2.7 Bandlimited Signals and Noise

Bandlimited Waveform

DEFINIATION. A waveform ***w(t)*** is said to be (absolutely) bandlimited to *B* hertz, if

$$W(f) = \mathfrak{I}[w(t)] = 0 \qquad \text{for } |f| \geq B$$

DEFINIATION. A waveform ***w(t)*** is said to be (absolutely) time limited if

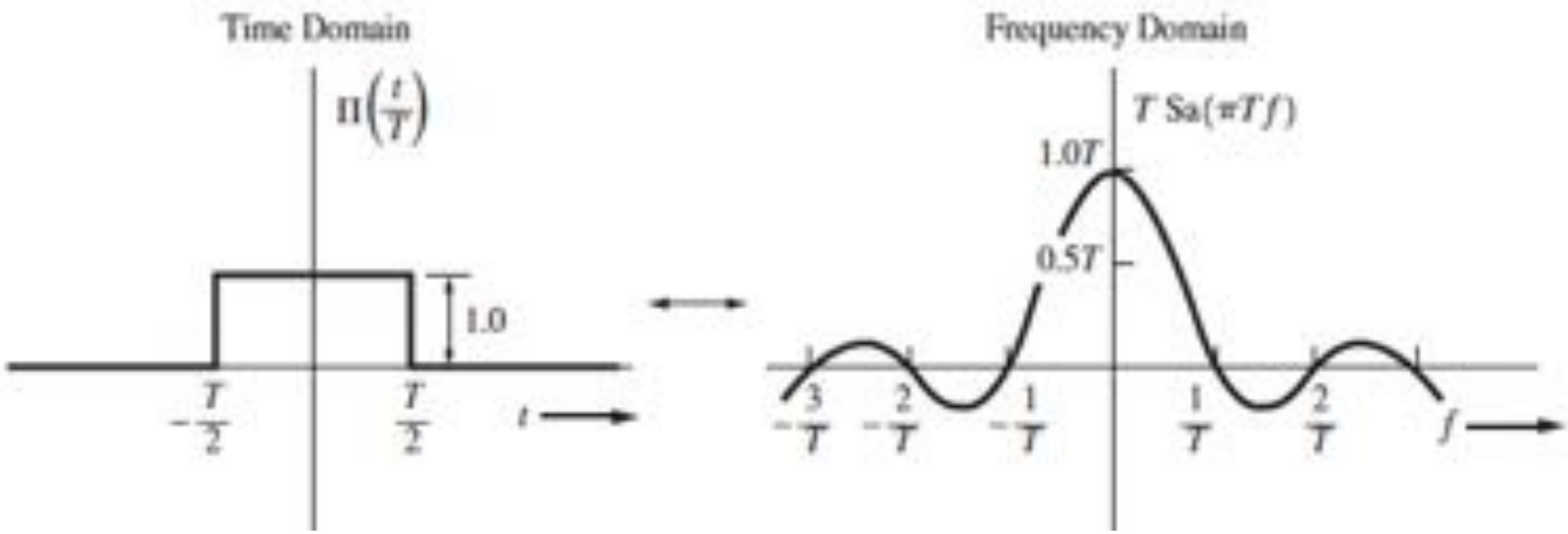
$$w(t) = 0, \qquad \text{for } |t| \geq T$$

2.7 Bandlimited Signals and Noise

Bandlimited Waveform

THEOREM. A absolutely bandlimited waveform cannot be absolutely time limited, and vice versa.

A physical waveform that is time limited, may not be absolutely bandlimited, but it may be bandlimited for all practical purposes in the sense that the amplitude spectrum has a negligible level above a certain frequency.



2.7 Bandlimited Signals and Noise

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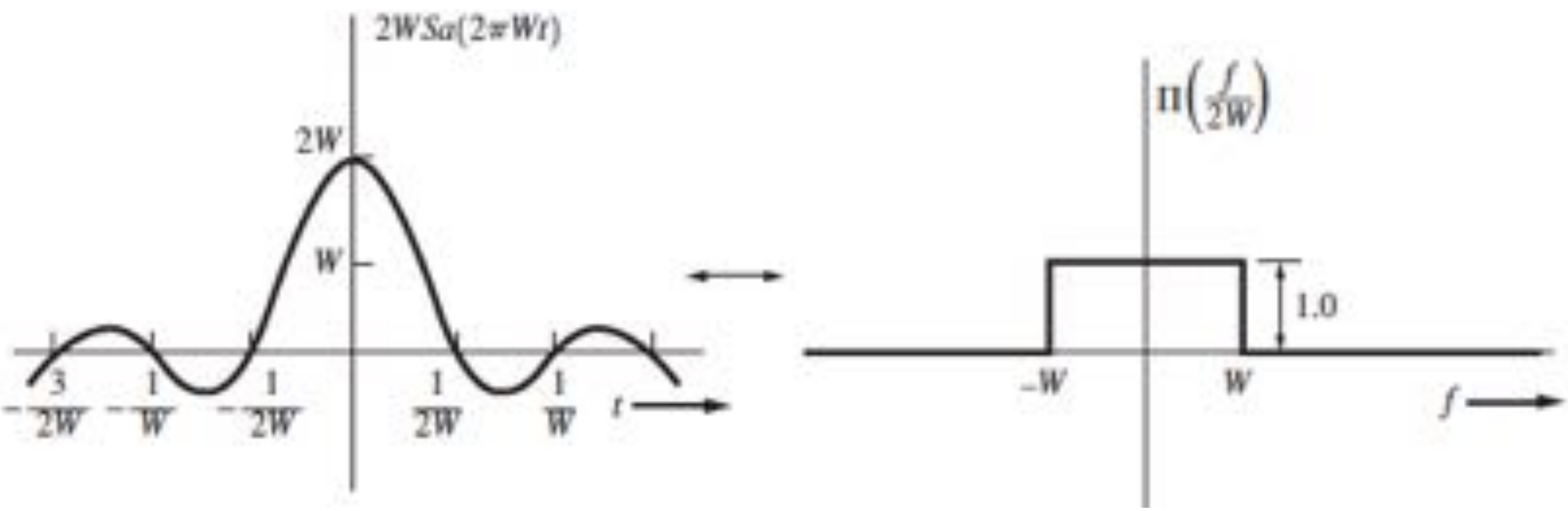


