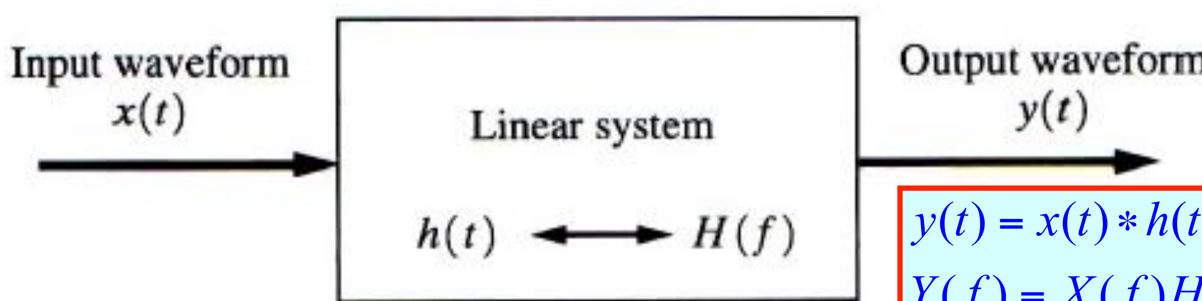
2.6 REVIEW OF LINEAR SYSTEMS

Linear Time-Invariant System

An electronic filter or system is ***Linear*** when ***Superposition*** holds, that is when,

$$y(t) = L[a_1 x_1(t) + a_2 x_2(t)] = a_1 L[x_1(t)] + a_2 L[x_2(t)]$$

- Where y(t) is the output and $x(t) = a_1 x_1(t) + a_2 x_2(t)$ is the input.
- *L*[.] denotes the linear system operator acting on [.].



$$y(t) = x(t) * h(t)$$
$$Y(f) = X(f)H(f)$$

2.6 REVIEW OF LINEAR SYSTEMS

Linear Time-Invariant System

Example. Are the following system is linear system or not?

1.) $y(t) = 5x(t)$

2.) $y(t) = 5x(t) + 3$

2.6 REVIEW OF LINEAR SYSTEMS

Linear Time-Invariant System

Conditions for time-invariance

$$Sys\{x(t)\} = y(t) \quad \text{implies that} \quad Sys\{x(t-\tau)\} = y(t-\tau)$$

- If the system is time invariant for any delayed input $x(t - t_0)$, the output is delayed by just the same amount $y(t - t_0)$.
- That is, the shape of the response is the same no matter when the input is applied to the system.

2.6 REVIEW OF LINEAR SYSTEMS

Linear Time-Invariant System

Example. Are the following system is time-invariant system or not?

1.) $y(t) = 5x(t)$

2.) $y(t) = 5x(t) + 3$

2.6 REVIEW OF LINEAR SYSTEMS

Impulse Response

The impulse response is the solution to the differential equation when the forcing function is a Dirac delta function. That is

***y*(*t*) = *h*(*t*) when *x*(*t*) = *δ*(*t*).**

A general waveform at the input may be approximated by

$$x(t) = \sum_{n=0}^{\infty} x(n\Delta t)\left[\delta(t - n\Delta t)\right]\Delta t$$

The output may be approximated by

$$y(t) = \sum_{n=0}^{\infty} x(n\Delta t)\left[h(t - n\Delta t)\right]\Delta t$$

This expression becomes the exact result as Δ*t* becomes zero, letting nΔ*t* = λ, we obtain

Convolution $$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda = x(t) * h(t)$$

2.6 REVIEW OF LINEAR SYSTEMS

Transfer Function

- The output waveform for a time-invariant network can be obtained by convolving the input waveform with the impulse response of the system.
- The spectrum of the output signal is obtained by taking the Fourier transform of both sides. Using the convolution theorem, we get

$$Y(f) = X(f)H(f)$$

or

$$H(f) = \frac{Y(f)}{X(f)}$$

or $H(f) = \Im\left[h(t)\right]$ is said to be the ***transfer function*** or ***frequency response*** of the network.

