it is the same as the arrival time a_k^1. If we assume the output is the customer arrival time of the second server, then we can write the system equation using max-plus algebra as

$$x_{k+1}=Ax_k\oplus Bu_k\oplus Sd_k,$$
$$y_k=Cx_k,$$

where the system matrices are

$$A=\begin{bmatrix} k_1 & k_2 & \epsilon & \epsilon & \epsilon\\ k_1 & k_2 & \epsilon & \epsilon & \epsilon\\ \epsilon & k_2 & \epsilon & \epsilon & \epsilon\\ \epsilon & \epsilon & k_3 & \epsilon & k_4\\ \epsilon & \epsilon & k_3 & \epsilon & k_4\end{bmatrix},\quad B=\begin{bmatrix}\epsilon\\ \epsilon\\ e\\ \epsilon\\ \epsilon\end{bmatrix} S=\begin{bmatrix}\epsilon\\ \epsilon\\ \epsilon\\ \epsilon\\ e\end{bmatrix}\quad\text{and}$$

$$C=\begin{bmatrix}\epsilon & \epsilon & e & \epsilon & \epsilon\end{bmatrix}.$$

The transfer function $G(z)=CBz^{-1}\oplus CABz^{-2}\oplus\cdots=z^{-1}$. Therefore, $G^{-1}(\Omega Y)=u_{\text{poly}}=u_1z\oplus u_2z^2\oplus\cdots$ for any $u_i\in U$. The controllability matrix of this system is

$$\begin{bmatrix}B & AB & A^2B & \cdots\end{bmatrix} = \begin{bmatrix} \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \cdots\\ \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \cdots\\ e & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \cdots\\ \epsilon & e & e & e & e & e & e & e & \cdots\\ \epsilon & \epsilon & e & e & e & e & e & e & \cdots\end{bmatrix},$$

Therefore, the proper image of the reachability space is generated by the following basis elements

$$\mathbf{e_3}=\begin{bmatrix}\epsilon\\ \epsilon\\ e\\ \epsilon\\ \epsilon\end{bmatrix},\quad \mathbf{e_{45}}=\begin{bmatrix}\epsilon\\ \epsilon\\ \epsilon\\ e\\ e\end{bmatrix}.$$

In another word, the vector can have an arbitrary third row element but the fourth and fifth row elements have to be the same. Therefore, $\operatorname{Ker} C\cap \operatorname{Im}_{\text{proper}}\mathcal{R}$ is a semimodule consisting the states in this form:

$$\mathbf{x}=\begin{bmatrix}\epsilon\\ \epsilon\\ \epsilon\\ x_4\\ x_5\end{bmatrix},$$

where x_4 and x_5 are the same.

In order to calculate the supremal (A,B)-invariant sub-semimodule $\mathcal{V}^*$, we can either use the algorithm in Equaion (13) or the mapping $p_1(G^{-1}(\Omega Y))$ in Proposition 4.4. Using the algorithm in Equaion (13), we obtain that

$$\begin{aligned} V_0 &= \mathcal{K} = \operatorname{Ker} C\cap \operatorname{Im}_{\text{proper}}\mathcal{R}\\ V_1 &= V_0\cap A^{-1}(V_0\,\widehat{\oplus}\,\mathcal{B}) = \operatorname{Im}_{\text{proper}}[\mathbf{e_{45}}]\\ V_2 &= V_1\cap A^{-1}(V_1\,\widehat{\oplus}\,\mathcal{B}) = \operatorname{Im}_{\text{proper}}[\mathbf{e_{45}}]\\ &\vdots\\ V_k &= \operatorname{Im}_{\text{proper}}[\mathbf{e_{45}}] = \mathcal{V}^*.\end{aligned}$$

We can also use the mapping $p_1(G^{-1}(\Omega Y))$ to a supremal controlled invariant sub-semimodule in $\operatorname{Ker} C\cap \operatorname{Im}_{\text{proper}}\mathcal{R}$.

$$\begin{aligned} p_1(G^{-1}(\Omega Y)) &= \widetilde{B}u_{\text{poly}}\\ &= \widetilde{B}(u_1z\oplus u_2z\oplus\cdots)\\ &= ABu_1\oplus A^2u_2\oplus\cdots\\ &= \operatorname{Im}_{\text{proper}}[\mathbf{e_{45}}]\\ &= \mathcal{V}^*.\end{aligned}$$