2.7 Bandlimited Signals and Noise

Sampling Theorem

Sampling Theorem. ***Any physical*** waveform may be represented over the interval $-\infty < t < \infty$ by

$$w(t) = \sum_{n=-\infty}^{\infty} a_n \frac{\sin\left\{\pi f_s \left[t-(n/f_s)\right]\right\}}{\pi f_s \left[t-(n/f_s)\right]}$$

where

$$a_n = f_s \int_{-\infty}^{\infty} w(t) \frac{\sin\left\{\pi f_s [t-(n/f_s)]\right\}}{\pi f_s \left[t-(n/f_s)\right]} dt$$

And f_s is a parameter that is assigned some convenient value greater than zero. Furthermore, if ***w(t)*** is bandlimited to *B* hertz and $f_s \geq 2B$, then previous equation becomes the sampling function representation, where

$$a_n = w(n/f_s)$$

That is, for $f_s \geq 2B$, the orthogonal series coefficients are simply the values of the waveform that are obtained when the waveform is sampled very $1/f_s$ seconds.

2.7 Bandlimited Signals and Noise

Sampling Theorem

- ✧ The MINIMUM SAMPLING RATE allowed for reconstruction without error is called the NYQUIST FREQUENCY or the Nyquist Rate. $(f_s)_{Min} = 2B$

- ✧ Suppose we are interested in reproducing the waveform over a T_0-sec interval, the minimum number of samples that are needed to reconstruct the waveform is:

$$N = \frac{T_0}{1/f_s} = f_s T_0 \geq 2BT_0$$

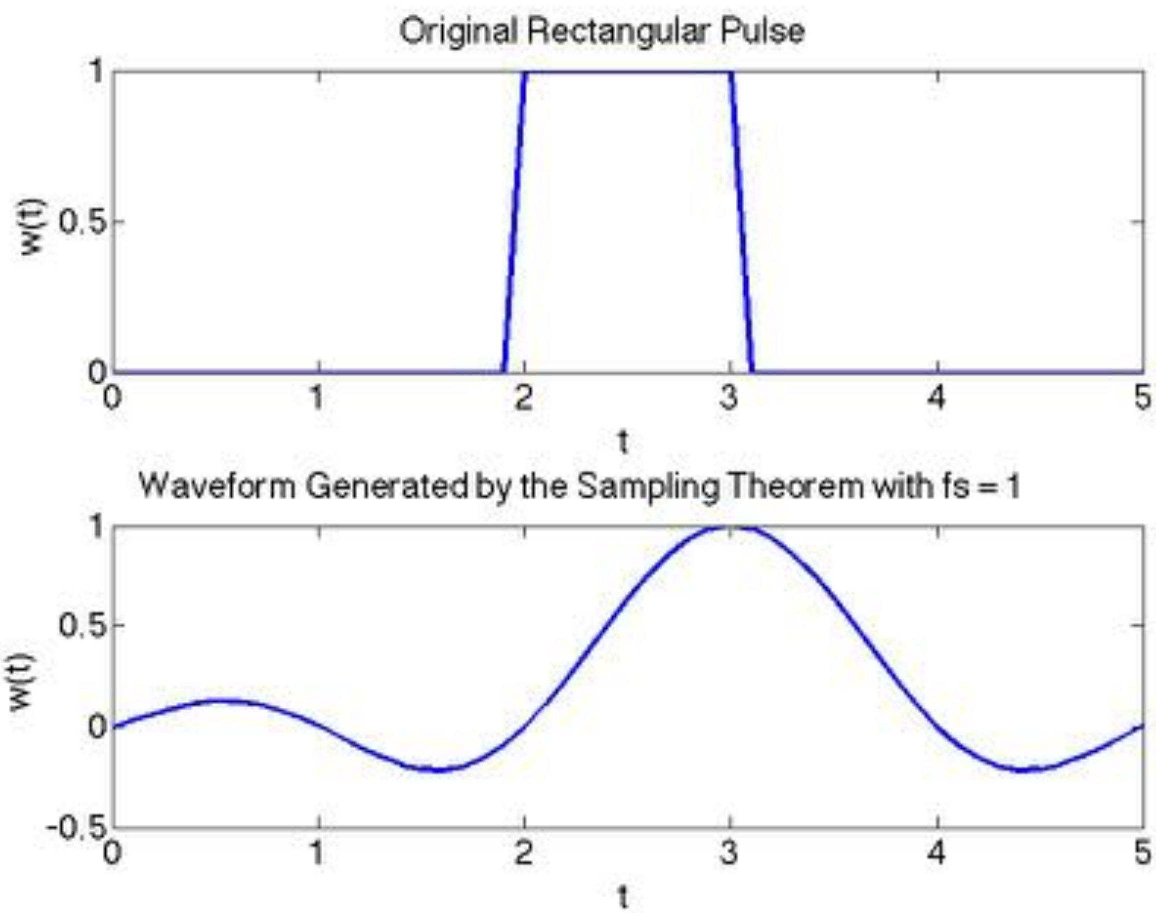
- There are *N* orthogonal functions in the reconstruction algorithm. We can say that *N* is the Number of Dimensions needed to reconstruct the T_0-second approximation of the waveform.

