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An Observer for Tropical Linear Event-Invariant Dynamical Systems

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Abstract— **This paper presents a sufficient condition to solve the observation problem in tropical linear event-invariant dynamical systems, where a linear functional of the states can be observed in a finite number of steps using only the information from inputs and outputs. Using the residuation theory, this solvability condition can be easily implemented in polynomial time. Moreover, the main results are applied to state feedback control using only the observed states based on the measurements of the original states in the system. Furthermore, the main results are implemented in the perturbation observation problem for tropical linear event-invariant dynamical systems, where the system matrices are perturbed in intervals.**

I. INTRODUCTION

A. State of the art

Control theory for linear time invariant systems (time sampled, in this case)

$$\mathbf{x}[k+1] = A\mathbf{x}[k] + B\mathbf{u}[k] \tag{I.1}$$

was largely studied. Their results, based in strong and elegant concepts of linear algebra, are ubiquitous in curricular grades of system engineers. Its importance is undoubtable, either being a direct application or as a basis for more general results (non-linear theory).

Some discrete event systems, specifically Timed Event Graphs (a subclass of Petri nets in which all places have a single transition upstream and a single one downstream, see [1]), TEGs henceforth, admits a representation in state space curiously similar to the one in Equation (I.1) when the timings are event-invariant (that is, the timing of the transitions/places never change with each firing)

$$\mathbf{x}[k+1] = A\mathbf{x}[k] \oplus B\mathbf{u}[k] \tag{I.2}$$

in which $x_i[k]$, $u_i[k]$ represent the time of the k^{th} firing of the i^{th} transition of state and controller, respectively. However, in the context of Equation (I.2), the matricial sums and products are performed in a different algebra, the so-called Tropical Algebra [1].

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[1]Also known as Max-Plus Algebra. The algebra received this name in the honor of the hungarian-born brazilian Imre Simon, who introduced it in order to solve a problem in formal languages (see [2]).

Recently, a significative effort was put in the pursuit of a solid theory for tropical linear event-invariant dynamical systems as in Equation (I.2). Due to the peculiar structure of the Tropical Algebra, few results of the well developed conventional linear dynamical system theory can be easily transposed for this new algebra. One important problem is the one of observability. The observability problem arises frequently, in particular when it is necessary to implement a state feedback controller $\mathbf{u}[k] = F\mathbf{x}[k]$ and not all states are measured. The alternative is to design a state feedback control $\mathbf{u}[k] = F\hat{\mathbf{x}}[k]$ using the observation of the state $\mathbf{x}$. In the case of traditional systems, this problem is very well understood, but in the tropical setting it is not the case.

To the authors' knowledge, only two papers studied problems related to observability in tropical setting. [3] construct an observer for a descriptor system (which can model uncertainties in the parameters). Hence, given that the initial condition of the system $\mathbf{x}[0]$ is known, it is possible to discover at a given step k all the possible values of the state $\mathbf{x}[k]$ that could be reached by the system with uncertainties. The second one, [4], used transfer series methods to construct a Luenberger-like observer that reconstructs the greatest state estimation $\hat{\mathbf{x}}[k]$ that is less than or equal to the real state $\mathbf{x}[k]$.

This paper is interested, specifically, in the following problem (it will be posed formally latter): using only the system outputs $\mathbf{y}[k]$ and inputs $\mathbf{u}[k]$, construct a sequence $\hat{\mathbf{x}}[k]$ that converges in a finite number of steps to a linear functional $W\mathbf{x}[k]$, for a given matrix W, no matter what the initial condition $\mathbf{x}[0]$ of the system is. In principle, the approach described in [4] could be used: observing $\mathbf{x}[k]$ and then computing $W\mathbf{x}[k]$. However, the conditions that ensure strict equality can be quite restrictive. Computing the form $W\mathbf{x}[k]$ indirectly, as it will be proposed in this paper, can be handy.

As it will be clear, using only the measurements of the inputs and output, this formulation is crucial for solving a class of control problems in tropical setting. For instance, the main results are applied to state feedback control using only the observed states based on the measurements of the original states in the system. Furthermore, the main results are implemented in the perturbation observation problem for tropical linear event-invariant dynamical system, where the system matrices are perturbed in intervals.

B. Definitions

Tropical Algebra is the idempotent semiring (or dioid)

$$\mathbb{T}_{\texttt{max}} \equiv \{\mathbb{Z} \cup \{-\infty\}\ ,\ \oplus\ ,\otimes\} \tag{I.3}$$

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in which $\oplus$ is the maximum and $\otimes$ is the traditional sum. It is usual, as well, to denote the neutral element of the sum, $-\infty$, as $\perp$. As in the traditional algebra, the symbol $\otimes$ is usually omitted.