2.6 REVIEW OF LINEAR SYSTEMS

Transfer Function

The impulse response and frequency response are a Fourier transform pair

$$h(t) \longleftrightarrow H(f)$$

Of course, the transfer function H(f) is, in general, a complex quantity and can be written in polar form as

$$H(f) = |H(f)| e^{j\angle H(f)}$$

where $|H(f)|$ is the ***amplitude*** (or ***magnitude***) response and

$$\theta(f) = \angle H(f) = \tan^{-1}\left[\frac{\mathrm{Im}\{H(f)\}}{\mathrm{Re}\{H(f)\}}\right]$$

is the ***phase*** response of the network.

2.6 REVIEW OF LINEAR SYSTEMS

Transfer Function

Since $h(t)$ is a real function of time (for real networks), it follows

- ✧ $|H(f)|$ is an even function of frequency ($|H(-f)| = |H(f)|$)
- ✧ $\vartheta(f)$ is an odd function of frequency. ($\vartheta(-f) = -\vartheta(f)$)

Example

Input

$$x(t) = A\cos w_0 t$$

$$X(f) = \sum_{n=-\infty}^{\infty} c_n \delta(f - nf_0)$$

$$H(f_0) = |H(f_0)| e^{j\angle H(f_0)}$$

Output

$$y(t) = A|H(f_0)|\cos[w_0 t + \angle H(f_0)]$$

$$Y(f) = \sum_{n=-\infty}^{\infty} c_n H(nf_0)\delta(f - nf_0)$$

2.6 REVIEW OF LINEAR SYSTEMS

Power Transfer Function

We also can obtain the relationship between the ***power spectral density*** (***PSD***) at the input, and the output for a linear time-invariant network as:

$$P_y(f) = \lim_{T \to \infty} \frac{1}{T} \left| Y_T(f) \right|^2$$

using $$Y(f) = X(f)H(f)$$

we get $$P_y(f) = \left| H(f) \right|^2 \lim_{T \to \infty} \frac{1}{T} \left| X_T(f) \right|^2$$

or $$P_y(f) = \left| H(f) \right|^2 P_x(f)$$

Power transfer function $$G_h(f) = \frac{P_y(f)}{P_x(f)} = \left| H(f) \right|^2$$

2.6 REVIEW OF LINEAR SYSTEMS

Distortionless Transmission

In communication systems, a distortionless channel is often desired. This implies that the channel output is just proportional to a delayed version of the input

$$y(t) = A(x - T_d)$$

Where *A* is the **gain** (which may be less than unity) and ***T_d*** is the **delay**

In the frequency domain

$$Y(f) = AX(f)e^{-j2\pi f T_d}$$

The transfer function of the channel is

$$H(f) = \frac{Y(f)}{X(f)} = Ae^{-j2\pi f T_d}$$

2.6 REVIEW OF LINEAR SYSTEMS

Distortionless Transmission

For a linear time-invariant system, two requirements are needed as Distortionless system.

✧ The ***amplitude response*** if flat. That is.

$$|H(f)| = \text{constant} = A$$

No Amplitude Distortion

✧ The ***phase response*** is a linear function of frequency. That is

$$\theta(f) = \angle H(f) = -2\pi f T_d$$

No Phase Distortion

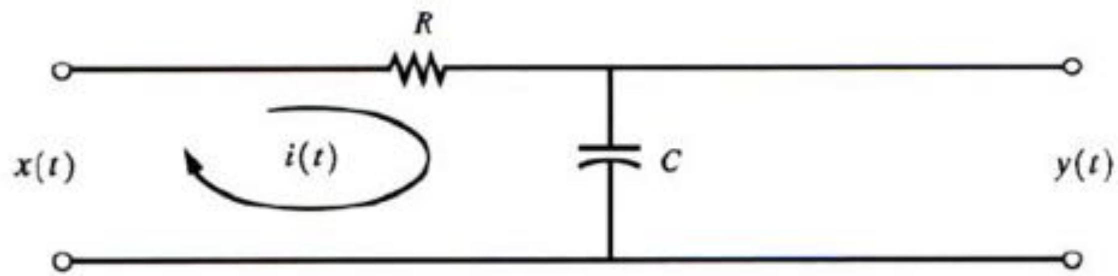
We define the ***time delay*** of the system as

