2.2 Fourier transform and spectra

DEFNITION. The ***Fourier Transform*** (**FT**) of a waveform ***w(t)*** is

$$W(f) = \mathfrak{f}[w(t)] = \lim_{T->\infty} \int_{-\infty}^{\infty} [w(t)] e^{-j2\pi ft}\, dt$$

Where f[*] denotes the ***Fourier transform*** of [*], and ***f*** is the ***frequency*** parameter with units of hertz (i.e., 1/s). This defines the term frequency. It is the parameter ***f*** in the Fourier transform.

W(f) is also called a ***two-sided spectrum*** of ***w(t)***, because both positive and Negative frequency components are obtained from previous equation.

What is Fourier and Fourier Transform???

2.2 Fourier transform and spectra

What is Fourier and Fourier Transform ??

Fourier is a man, a genius

Name: Jean Baptiste Joseph Fourier

Year: 1768-1830

Nationality: French

Fields: Mathematician, physicist, historian

2.2 Fourier transform and spectra

Fourier Series and Fourier Transformer

A weighted summation of *Sines* and *Cosines* of different frequencies can be used to represent periodic (***Fourier Series***), or non-periodic (***Fourier Transform***) functions.

Is this true?

People didn't believe that, including Lagrange, Laplace, Poisson, and other big wigs.

But, yes, this is true?

Possibly the greatest tool used in Engineering, one of the the fundaments of modern communication, control, signal processing, and etc.

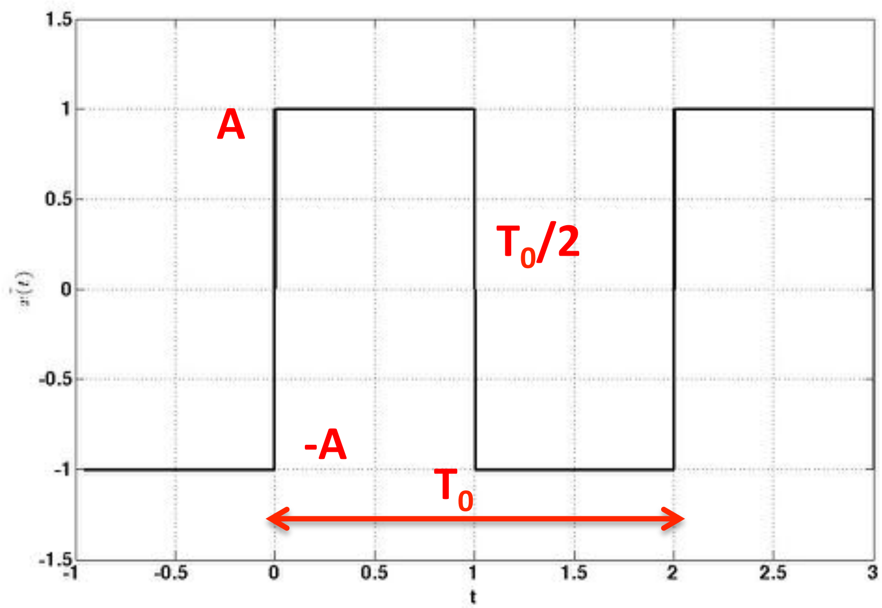
2.2 Fourier transform and spectra

Fourier Series

Approximating a periodic signal with trigonometric functions

For a periodic signal $\tilde{x}(t)$ which is periodic with period T_0 has the property