- The sample values may be saved, for example in the memory of a digital computer, so that the waveform may be reconstructed later, or the values may be transmitted over a communication system for waveform reconstruction at the receiving end.

2.7 Bandlimited Signals and Noise

Sampling Theorem

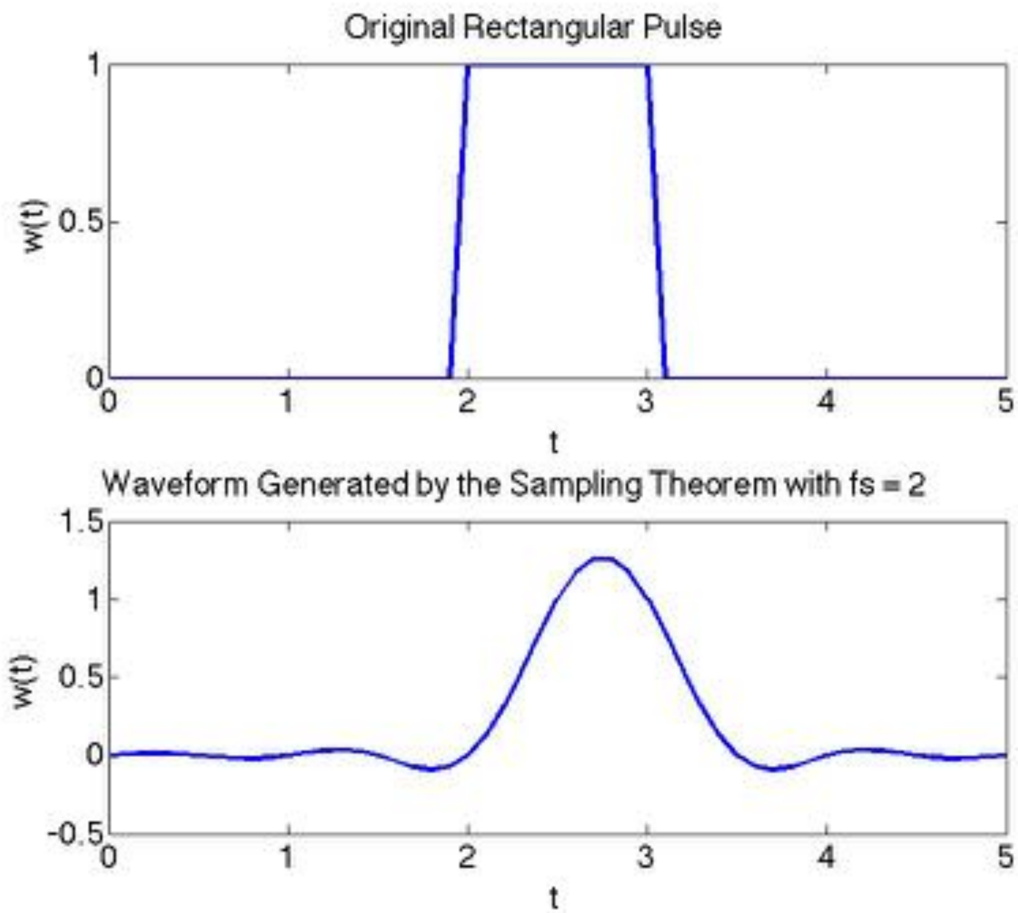
Example 2-19. Sampling theorem for a rectangular pulse



2.7 Bandlimited Signals and Noise

Sampling Theorem

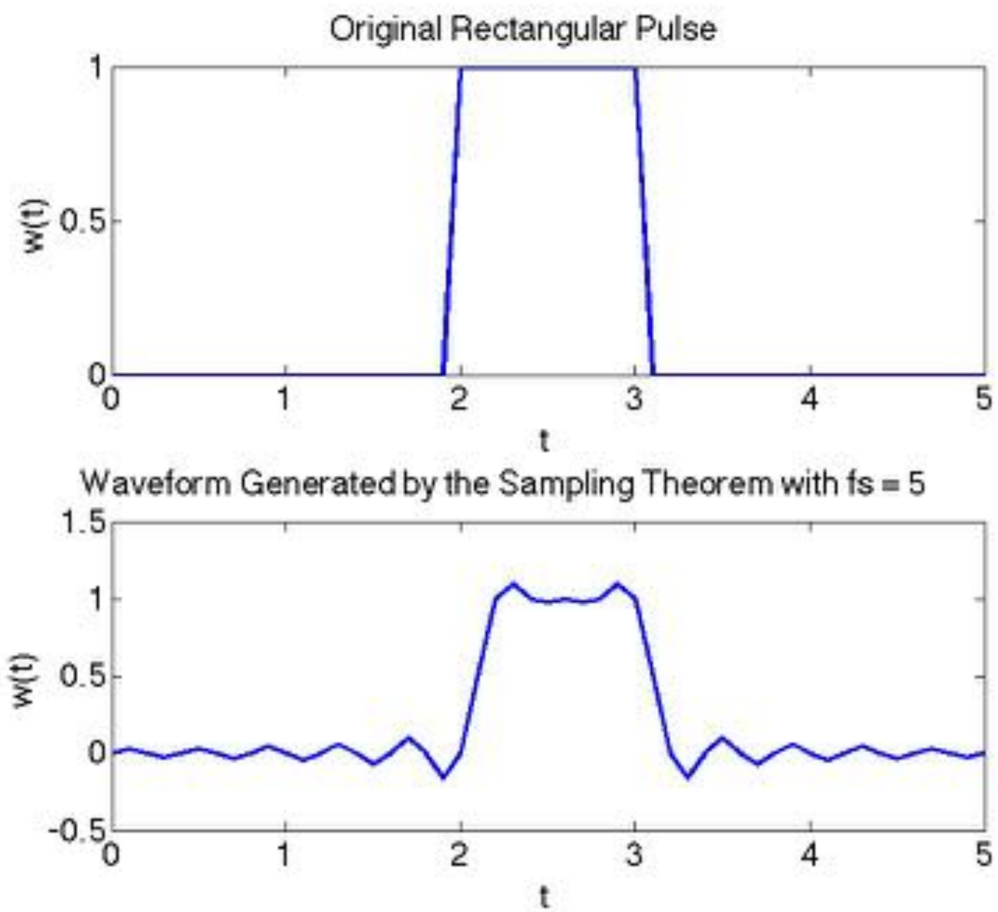
Example 2-19. Sampling theorem for a rectangular pulse