The tropical identity matrix of appropriate order is denoted by I. $\diagdown\!\!\circ$, $\diagup\!\!\circ$ are used to denote the left and right residuation of the product, respectively:

$$M \backslash N = \bigoplus \{X | MX \preceq N\} \text{ and}$$
$$N / M = \bigoplus \{X | XM \preceq N\}.$$

$+$ and $-$ will have their usual meaning (traditional sum and subtraction/opposite) while $\cdot$ is the traditional product. M^T is the transpose of M. For a natural number n and a square matrix M, M^n (the n^{th} power of M) is defined recursively as $M^0 \equiv I$ and $M^n = MM^{n-1}$. If α is a scalar $\neq \perp$, $\alpha^{-1} \equiv -\alpha$. The Kleene Closure of a square matrix M is defined as $M^* = \bigoplus_{i=0}^{\infty} M^i$. $\rho(M)$ is the largest eigenvalue of M. It is also defined a matrix composed entirely of $\perp$ as $\perp$. If M is a matrix, the entry on the i^{th} row and j^{th} columns is denoted as M_{ij} or $\{M\}_{ij}$, whichever is more convenient. The dimension of the matrices will only be specified when necessary. All vectors are column vectors and written in bold, but the bold font is dropped when indexing (so x_i is the i^{th} component of the vector $\mathbf{x}$). *Im*$\{M\}$ is the semimodule generated by linear combinations of the columns of M.

II. The Problem

A. Problem Statement

Consider the system

$$\begin{aligned} S \ : \mathbf{x}[k+1] &= A\mathbf{x}[k] \oplus B\mathbf{u}[k]; \\ \mathbf{y}[k] &= C\mathbf{x}[k] \end{aligned} \tag{II.1}$$

for $A \in \mathbb{T}_{\max}^{n\times n}$, $B \in \mathbb{T}_{\max}^{n\times m}$, $C \in \mathbb{T}_{\max}^{d\times n}$. For a given matrix $W \in \mathbb{T}_{\max}^{s\times n}$, using the inputs $\mathbf{u}[k]$ and outputs $\mathbf{y}[k]$, construct a sequence $\mathbf{s}[k]$ such that there exists a finite l in which $\mathbf{s}[k] = W\mathbf{x}[k]\ \forall k \geq l$.

It is worthy mentioning that the usual observation problem takes $W = I$. However, the choice of another matrix can substantially weaken the problem. As it will be shown in the application section, sometimes it is sufficient to observe only a linear functional of the states, not every one of them (the aforementioned choice $W = I$).

B. Methodology

This problem obviously arises when one tries to implement a feedback controller $F\mathbf{x}[k]$ (set $W = F$) using only the outputs and inputs. This paper will present a sufficient condition for solving this problem. For that, it is necessary the following definition, borrowed from the traditional linear system theory.

Definition 2.1: (Controllability and observability matrix) For a given system S as in Equation (II.1) and a natural number r, the r^{th} *controllability matrix* $\mathcal{C}_S[r] \in \mathbb{T}_{\max}^{(r+1)\times n\cdot m}$ is defined recursively as

$$\mathcal{C}_S[r] \equiv \begin{pmatrix} A\mathcal{C}_S[r-1] & B \end{pmatrix} \tag{II.2}$$

in which $\mathcal{C}_S[0] \equiv B$. The r^{th} *observability matrix*, $\mathcal{O}_S[r] \in \mathbb{T}_{\max}^{(r+1)\cdot d\times n}$, is defined recursively as

$$\mathcal{O}_S[r] \equiv \begin{pmatrix} C \\ \mathcal{O}_S[r-1]A \end{pmatrix} \tag{II.3}$$

in which $\mathcal{O}_S[0] \equiv C$. ■

Notice that the controllability matrix in Equation (II.2) is in reverse order compared to the controllability matrix in traditional linear time-invariant systems, purely for computational purpose in this paper. It will not affect the properties of the controllability.

In the traditional algebra setting, if $A \in \mathbb{R}^{n\times n}$ then (for the analogous r^{th} controllability matrix) *Im*$\{\mathcal{C}_S[r]\}$ = *Im*$\{\mathcal{C}_S[n-1]\}$ $\forall r \geq n$. An analogous result holds for the observability matrix. These affirmations are true because the Cayley-Hamilton Theorem guarantees that A^n can be written as a linear combination of the previous powers A^i, $i = 0, 1, 2, ..., n-1$. In the tropical setting, this is not the case. Indeed, it can be that *Im*$\{\mathcal{C}_S[r]\} \subset$ *Im*$\{\mathcal{C}_S[r+1]\}$ (note the strict inclusion) for all r (see [2]).