$$\tilde{x}(t+T)=\tilde{x}(t)$$



Periodic square-wave signal

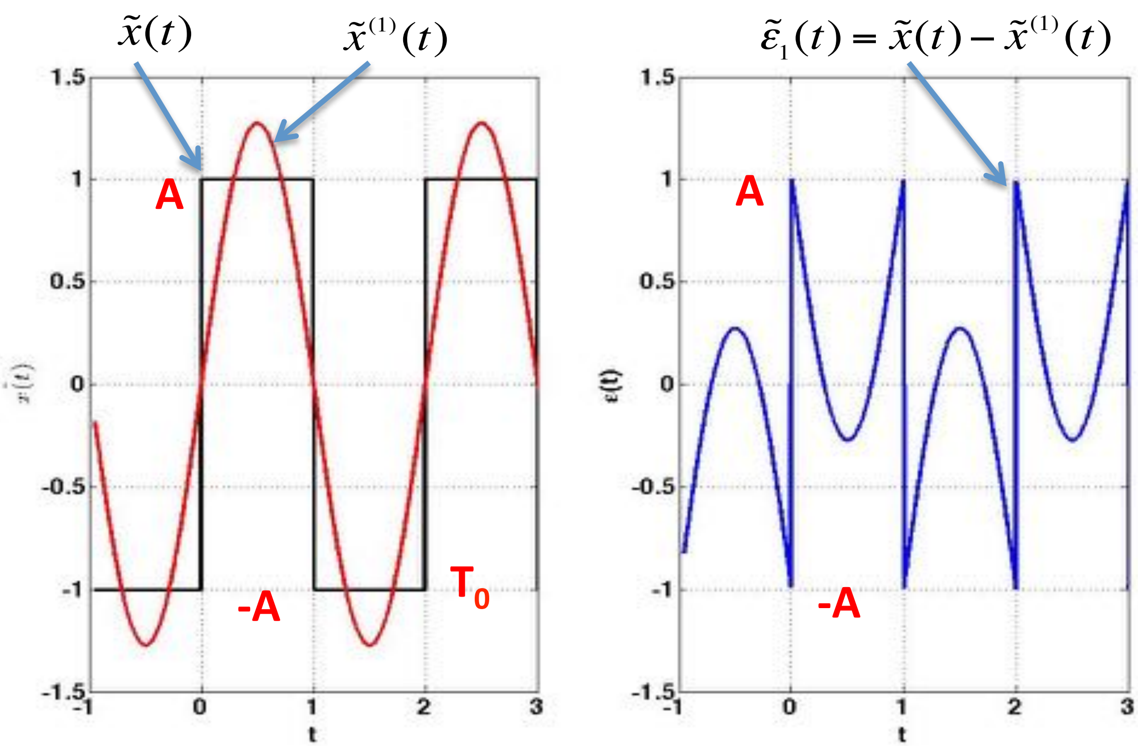
2.2 Fourier transform and spectra

Fourier Series

Approximating a periodic signal with trigonometric functions

The best approximation to $\tilde{x}(t)$ using only one trigonometric function is

$$\tilde{x}^{(1)}(t) = \frac{4A}{\pi}\sin(\omega_0 t)$$



2.2 Fourier transform and spectra

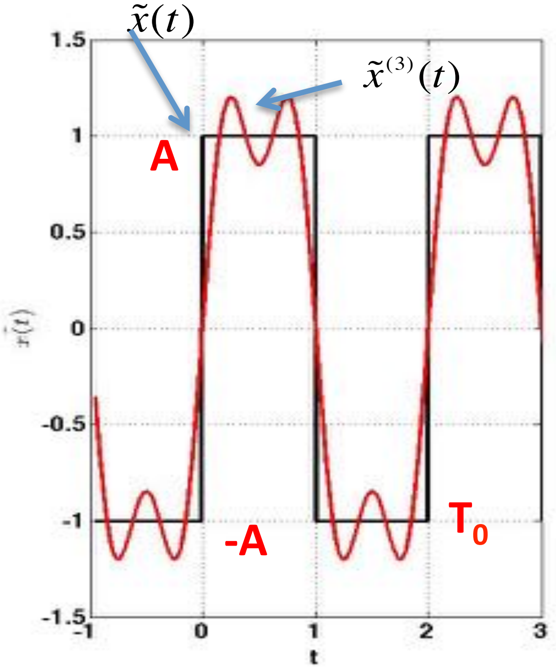
Fourier Series

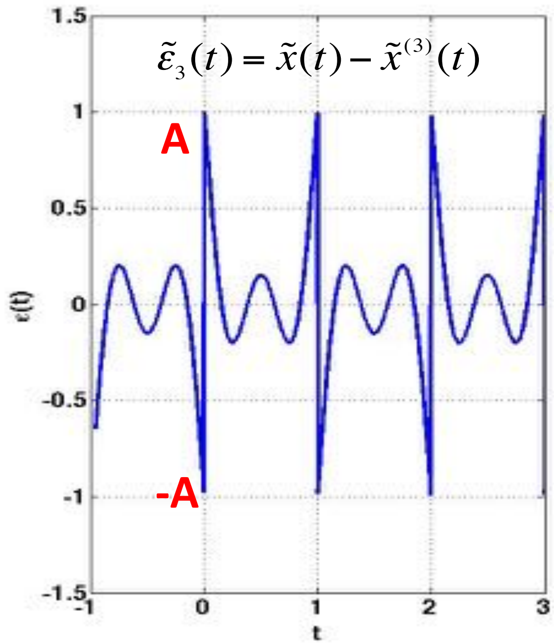
Approximating a periodic signal with trigonometric functions