2.7 Bandlimited Signals and Noise

Sampling Theorem

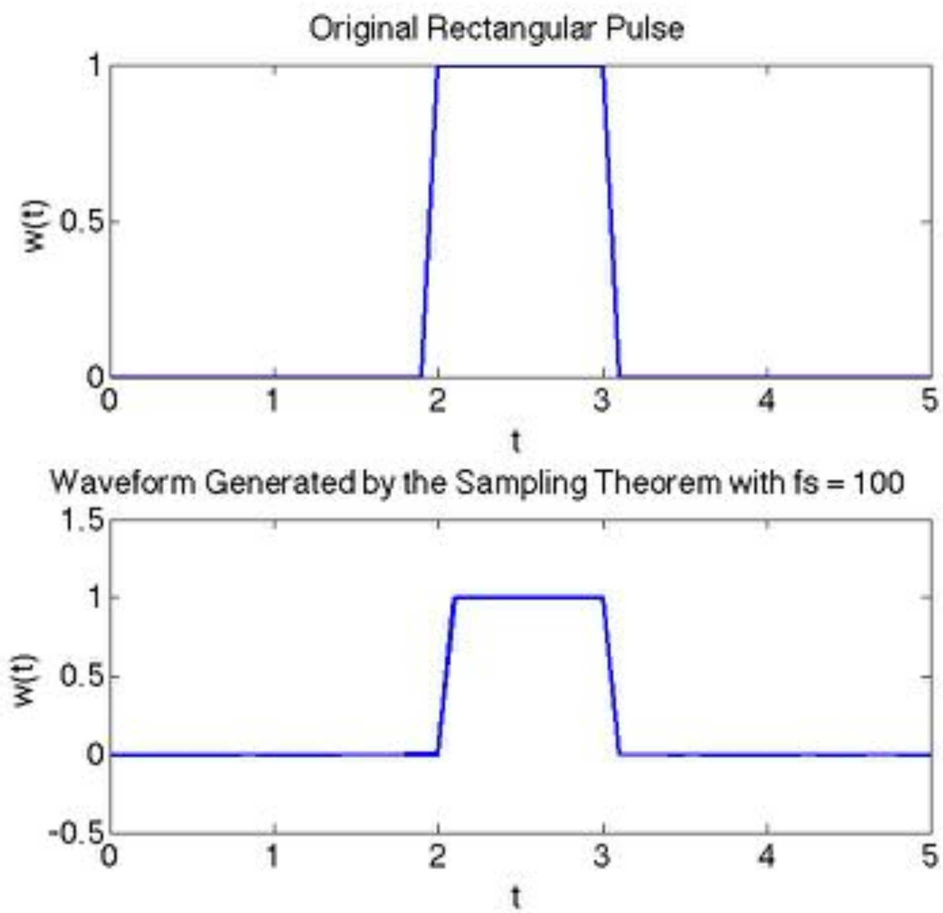
Example 2-19. Sampling theorem for a rectangular pulse



2.7 Bandlimited Signals and Noise

Sampling Theorem

Example 2-19. Sampling theorem for a rectangular pulse

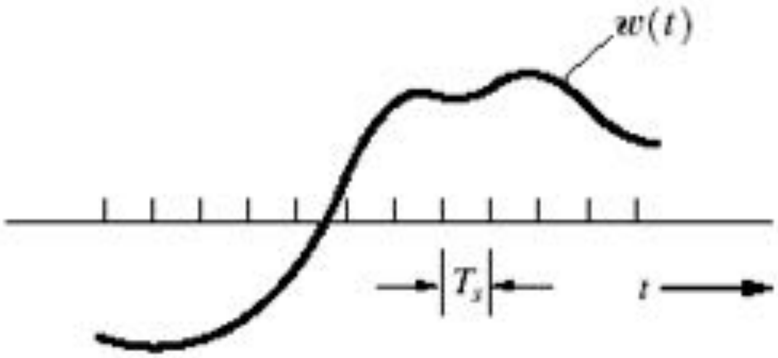


2.7 Bandlimited Signals and Noise

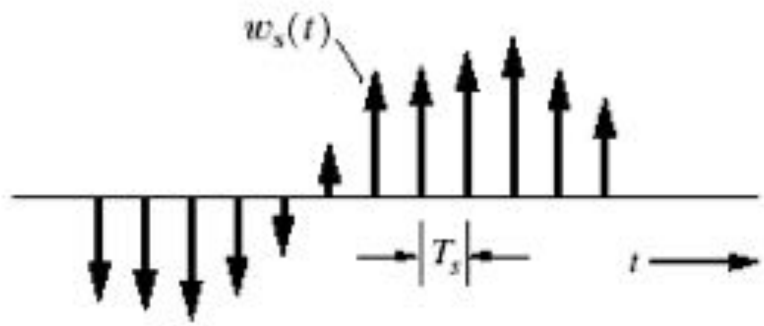
Impulse Sampling and Digital Signal Processing