And then, two definitions are necessary.

Definition 2.2: ($\mathcal{H}$ matrix) For a given system S as in Equation (II.1) and a natural number r, the $\mathcal{H}$ matrix, $\mathcal{H}_S[r] \in \mathbb{T}_{\max}^{(r+1)\cdot d\times r\cdot m}$, is defined recursively as

$$\mathcal{H}_S[r] \equiv \begin{pmatrix} \mathcal{H}_S[r-1] & \perp \\ C\mathcal{C}_S[r-1] & \end{pmatrix} \tag{II.4}$$

with $\mathcal{H}_S[0] = (\emptyset)$. ■

Definition 2.3: (Extended vector) For an integer k and a natural number r, the vector $\hat{\mathbf{u}}[r,k] \in \mathbb{T}_{\max}^{r\cdot m}$ is defined recursively as

$$\hat{\mathbf{u}}[r+1,k] = (\hat{\mathbf{u}}[r,k]^T \ \mathbf{u}[k+r+1]^T)^T \tag{II.5}$$

with $\hat{\mathbf{u}}[0,k] = \mathbf{u}[k]$. ■

Then, the following lemma must be derived.

Lemma 2.1: (Iterated equation) Consider a system as in Equation (II.1). Then, for any real r

$$\mathbf{x}[k+r+1] = A^{r+1}\mathbf{x}[k] \oplus \mathcal{C}_S[r]\hat{\mathbf{u}}[r,k]. \tag{II.6}$$

Proof: The lemma is verified easily by r iterations of Equation (II.1). ■

Then, the principal result of this paper can be stated.

Proposition 2.1: (Steady state observer) If there is a natural number r and a matrix $R = (R[0]\ R[1]\ ...\ R[r]) \in \mathbb{T}_{\max}^{s\times(r+1)\cdot d}$ such that the system of equations

$$\begin{aligned} (i) \ &: \ WA^{r+1} = R\mathcal{O}_S[r]; \\ (ii) \ &: \ WA\mathcal{C}_S[r-1] = R\mathcal{H}_S[r] \end{aligned} \tag{II.7}$$

(with $\mathcal{C}_S[-1] \equiv (\emptyset)$) has a solution, then the proposed problem in Subsection II-A has a solution with $l = r+1$ and

$$\mathbf{s}[k+1] = WB\mathbf{u}[k] \oplus \bigoplus_{i=0}^{r} R[i]\mathbf{y}[k-r+i]. \tag{II.8}$$

Proof:

For simplicity, the proof is split in three parts.

Part I: According to Equation (II.7)-(i)

$$WA^{r+1} = \bigoplus_{i=0}^{r} R[i]CA^i. \tag{II.9}$$

Post multiply by $\mathbf{x}[k]$

$$WA^{r+1}\mathbf{x}[k] = \bigoplus_{i=0}^{r} R[i]CA^i\mathbf{x}[k]. \tag{II.10}$$

Add $W\mathcal{C}_S[r]\hat{\mathbf{u}}[k,r]$ in both sides

$$\begin{aligned}&WA^{r+1}\mathbf{x}[k] \oplus W\mathcal{C}_S[r]\hat{\mathbf{u}}[k,r] = \\ &\bigoplus_{i=0}^{r} R[i]CA^i\mathbf{x}[k] \oplus W\mathcal{C}_S[r]\hat{\mathbf{u}}[k,r].\end{aligned} \tag{II.11}$$

Factoring the matrix W at the left and using Lemma 2.1, one can obtain

$$\begin{aligned}&W\mathbf{x}[k+r+1] = \bigoplus_{i=0}^{r} R[i]CA^i\mathbf{x}[k] \oplus W\mathcal{C}_S[r]\hat{\mathbf{u}}[k,r] = \\ &R[0]\mathbf{y}[k] \oplus \bigoplus_{i=0}^{r-1} R[i+1]CA^{i+1}\mathbf{x}[k] \oplus W\mathcal{C}_S[r]\hat{\mathbf{u}}[k,r]\end{aligned} \tag{II.12}$$

in which in the right the sum from $i=0$ to r is split in the value for $i=0$ and a sum from $i=1$ to r, and the latter is reposed as a sum from $i=0$ to $r-1$.