Let's try a three-frequency approximation to $\tilde{x}(t)$ and see if the ***approximate error*** can be reduced.

$$\tilde{x}^{(3)}(t) = b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + b_3 \sin(3\omega_0 t)$$

$$\tilde{\varepsilon}_3(t) = \tilde{x}(t) - \tilde{x}^{(3)}(t) = \tilde{x}(t) - b_1 \sin(\omega_0 t) - b_2 \sin(2\omega_0 t) - b_3 \sin(3\omega_0 t)$$





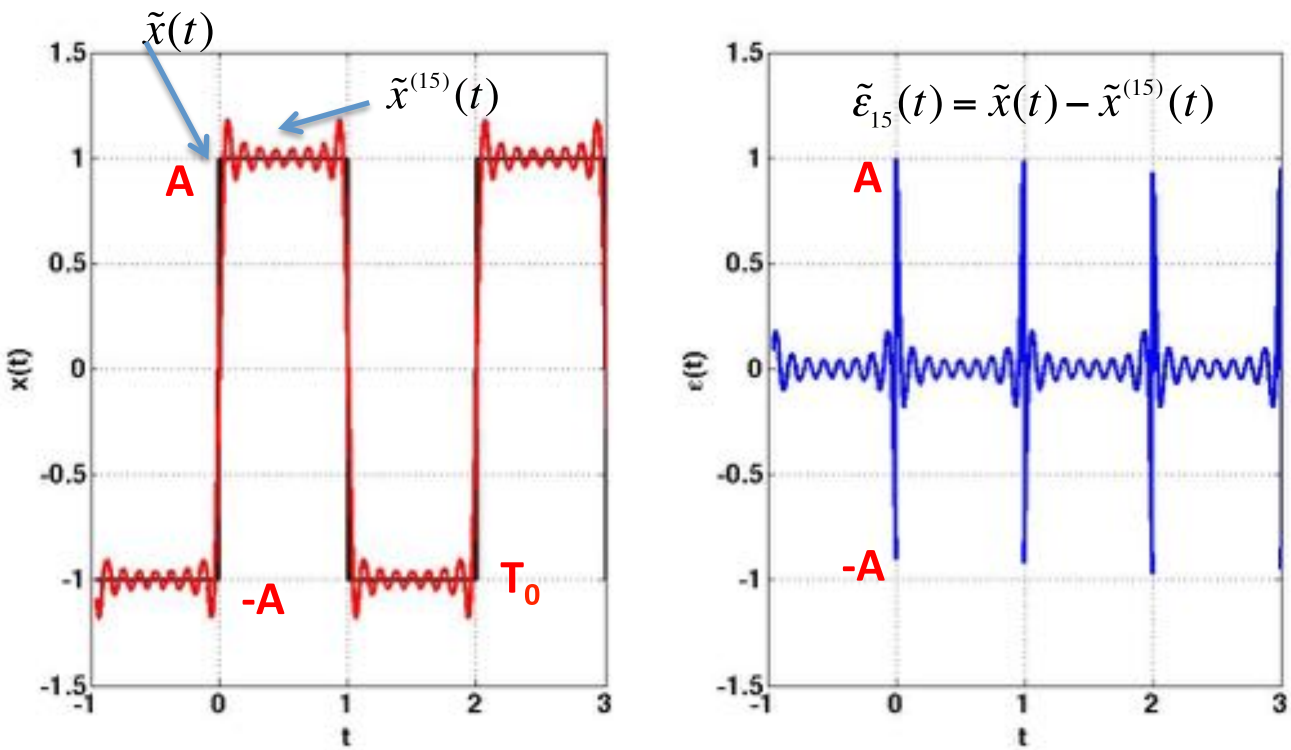
2.2 Fourier transform and spectra

Fourier Series

Approximating a periodic signal with trigonometric functions

Let's try a 15-frequency approximation to $\tilde{x}(t)$ and see if the ***approximate error*** can be reduced.

$$\tilde{x}^{(15)}(t) = b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + \quad . \quad . \quad . \quad + b_{15} \sin(15\omega_0 t)$$



2.2 Fourier transform and spectra

Fourier Series

Trigonometric Fourier Series (TFS)

$$\tilde{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t)$$

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

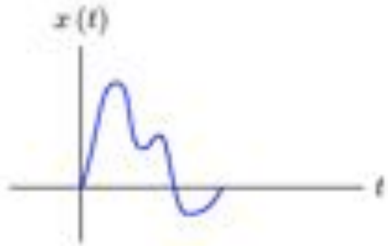
Exponential Fourier Series (EFS)