The ***impulse-sampled series*** is another **orthogonal series**. It is obtained when the (sin *x*) / *x* orthogonal functions of the sampling theorem are replaced by an orthogonal ***set of delta (impulse) functions***. The impulse-sampled series is identical to the impulse-sampled waveform ***$w_s(t)$***: both can be obtained by multiplying the unsampled waveform by a unit-weight impulse train, yielding

$$w_s(t) = w(t)\sum_{n=-\infty}^{\infty}\delta(t - nT_s) = \sum_{n=-\infty}^{\infty} w(nT_s)\delta(t - nT_s)$$



Waveform



Impulse sampled waveform

2.7 Bandlimited Signals and Noise

Impulse Sampling and Digital Signal Processing

$$w_s(t) = w(t)\sum_{n=-\infty}^{\infty}\delta(t - nT_s) \quad = \sum_{n=-\infty}^{\infty} w(nT_s)\delta(t - nT_s)$$

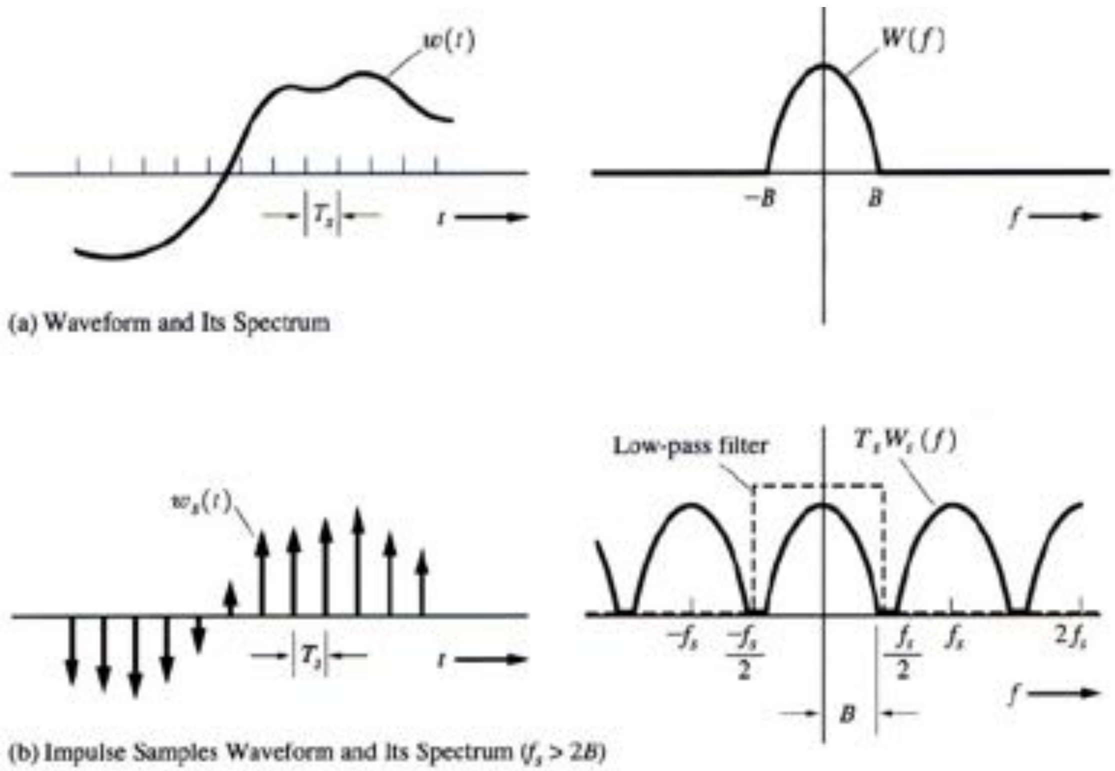
Take the Fourier transform on both sides of this equation:

$$W_s(f) = \frac{1}{T_s}W(f) * \Im\left[\sum_{n=-\infty}^{\infty} e^{jnw_s t}\right] = \frac{1}{T_s}W(f) * \sum_{n=-\infty}^{\infty}\Im[e^{jnw_s t}]$$

$$= \frac{1}{T_s}W(f) * \sum_{n=-\infty}^{\infty}\delta(f - nf_s)$$

Or $$= \frac{1}{T_s}\sum_{n=-\infty}^{\infty} W(f - nf_s)$$

2.7 Bandlimited Signals and Noise



➢ The spectrum of the impulse sampled signal is the spectrum of the unsampled signal that is repeated every f_s Hz, where f_s is the sampling frequency (samples/sec).

➢ This is quite significant for *digital signal processing* (DSP).

➢ This technique of impulse sampling maybe be used to translate the spectrum of a signal to another frequency band that is centered on some harmonic of the sampling frequency.

2.7 Bandlimited Signals and Noise

Dimensionality Theorem

THEOREM: *When BT_0 is large, a real waveform may be completely specified by*

$$N=2BT_0$$

independent pieces of information that will describe the waveform over a T_0 interval. N is said to be the number of dimensions required to specify the waveform, and B is the absolute bandwidth of the waveform.

- The information which can be conveyed by a bandlimited waveform or a bandlimited communication system is proportional to the product of the bandwidth of that system and the time allowed for transmission of the information.

- The dimensionality theorem has profound implications in the design and performance of all types of communication systems.

2.8 Discrete Fourier Transform

DEFINIATION: The ***discrete Fourier transform*** (**DFT**) is defined by

$$X(n) = \sum_{k=0}^{k=N-1} x(k) e^{-j(2\pi/N)nk}$$

Where *n = 0, 1, 2, …, N-1*, and the **inverse discrete Fourier transform** (**IDFT**)
Is defined by

$$x(k) = \frac{1}{N} \sum_{k=0}^{k=N-1} X(n) e^{j(2\pi/N)nk}$$

Where *k = 0, 1, 2, …, N-1,*

- ✧ The definition could be different according to different authors, which Only effect the "scale factor" and "frequency factor".