Part II: Now consider Equation (II.7)-(ii), which reads as

$$WA^{r-j}B = \bigoplus_{i=j}^{r-1} R[i+1]CA^{i-j}B \ , \ j = 0,1,...,r-1. \tag{II.13}$$

Post multiply by $\mathbf{u}[k+j]$ and sum the resulting equation for $j=0,1,...,r-1$. Finally, add $WB\mathbf{u}[k+r]$ in both sides to arrive in

$$\begin{aligned}&\bigoplus_{j=0}^{r-1} WA^{r-j}B\mathbf{u}[k+j] \oplus WB\mathbf{u}[k+r] = \\ &WB\mathbf{u}[k+r] \oplus \bigoplus_{j=0}^{r-1}\bigoplus_{i=j}^{r-1} R[i+1]CA^{i-j}B\mathbf{u}[k+j].\end{aligned} \tag{II.14}$$

After factoring the matrix W, use the definition of the controllability matrix in the left side to conclude that

$$\begin{aligned}&W\mathcal{C}_S\hat{\mathbf{u}}[k,r] = \\ &WB\mathbf{u}[k+r] \oplus \bigoplus_{j=0}^{r-1}\bigoplus_{i=j}^{r-1} R[i+1]CA^{i-j}B\mathbf{u}[k+j].\end{aligned} \tag{II.15}$$

Exchange the order of summation using the rule $\bigoplus_{j=0}^{r-1}\bigoplus_{i=j}^{r-1} f(i,j) = \bigoplus_{i=0}^{r-1}\bigoplus_{j=0}^{i} f(i,j)$

$$\begin{aligned}&W\mathcal{C}_S\hat{\mathbf{u}}[k,r] = \\ &WB\mathbf{u}[k+r] \oplus \bigoplus_{i=0}^{r-1}\bigoplus_{j=0}^{i} R[i+1]CA^{i-j}B\mathbf{u}[k+j] = \\ &WB\mathbf{u}[k+r] \oplus \bigoplus_{i=0}^{r-1} R[i+1]C\left(\bigoplus_{j=0}^{i} A^{i-j}B\mathbf{u}[k+j]\right) = \\ &WB\mathbf{u}[k+r] \oplus \bigoplus_{i=0}^{r-1} R[i+1]C\mathcal{C}_S[i]\hat{\mathbf{u}}[i,k].\end{aligned} \tag{II.16}$$

Part III: Using the result obtained in Equation (II.16) in Equation (II.12)

$$\begin{aligned}&W\mathbf{x}[k+r+1] = \\ &R[0]\mathbf{y}[k] \oplus \bigoplus_{i=0}^{r-1} R[i+1]CA^{i+1}\mathbf{x}[k] \oplus W\mathcal{C}_S[r]\hat{\mathbf{u}}[k,r] = \\ &WB\mathbf{u}[k+r] \oplus R[0]\mathbf{y}[k] \oplus \\ &\bigoplus_{i=0}^{r-1} R[i+1]CA^{i+1}\mathbf{x}[k] \oplus \bigoplus_{i=0}^{r-1} R[i+1]C\mathcal{C}_S[i]\hat{\mathbf{u}}[i,k] = \\ &WB\mathbf{u}[k+r] \oplus \bigoplus_{i=0}^{r} R[i]C(A^{i+1}\mathbf{x}[k] \oplus \mathcal{C}_S[i]\hat{\mathbf{u}}[i,k]) = \\ &WB\mathbf{u}[k+r] \oplus \bigoplus_{i=0}^{r} R[i]\mathbf{y}[k+i]\end{aligned} \tag{II.17}$$

in which Lemma 2.1 was used.

Based on the iterations of $\mathbf{s}[k+1]$ in Equation (II.8), if one replaces the $(k+1)^{th}$ by the $(k+r+1)^{th}$ iteration, then one obtains the following equality:

$$\begin{aligned}\mathbf{s}[k+r+1] &= WB\mathbf{u}[k+r] \oplus \bigoplus_{i=0}^{r} R[i]\mathbf{y}[k+i] \\ &= W\mathbf{x}[k+r+1], \text{ due to (II.17)}.\end{aligned}$$

This means for any $k \geq r+1$, the equality between $\mathbf{s}[k]$ and $W\mathbf{x}[k]$ holds. And the proposition is proved. ∎