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jkw_0 t}$$

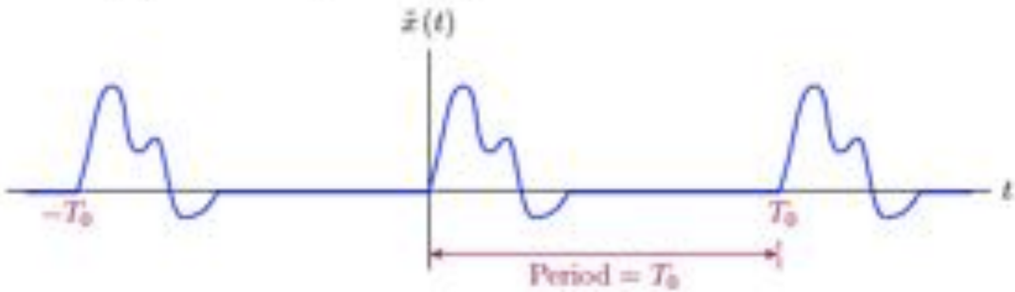
2.2 Fourier transform and spectra

Fourier Transform

A non-periodic signal $x(t)$:



Periodic extension $\tilde{x}(t)$ of the signal $x(t)$:



$$\tilde{x}(t) = \ldots + x(t + T_0) + x(t) + x(t - T_0) + x(t - 2T_0) + \ldots$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(t - kT_0)$$

2.2 Fourier transform and spectra

Fourier Transform for continuous-time signals

Fourier Transform (Forward Transform)

$$W(f) = \Im[w(t)] = \int_{-\infty}^{\infty} [w(t)] e^{-j2\pi ft} \, dt$$

Inverse Fourier Transform (Inverse Transform)

$$w(t) = \Im^{-1}[w(t)] = \int_{-\infty}^{\infty} [W(f)] e^{j2\pi ft} \, dt$$

2.2 Fourier transform and spectra

Alternative Evaluation Techniques for FT Integral

- ✧ Direct integration.
- ✧ Tables of Fourier transforms or Laplace transforms.
- ✧ FT theorems.
- ✧ Superposition to break the problem into two or more simple problems.
- ✧ Differentiation or integration of $w(t)$.
- ✧ Numerical integration of the FT integral on the PC via MATLAB or MathCAD integration functions.
- ✧ Fast Fourier transform (FFT) on the PC via MATLAB or MathCAD FFT functions.

2.2 Fourier transform and spectra

DEFNITION. The ***Fourier Transform*** (**FT**) of a waveform ***w(t)*** is

$$W(f) = \Im[w(t)] = \lim_{T->\infty} \int_{-\infty}^{\infty} [w(t)] e^{-j2\pi f t}\, dt$$

W(f) is a complex function of frequency, and can therefore be represented in as

Quadrature / Cartesian

$$W(f) = X(f) + jY(f)$$

$$|W(f)| = \sqrt{X^2(f) + Y^2(f)}$$

$\Longleftrightarrow$

Magnitude-Phase / Polar

$$W(f) = |W(f)| e^{j\theta(f)}$$

$$\theta(f) = \tan^{-1}\left(\frac{Y(f)}{X(f)}\right)$$

2.2 Fourier transform and spectra

DEFNITION. The ***Inverse Fourier Transform*** (**FT**) of a waveform ***w(t)*** is

$$w(t) = \int_{-\infty}^{\infty} W(f) e^{j2\pi ft}\, df$$

The functions ***w(t)*** and ***W(f)*** constitute a ***Fourier transform pair***

$$w(t) \underset{\text{Inverse Fourier transform}}{\overset{\text{Fourier transform}}{\rightleftarrows}} W(f)$$

Time domain **Frequency domain**

2.2 Fourier Transform and Spectra

The waveform $w(t)$ is Fourier transformable if it satisfies both **Dirichlet conditions:**

✧ Over any time interval of finite length, the function ***w*(*t*) is single valued with a finite number of maxima and minima**, and the number of discontinuities (if any) is finite.

✧ ***w*(*t*) is absolutely integrable.** That is, $\int_{-\infty}^{\infty} |w(t)|\, dt < \infty$

Above conditions are **sufficient**, but not **necessary**

2.2 Fourier Transform and Spectra

A weaker **sufficient** condition for the existence of the Fourier transform is:

$$E = \int_{-\infty}^{\infty} |w(t)|^2 \, dt < \infty$$

Finite Energy

Where ***E*** is the **normalized energy**.