Next, we will demonstrate Proposition 4.5 and Proposition 4.6 by calculating the supremal controlled invariant semimodule in the equivalence kernel of C in $\operatorname{Im}_{\text{proper}}\mathcal{R}$. Given that

$$\mathbf{x_1}=\begin{bmatrix}\epsilon\\ \epsilon\\ x_3^1\\ x_4^1\\ x_5^1\end{bmatrix},\quad\text{and}\quad \mathbf{x_2}=\begin{bmatrix}\epsilon\\ \epsilon\\ x_3^2\\ x_4^2\\ x_5^2\end{bmatrix}$$

contained in the supremal (A,B)-invariant sub-semimodule in $\frac{\operatorname{Im}_{\text{proper}}\mathcal{R}}{\operatorname{Ker}_{\text{eq}}C}$, then $C\mathbf{x}_1=\mathbf{C}\mathbf{x}_2$, it implies that $x_3^1=x_3^2$, $x_4^1=x_5^1$, and $x_4^2=x_5^2$. Moreover, there exists a pair of control signals (u_1,u_2) such that $C(A\mathbf{x}_1\oplus Bu_1)=C(A\mathbf{x}_2\oplus Bu_2)$, the pair of states $\mathbf{x}_1$ and $\mathbf{x}_2$ satisfy the invariant condition. We can find two elementals in $p_1(G^{-1}(\Omega Y))$ such that

$$\begin{bmatrix}\epsilon\\ \epsilon\\ x_3^1\\ x_4^1\\ x_5^1\end{bmatrix} \oplus \begin{bmatrix}\epsilon\\ \epsilon\\ \epsilon\\ x_4^2\\ x_5^2\end{bmatrix} = \begin{bmatrix}\epsilon\\ \epsilon\\ x_3^2\\ x_4^2\\ x_5^2\end{bmatrix} \oplus \begin{bmatrix}\epsilon\\ \epsilon\\ \epsilon\\ x_4^1\\ x_5^1\end{bmatrix}$$
$$\Longrightarrow \mathbf{x_1} \equiv_{p_1(G^{-1}(\Omega Y))} \mathbf{x_2}.$$

Pick $F=[f_1, f_2, f_3, \epsilon, \epsilon]$ for any $f_i \in \mathbb{R}_{\text{Max}}$, then for any initial state $x(0)$ and any time instant n, $(A \oplus BF)^n x(0)$ and

$$(A \oplus BF)^n x(0) \oplus \langle A \oplus BF | \text{Im}_{\text{proper}} S \rangle_n,$$

are equivalent by the equivalent kernel of C for any integer $n \geq 0$. Hence, the DDP is solvable for this queueing network. If we use Equaion (16) in Proposition 5.3, the equation is also satisfied. $\mathcal{V}_{FB}$ is the same as $\text{Im}_{\text{proper}} \mathcal{R}$ in this example, $\langle A \oplus BF | \text{Im}_{\text{proper}} S \rangle$ is the range of the basis $\mathbf{e_{45}}$, therefore, for any element $\mathbf{x} \in \text{Im}_{\text{proper}} \mathcal{R}$,

$$C(\mathbf{x}) = C\begin{bmatrix}\epsilon\\ \epsilon\\ x_3\\ x_4\\ x_5\end{bmatrix} = x_3$$

$$C(\mathbf{x} \oplus \langle A \oplus BF | \text{Im}_{\text{proper}} S \rangle) = C\begin{bmatrix}\epsilon\\ \epsilon\\ x_3\\ x_4\\ x_5\end{bmatrix} \oplus C\begin{bmatrix}\epsilon\\ \epsilon\\ \epsilon\\ x_4\\ x_5\end{bmatrix} = x_3.$$

Moreover,

$$C((A \oplus BF)\mathbf{x}) = C\begin{bmatrix}\epsilon\\ \epsilon\\ f_3x_3 \oplus f_4x_4 \oplus f_5x_5\\ k_4x_5\\ k_4x_5\end{bmatrix} = f_3x_3 \oplus f_4x_4 \oplus f_5x_5,$$

$$C((A \oplus BF)\mathbf{x} \oplus \langle A \oplus BF | \text{Im}_{\text{proper}} S \rangle) = C\begin{bmatrix}\epsilon\\ \epsilon\\ f_3x_3 \oplus f_4x_4 \oplus f_5x_5\\ k_4x_5\\ k_4x_5\end{bmatrix} \oplus C\begin{bmatrix}\epsilon\\ \epsilon\\ \epsilon\\ x_4\\ x_5\end{bmatrix} = f_3x_3 \oplus f_4x_4 \oplus f_5x_5.$$

It also verifies that the conditions in Proposition 5.3 are satisfied, hence the DDP is solvable.

6. Conclusion

This article generalises R.E. Kalman's abstract realisation theory for systems over fields to max-plus linear systems. The new generalised version of Kalman's abstract realisation theory not only provides a more concrete state space representation other than just a 'set-theoretic' representation for a canonical realisation of a transfer function, but also leads to the computational methods for the controlled invariant semimodules in the kernel and the equivalence kernel of the output map. These controlled invariant semimodules play key roles in the standard geometric control problems, such as disturbance decoupling problem and block decoupling problem. A queueing network is used to illustrate the main results in this article. Future research directions are expanding the results to the block decoupling problem and the model matching problem towards to a systemic geometric control theory for max-plus linear systems.

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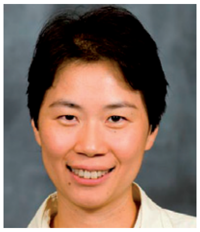
Notes on contributor



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