- ✧ The fast Fourier transform (FFT) is a fast algorithm for evaluating the DFT

2.8 Discrete Fourier Transform

Using the DFT to Compute the Continuous Fourier Transform

In digital signal processing, we use DFT (approximation)to represent CFT (truth) through three steps.

✧ **Step 1**: the time waveform is first windowed (truncated) over the interval (***0, T***) so that only a finite number of samples ***N*** are needed

$$w_w(t) = \begin{cases} w(t), & 0 \le t \le T \\ 0, & otherwise \end{cases} = w(t)\prod\left(\frac{t-(T/2)}{T}\right)$$

✧ **Step 2**: do the Fourier transform on the windowed waveform

$$W_w(f) = \int_{-\infty}^{\infty} w_w(t) e^{-j2\pi ft}\, dt = \int_0^T w(t) e^{-j2\pi ft}\, dt$$

2.8 Discrete Fourier Transform

Using the DFT to Compute the Continuous Fourier Transform

In digital signal processing, we use DFT (approximation)to represent CFT (truth) through three steps.

✧ **Step 3**: approximate the CFT by using a finite series to represent the integral

$$W_w(f)\Big|_{f=n/T} \approx \sum_{k=0}^{N-1} w(k\Delta t)e^{-j(2\pi/N)nk}\Delta t$$

Where $t = k\Delta t, f = n / T, dt = \Delta t, \quad and \quad \Delta t = T / N$

The relation between CFT and DFT is

$$W_w(f)\Big|_{f=n/T} \approx \Delta t X(n)$$

Where

$$X(n) = \sum_{k=0}^{k=N-1} x(k)e^{-j(2\pi/N)nk}$$

2.8 Discrete Fourier Transform

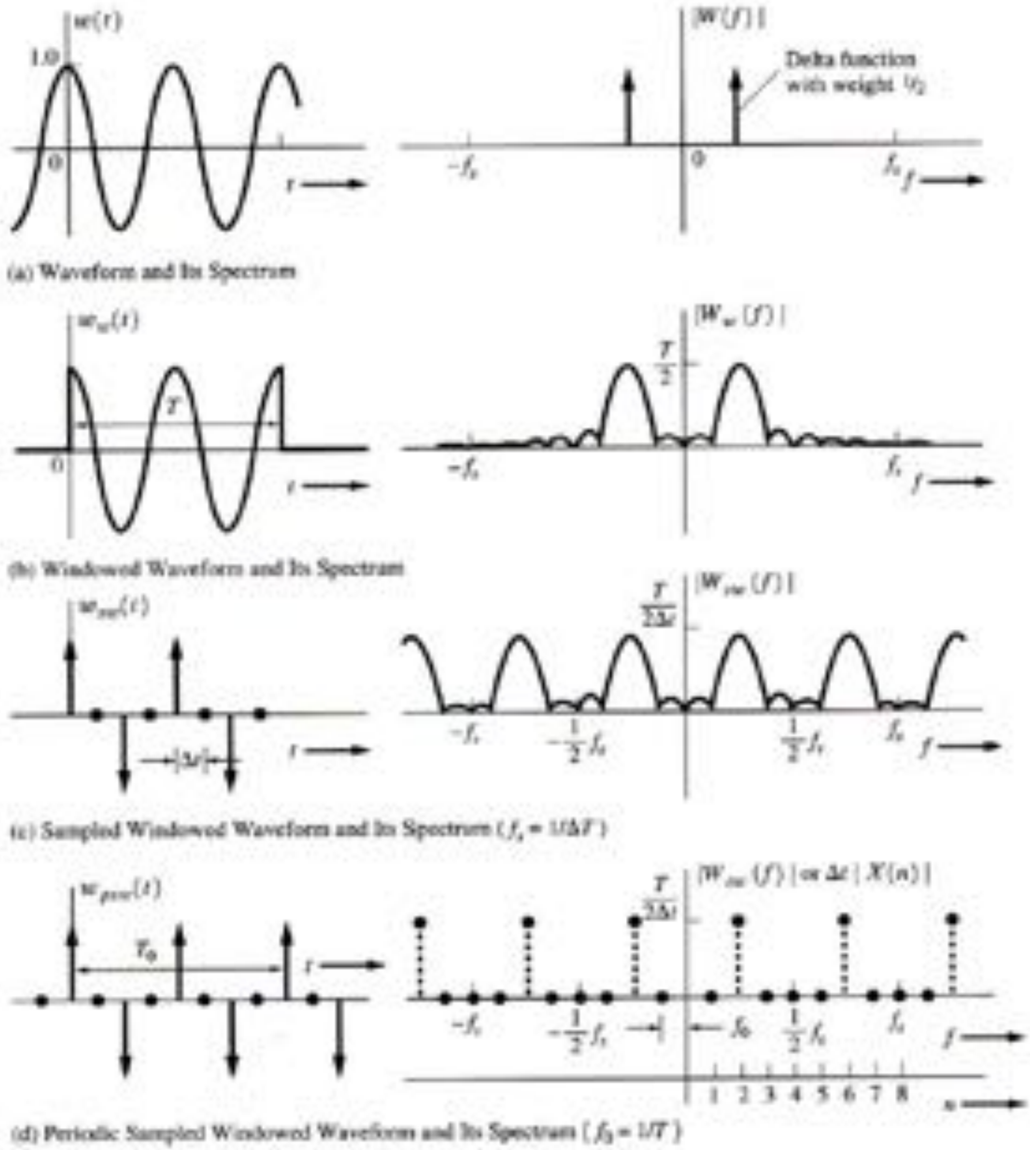


Figure 2–20 Comparison of CFT and DFT spectra.