This is the **finite-energy condition** that is satisfied by all physically realizable forms.

Conclusion: All physical waveforms encountered in engineering practice are **Fourier transformable.**

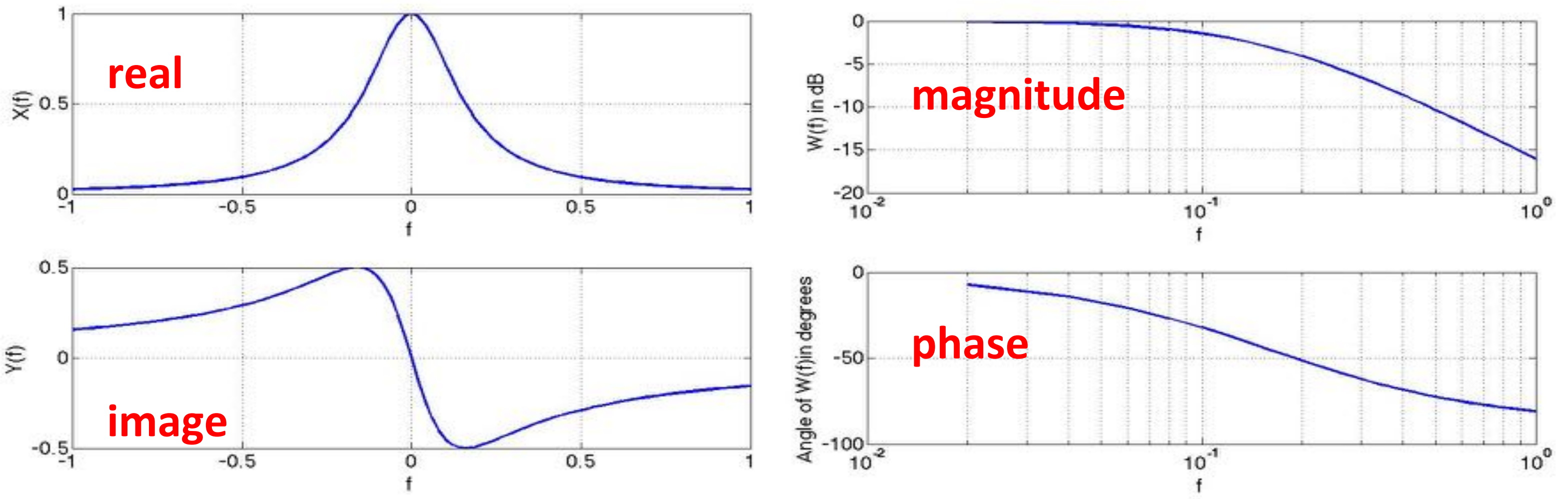
2.2 Fourier Transform and Spectra

Example 2-3. Spectrum of an exponential pulse

Let *w*(*t*) be a decaying exponential pulse that is switched on at *t* = *0*. That is

$$w(t) = \begin{cases} e^{-t} & if \quad t > 0 \\ 0 & if \quad t < 0 \end{cases}$$

find its spectrum?



2.2 Fourier transform and spectra

Properties of Fourier Transforms

THEOREM. ***Spectral symmetry of real signals***. If $w(t)$ is real, then

$$W(-f) = W^*(f)$$

The superscript asterisk denotes the **conjugate** operation.

$$x(t) = a + bj \quad \Longrightarrow \quad x^*(t) = a - bj$$

Properties of the Fourier transform:

- *f*, called frequency and having units of hertz, specifies the specific frequency in the waveform $w(t)$.
- The FT looks for the frequency ***f*** in the ***w*(*t*)** over all time. That is, over

$$-\infty < t < \infty$$

- ***W*(*f*)** can be complex, even though ***w*(*t*)** is **real**
- If ***w*(*t*)** is **real**, then ***W*(-*f*)=*W**(*f*)**