It is important to stress that Equation (II.7) can be written in matricial form as

$$W\underbrace{\begin{pmatrix} A^{r+1} & A\mathcal{C}_S[r-1] \end{pmatrix}}_{U} = R\underbrace{\begin{pmatrix} \mathcal{O}_S[r] & \mathcal{H}_S[r] \end{pmatrix}}_{V} \tag{II.18}$$

which is an equation of the form $WU = RV$, with U, V, W known and R unknown. It is a well known fact that this kind of equation has solution if and only if $((WU) \not\phi V)V = WU$ and that in this case $R = (WU) \not\phi V$ is the greatest solution (see [1]). Thus, the sufficient condition can be checked (and the parameters R computed) very easily in polynomial time. The observer will need initial conditions $\mathbf{y}[-1]$,...,$\mathbf{y}[-(r+1)]$ and $\mathbf{u}[-1]$. These initial conditions can be any vectors, because in at most r steps the correct value of $W\mathbf{x}[k]$ is recovered.

III. Application: Feedback Control

A. Methodology

As mentioned, one of the possible (and perhaps the main) applications of the proposed methodology is in the implementation of a state feedback control law of the form $\mathbf{u}[k] = F\mathbf{x}[k]$ in situations in which the matrix controller F is known (previously designed) but the state $\mathbf{x}[k]$ is not. In this case, the problem can be overcome by estimating the amount $F\mathbf{x}[k]$. So this problem can be easily recast as the (generalized) observation problem presented by choosing $W = F$. However, the observer takes some iterations to achieve the correct value and this implies perturbations in the system. Hence, in order to the proposed methodology to be useful, the feedback controller must be robust in the sense that it can reject any kind of eventual perturbations. In principle, the implementation of the observer could degrade the controller performance, and this is much as true as larger is the value of r in Equation (II.7). This will be, indeed, observed in simulations.

In order to illustrate the methodology, consider the problem considered in [5] which models a small traffic light. The matrices A and B are (see Figure 1, refer to [1] to see how to write the dynamics of a TEG as a tropical linear dynamical system)

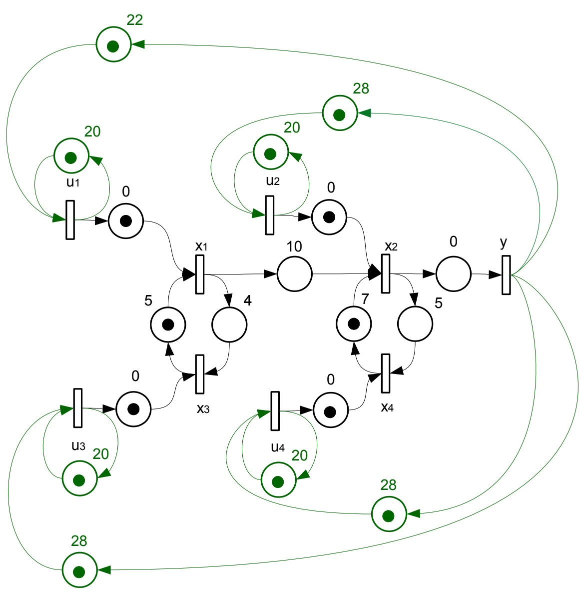

Fig. 1. The TEG for a small traffic light. In green, the implementation of the output feedback controller.

$$A = \begin{pmatrix} 0 & \perp & 5 & \perp \\ 10 & 0 & 15 & 7 \\ 4 & \perp & 9 & \perp \\ 15 & 5 & 20 & 12 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \perp & \perp & \perp \\ 10 & 0 & \perp & \perp \\ 4 & \perp & 0 & \perp \\ 15 & 5 & \perp & 0 \end{pmatrix}. \tag{III.1}$$

The constraint is

$$\mathbf{x}[k] = \underbrace{\begin{pmatrix} 0 & -15 & -15 & -30 \\ 10 & 0 & -5 & -15 \\ 6 & -11 & 0 & -26 \\ 15 & 5 & 0 & 0 \end{pmatrix}}_{D} \mathbf{x}[k]. \tag{III.2}$$

Using the methodology proposed in [5], it can be shown that the controller $\mathbf{u}[k] = F\mathbf{x}[k] = \boldsymbol{\mu}\boldsymbol{\zeta}^T\mathbf{x}[k]$ with

$$\begin{aligned} \boldsymbol{\mu} &= (5\ 11\ 11\ 11)^T; \\ \boldsymbol{\zeta} &= (0\ 0\ 0\ 0)^T \end{aligned} \tag{III.3}$$

solves the problem *robustly*: for any initial condition $\mathbf{x}[0]$, in at most one step the system complies with the desired constraints. Even if there is an arbitrary perturbation in the state (eventually driving $\mathbf{x}[k]$ out of the constraint semimodule), the controller will eventually drive the system back again to the required specifications. This happens because, due to the fact that the matrices A, B do not depend on k (event invariance), one can consider the evolution from the perturbed state as a new evolution, of the same system, but with a new initial condition which is exactly this perturbed state. Since the convergence is guaranteed for any initial condition, eventually the system will converge again to the desired set.