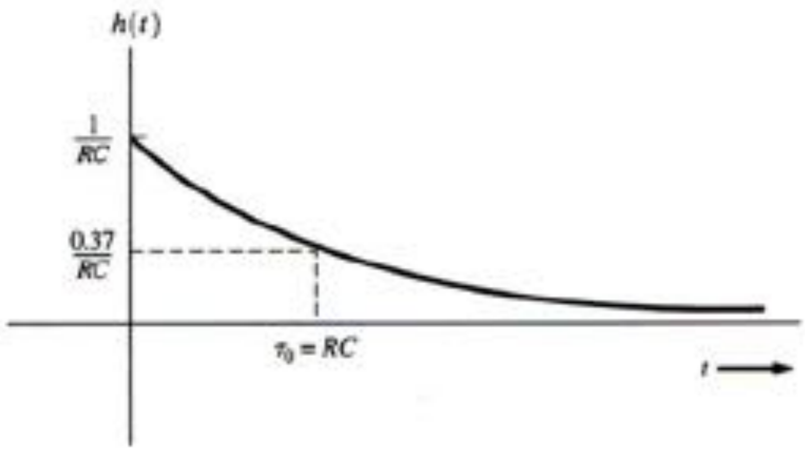
$$T_d(f) = -\frac{1}{2\pi f}\theta(f) = -\frac{1}{2\pi f}\angle H(f)$$

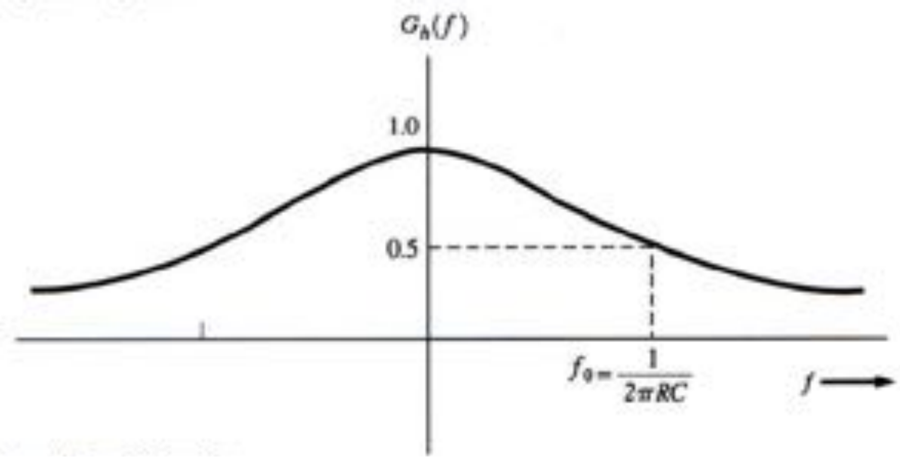
For distortionless system, $T_d(f)$ must be **constant.**