The DFT may give significant errors when it is used to approximate the CFT. The errors are due to a number of factors that may be categorized into three basic effects: ***leakage***, ***aliasing***, And the ***picket-fence effect***.

2.8 Discrete Fourier Transform

Using the DFT to Compute the Fourier Series

The DFT may be also used to evaluate the coefficients for the complex Fourier series.

From $$c_n = \frac{1}{T}\int_0^T w(t)e^{-j2\pi n f_0 t}\,dt$$

We approximate this integral by using a finite series

$$c_n \approx \frac{1}{T}\sum_{k=0}^{N-1} w(k\Delta t)e^{-j(2\pi/N)nk}\Delta t$$

where $t = k\Delta t, f = n/T, dt = \Delta t,\ \ and\ \ \Delta t = T/N$

The Fourier series coefficient is related to the DFT by

$$c_n \approx \frac{1}{N}X(n)$$

2.8 Discrete Fourier Transform

Using the DFT to Compute the Fourier Series

The Fourier series coefficient is related to the DFT by

$$c_n \approx \frac{1}{N} X(n)$$

For positive ***n***, we use

$$c_n = \frac{1}{N} X(n) \qquad 0 \le n < N/2$$

For negative ***n***, we use

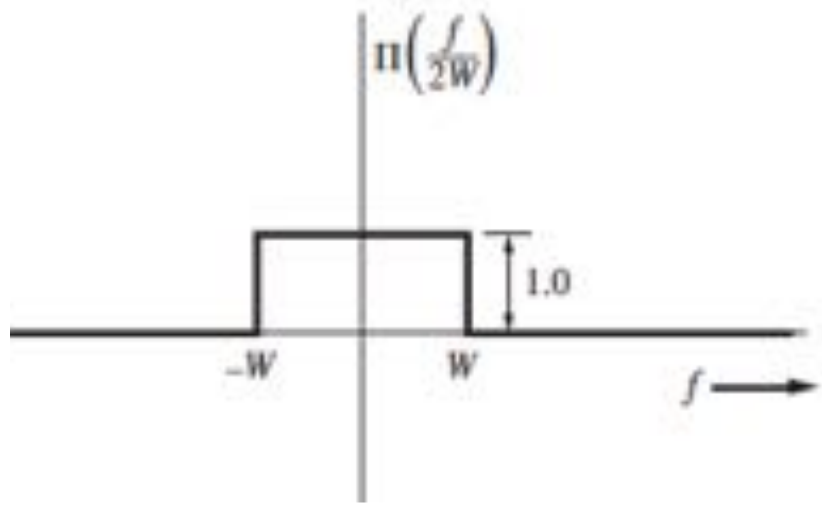
$$c_n = \frac{1}{N} X(N+n) \qquad -N/2 < n < 0$$

2.9 Bandwidth of Signals

In engineering definitions, the bandwidth is taken to be the width of positive frequency band.

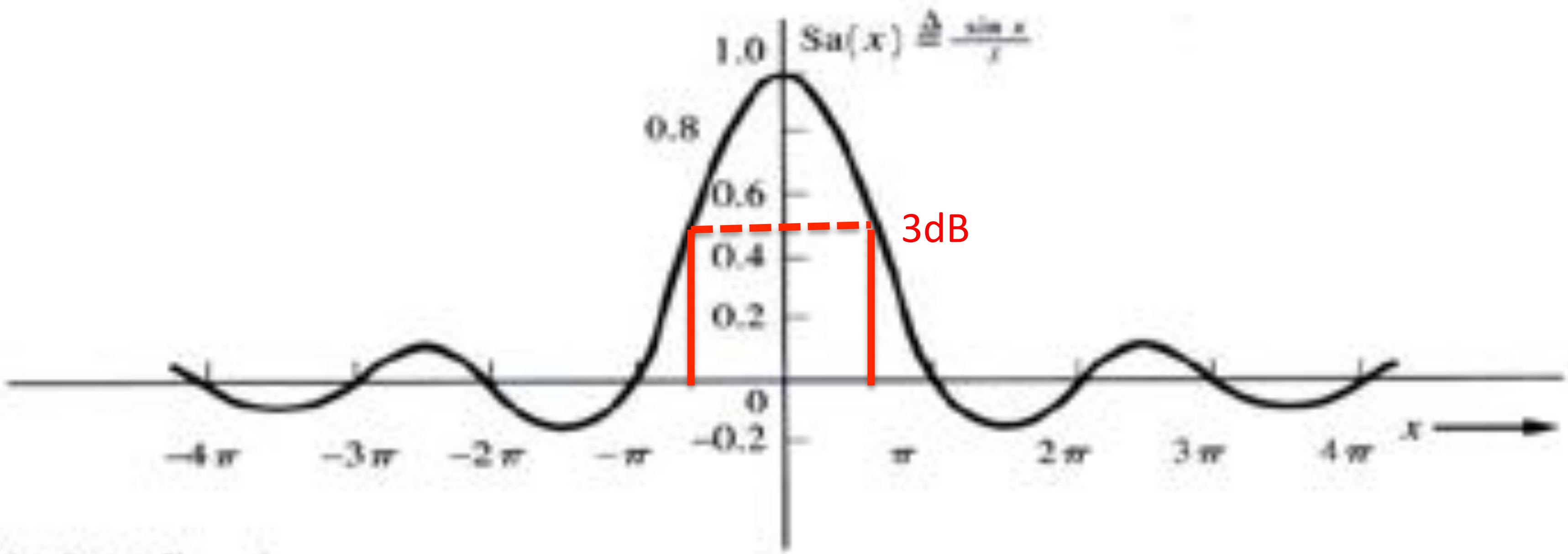
We will give six engineering definitions and one legal definition of Bandwidth that are often used.