All the controllers presented in [5] can be factored in the form $F = \boldsymbol{\mu}\boldsymbol{\zeta}^T$ for vectors $\boldsymbol{\mu}$ and $\boldsymbol{\zeta}$. This factorization is highly proficuous for the proposed methodology. Indeed, using this feedback

$$\mathbf{x}[k+1] = A\mathbf{x}[k] \oplus (B\boldsymbol{\mu})(\boldsymbol{\zeta}^T\mathbf{x}[k]) \tag{III.4}$$

Let $\mathbf{b} \equiv B\boldsymbol{\mu}$ then

$$\mathbf{x}[k+1] = A\mathbf{x}[k] \oplus \mathbf{b}(\boldsymbol{\zeta}^T\mathbf{x}[k]). \tag{III.5}$$

This implies that in order to implement this controller one can consider a new system with the same matrix A and the matrix B replaced by the vector $\mathbf{b}$. In this new system the control input $v[k]$ is scalar, and the original controller can be recovered if $v[k] = \boldsymbol{\zeta}^T\mathbf{x}[k]$. This implies that it is only necessary to observe a scalar functional $\boldsymbol{\zeta}^T\mathbf{x}[k]$ in this new system.

Suppose one only observes $x_2[k]$, that is $C = (\perp\ 0\ \perp\ \perp)$ and $y[k] = x_2[k]$ (see Figure 1). Then, the task is to implement the state feedback controller using only the outputs and the inputs. To this end, it is possible to solve Equation (II.7) (or equivalently Equation (II.18)), for the reduced system (with A and $\mathbf{b}$) for $r = 1$, and obtain $(R[0]\ R[1]) = (17\ 17)$. Hence $\boldsymbol{\zeta}^T\mathbf{x}[k]$ can be observed as

$$s[k+1] = 20v[k] \oplus 17y[k] \oplus 17y[k-1] = 20v[k] \oplus 17y[k] \tag{III.6}$$

since $y[k] \succeq y[k-1]$. As the control input will be chosen as $\boldsymbol{\zeta}^T\mathbf{x}[k]$, $v[k] = s[k]$ and hence one has the dynamical equation for the control action of the reduced system

$$v[k+1] = 20v[k] \oplus 17y[k] \tag{III.7}$$

in which the initial conditions $v[-1]$ and $y[-1]$ can be chosen in an arbitrary manner. Therefore, post-multiplying both sides of Equation (III.7) by $\boldsymbol{\mu}$, it is easy to see that the control input $\mathbf{u}[k] = \boldsymbol{\mu} v[k]$ of the original system can be computed according to the dynamical equation

$$\mathbf{u}[k+1] = 20\mathbf{u}[k] \oplus \begin{pmatrix} 22 \\ 28 \\ 28 \\ 28 \end{pmatrix} y[k] \tag{III.8}$$

in which the initial conditions $\mathbf{u}[-1]$ and $y[-1]$ can be chosen in an arbitrary manner. See Figure 1 for the implementation.

B. Simulation

The performance of both the state feedback controller and output controller will now be tested and compared. For fairness, in both cases the same initial condition was considered and the same perturbations were inflicted in them. 21 steps were simulated (from $k = 0$ to $k = 20$).

The perturbations are

- At $k = 4$, a delay of 20 time units was added at x_1 and 15 time units at x_3;
- At $k = 9$, a delay of 12 time units was added at x_2 and 20 time units at x_4;
- At $k = 14$, a delay of 8 time units was added at x_1 and 30 time units at x_2.

Consider the initial condition

$$\mathbf{x}[0] = (11\ 27\ 15\ 32)^T \tag{III.9}$$

chosen at random in *Im*$\{A\}$ (so it is *feasible*). The initial conditions $u[-1], y[-1]$ were chosen as $\perp$. In Figure 2, it is possible to see the average error from the constraint set ($\hat{e}[k] = \frac{1}{4}\sum_{i=1}^{4} e_i[k]$ in which $\mathbf{e}[k] = D\mathbf{x}[k] - \mathbf{x}[k]$). One can see that, clearly, the insertion of the observer degrades slightly the performance of the controller, since instead of only one step it takes two steps to totally reject the perturbation.