2.6 REVIEW OF LINEAR SYSTEMS

Example 2-18 RC Low-Pass Filter

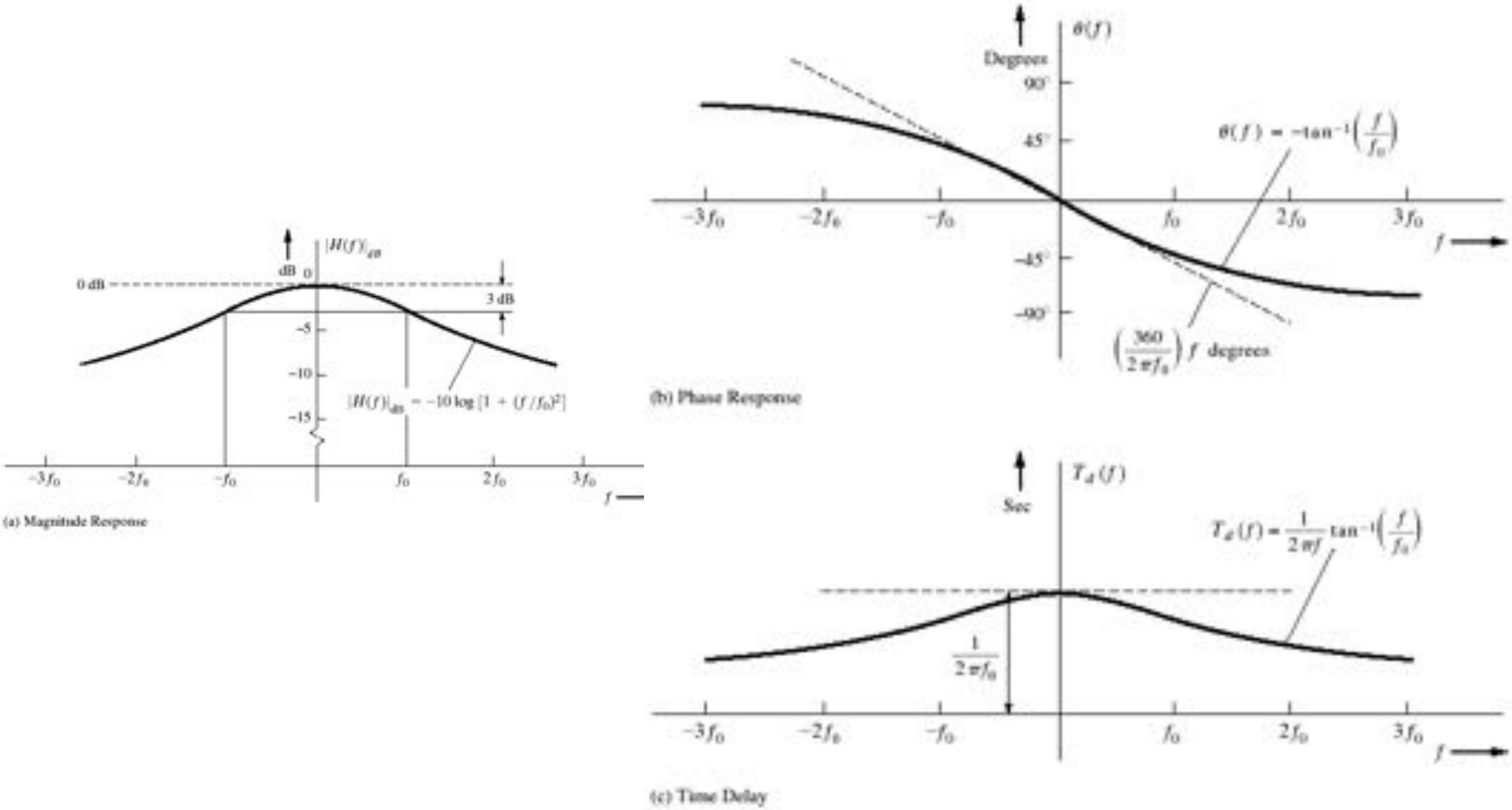






2.6 REVIEW OF LINEAR SYSTEMS

<u>*Example 2-18 Distortion caused by a filter*</u>



2.6 REVIEW OF LINEAR SYSTEMS

Effects of Distortion

Audio

Human ear more sensitive to amplitude distortion, but less so to phase distortion

Analog Video

Human visual system more sensitive to time delay errors, which result in smearing of edges, but less so to intensity variation

Digital Signals

Pulses smearing into other time slots – “Inter-Symbol Interference” (ISI)