✧ ***Absolute bandwidth*** is $f_2 - f_1$: where the spectrum is zero outside the Interval $f_1 < f < f_2$ along the positive frequency axis.



2.9 Bandwidth of Signals

- ***3-dB bandwidth (or half-power bandwidth)*** is $f_2 - f_1$: where for frequency inside the band $f_1 < f < f_2$, the magnitude spectra, say, $|H(f)|$, fall no lower than 1/√2 times the maximum value of $|H(f)|$, and the maximum value occurs at a frequency inside the band.

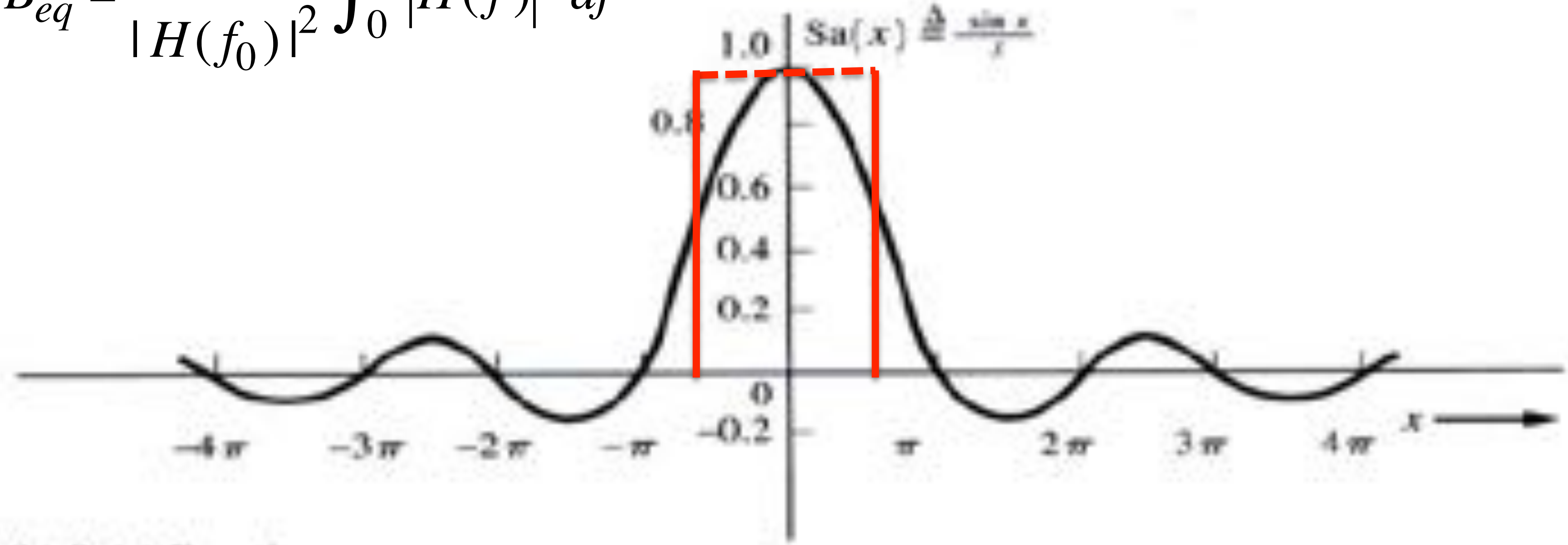


(b) Sa(x) Function

2.9 Bandwidth of Signals

✧ ***Equivalent noise bandwidth***: the width of a fictitious rectangular spectrum such that the power in that rectangular band is equal to the power associated with the actual spectrum over positive frequencies.

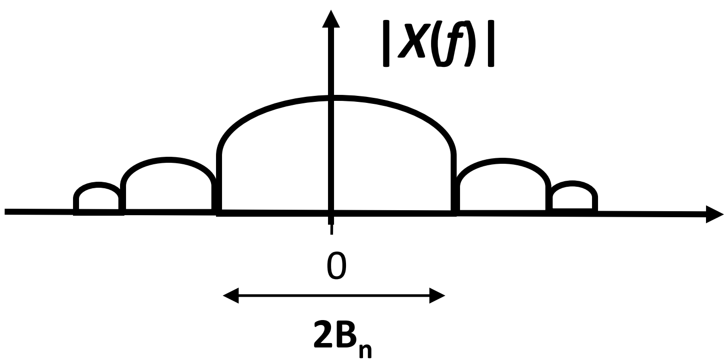
$$B_{eq} = \frac{1}{|H(f_0)|^2} \int_0^{\infty} |H(f)|^2 \, df$$



(b) Sa(x) Function

2.9 Bandwidth of Signals

✧ ***Null-to-null bandwidth (or zero-crossing bandwidth)*** is $f_2 - f_1$: where f_2 is the first null in the envelope of the magnitude spectrum above f_0 and, for bandpass system, f_1 is the first null in the envelope below f_0, where f_0 is the frequency where the magnitude spectrum is maximum. For baseband systems, f_1 is usually zero.



Null-to-null Bandwidth B_n

2.9 Bandwidth of Signals

✧ ***Bounded spectrum bandwidth*** is $f_2 - f_1$ such that outside the band $f_1 < f < f_2$, the PSD, which is proportional to $|H(f)|^2$, must be down by at least a certain amount, say 50 dB, below the maximum value of the power spectral density.

✧ ***Power bandwidth*** is $f_2 - f_1$ where $f_1 < f < f_2$ defines the frequency band in which 99% of the total power resides. This is similar to the FCC definition of occupied bandwidth, which states that the power above the the upper band edge f2 is 0.5% and the power below the lower band edge is 0.5%, leaving 99% of the total power within the occupied band.

✧ ***FCC bandwidth*** is an authorized bandwidth parameter assigned by the FCC to specifiy the spectrum allowed in communication systems.