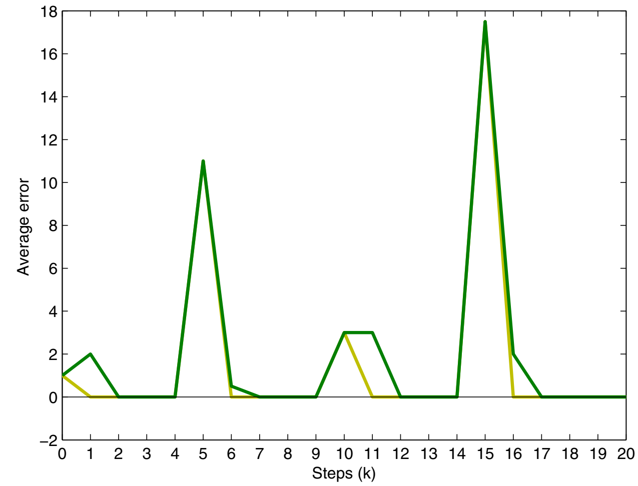

Fig. 2. Average error at each step. Yellow for the state feedback and green for the output feedback.

IV. Application: Perturbation observation problem

A. Methodology

Now, a problem of observation discussed in [3] and in [6] (the latter using the methodology proposed in [4]) will be discussed. It concerns a flowshop system for which the dynamics are modeled in the form $\mathbf{x}[k+1] = A\mathbf{x}[k]$ (no control inputs) and an output $\mathbf{y}[k] = C\mathbf{x}[k]$ is observed. Some timings of the system lie in an interval, and the challenge is to develop an observer with the outputs that is able to retrieve the perturbed states.

The matrices are (see Figure 3)

$$A = \begin{pmatrix} \perp & \perp & 4 & \perp & \perp & \perp & 2 & \perp & \perp \\ [1\ 7] & \perp & \perp & \perp & \perp & \perp & \perp & 3 & \perp \\ \perp & 5 & \perp & \perp & \perp & \perp & \perp & \perp & 1 \\ 4 & \perp & \perp & \perp & \perp & 3 & \perp & \perp & \perp \\ \perp & [3\ 5] & \perp & [1\ 3] & \perp & \perp & \perp & \perp & \perp \\ \perp & \perp & 5 & \perp & 4 & \perp & \perp & \perp & \perp \\ \perp & \perp & \perp & 4 & \perp & \perp & \perp & \perp & 3 \\ \perp & \perp & \perp & \perp & 3 & \perp & 5 & \perp & \perp \\ \perp & \perp & \perp & \perp & \perp & 2 & \perp & 4 & \perp \end{pmatrix}; \tag{IV.1}$$

$$C = \begin{pmatrix} \perp & \perp & 0 & \perp & \perp & \perp & \perp & \perp & \perp \\ \perp & \perp & \perp & \perp & \perp & 0 & \perp & \perp & \perp \\ \perp & \perp & \perp & \perp & \perp & \perp & \perp & 0 & \perp \end{pmatrix} \tag{IV.2}$$

with the intervals, so for instance the entry A_{21} can be any number between 1 and 7. As pointed by [6], perturbations in the matrix entries can be interpreted as (tropical) additive entries in a system in which a matrix is unperturbed. So, for instance, the system modeled by $\mathbf{x}[k] = A\mathbf{x}[k]$, in which some entries are in an interval, can be written as

$$\begin{aligned} \mathbf{x}[k+1] &= \underline{A}\mathbf{x}[k] \oplus P\mathbf{p}[k]; \\ \mathbf{y}[k] &= C\mathbf{x}[k]. \end{aligned} \tag{IV.3}$$

In which

$$\underline{A} = \begin{pmatrix} \perp & \perp & 4 & \perp & \perp & \perp & 2 & \perp & \perp \\ 1 & \perp & \perp & \perp & \perp & \perp & \perp & 3 & \perp \\ \perp & 5 & \perp & \perp & \perp & \perp & \perp & \perp & 1 \\ 4 & \perp & \perp & \perp & \perp & 3 & \perp & \perp & \perp \\ \perp & 3 & \perp & 1 & \perp & \perp & \perp & \perp & \perp \\ \perp & \perp & 5 & \perp & 4 & \perp & \perp & \perp & \perp \\ \perp & \perp & \perp & 4 & \perp & \perp & \perp & \perp & 3 \\ \perp & \perp & \perp & \perp & 3 & \perp & 5 & \perp & \perp \\ \perp & \perp & \perp & \perp & \perp & 2 & \perp & 4 & \perp \end{pmatrix}; \tag{IV.4}$$

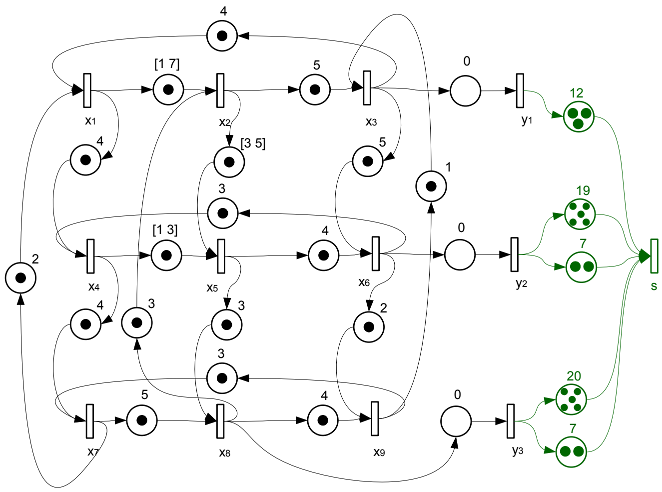

Fig. 3. The TEG for the flowshop. In green, the implementation of the observer for $x_7[k]$.

$$P^T = \begin{pmatrix} \perp & 0 & \perp & \perp & \perp & \perp & \perp & \perp & \perp \\ \perp & \perp & \perp & \perp & 0 & \perp & \perp & \perp & \perp \end{pmatrix}. \quad \text{(IV.5)}$$

The vector $\mathbf{p}[k] \in \mathbb{T}^2_{\max}$ can be chosen to achieve perturbations in the entry of the matrix (see [6] for details). Both papers [3], [6] focus in the transition $x_7[k]$ so, for comparison, the problem of observing this transition will be handled using the proposed methodology. Note that the system in Equation (IV.3) fits the one considered in Equation (II.1), with the matrix P playing the role of B and $\mathbf{p}[k]$ of $\mathbf{u}[k]$. The only difference is that, as opposed to $\mathbf{u}[k]$, the perturbation $\mathbf{p}[k]$ cannot be measured. However, if one chooses $W = (\perp\ \perp\ \perp\ \perp\ \perp\ \perp\ \ 0\ \ \perp\ \perp)$ (it is desirable to observe only $x_7[k]$) then $WP\ = \perp$, and the observer presented in Equation (II.8) will not require the measure of $\mathbf{p}[k]$.

Solving Equation (II.7) for $r = 4$, one can obtain $R[0] = (12\ 19\ 20)$, $R[1] = (12\ 7\ 7)$, $R[2] = (12\ 7\ 7)$, $R[3] = (\perp\ 7\ 7)$ and $R[4] = (\perp\ \perp\ \perp)$. Hence

$$\begin{aligned} s[k+1] =& 12y_1[k-4] \oplus 19y_2[k-4] \oplus 20y_3[k-4] \oplus \\ & 12y_1[k-3] \oplus 7y_2[k-3] \oplus 7y_3[k-3] \oplus \\ & 12y_1[k-2] \oplus 7y_2[k-2] \oplus 7y_3[k-2] \oplus \\ & 7y_2[k-1] \oplus 7y_3[k-1]. \end{aligned} \quad \text{(IV.6)}$$

Using the fact that the sequence $\mathbf{y}[k]$ is non-decreasing, some simplifications can be made in Equation (IV.6)

$$\begin{aligned} s[k+1] =& 19y_2[k-4] \oplus 20y_3[k-4] \oplus 12y_1[k-2] \oplus \\ & 7y_2[k-1] \oplus 7y_3[k-1] \end{aligned} \quad \text{(IV.7)}$$

(see Figure 3 for the implementation) and $s[k] = x_7[k]$ for $k \geq 4 + 1 = 5$, regardless which initial conditions $\mathbf{y}[-5]$, $\mathbf{y}[-4]$, $\mathbf{y}[-3]$, $\mathbf{y}[-2]$, $\mathbf{y}[-1]$ are chosen for the observer.

V. Conclusion

In this paper, a sufficient condition for solving a specific observation problem is presented. Using the residuation theory, the observer can be derived very easily in polynomial time. Applications of the methodology in a feedback control implementation and observation in perturbed systems are considered. The authors believe that the conditions presented in Proposition 2.1 can be weakened, and are working in that direction.

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