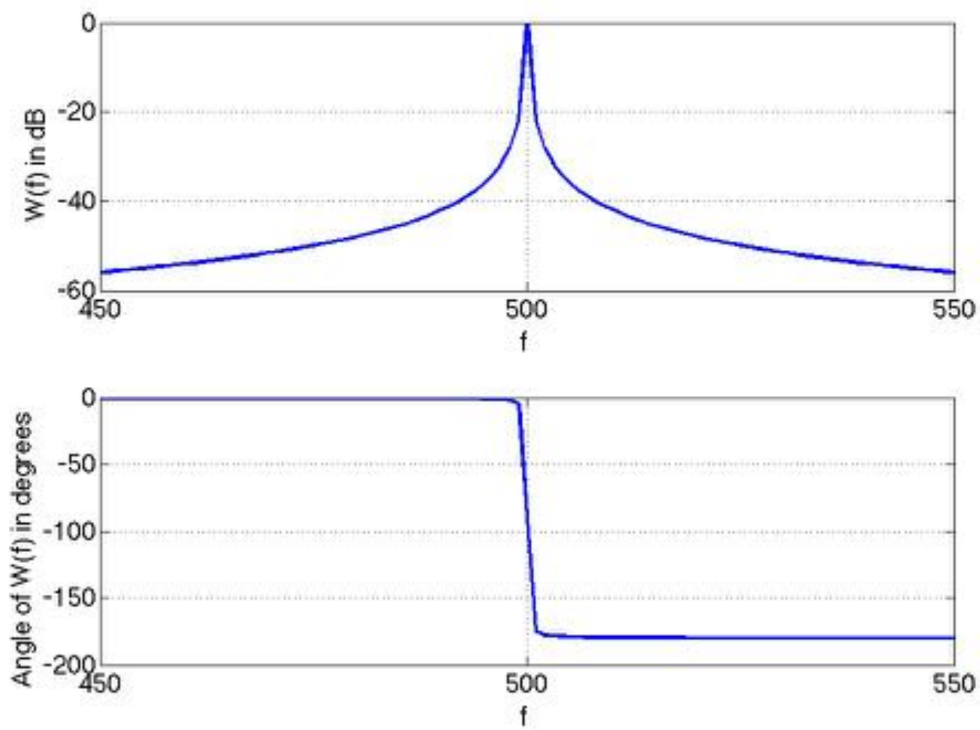
2.2 Fourier transform and spectra

Example 2-4. Spectrum of a damped sinusoid

Let damped sinusoid be given by

$$w(t) = \begin{cases} e^{-t/T} \sin \omega_0 t & \text{if} \quad t > 0, T > 0 \\ 0 & \text{if} \quad t < 0 \end{cases}$$

find its spectrum?



2.2 Fourier transform and spectra

Properties of Fourier Transforms

Parseval's Theorem:

$$\int_{-\infty}^{\infty} w_1(t) w_2^*(t)\, dt = \int_{-\infty}^{\infty} W_1(f) W_2^*(f)\, df$$

If $w_1(t)=w_2(t)=w(t)$, then the theorem reduces to

Rayleight's Energy Theorem:

$$\int_{-\infty}^{\infty} |w_1(t)|^2\, dt = \int_{-\infty}^{\infty} |W_1(f)|^2\, df$$

The energy calculated from the time domain is equal to the energy calculated from the frequency domain

2.2 Fourier transform and spectra

Parseval's Theorem and Energy Spectral Density

DEFNITION. The ***Energy Spectral Density*** (**ESD**) is defined for energy waveforms by

$$\mathcal{E}(f) = |W(f)|^2$$

where $w(t) \leftrightarrow W(f)$. $\mathcal{E}(f)$ has units of joules per hertz.

We can see that the total normalized energy is given by the area under ESD function

$$E = \int_{-\infty}^{\infty} \mathcal{E}(f)\, df$$

2.2 Fourier transform and spectra

Some Fourier Transform Theorems

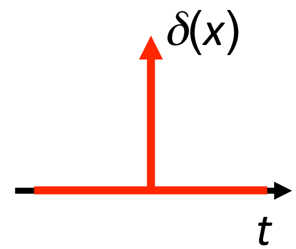
Operation	Function	Fourier Transform
Linearity	$a_1w_1(t) + a_2w_2(t)$	$a_1W_1(f) + a_2W_2(f)$
Time delay	$w(t - T_d)$	$W(f)\,e^{-j\omega T_d}$
Scale change	$w(at)$	$\frac{1}{\lvert a\rvert}W\left(\frac{f}{a}\right)$
Conjugation	$w^*(t)$	$W^*(-f)$
Duality	$W(t)$	$w(-f)$
Real signal frequency translation [$w(t)$ is real]	$w(t)\cos(\omega_c t + \theta)$	$\frac{1}{2}[e^{j\theta}W(f - f_c) + e^{-j\theta}W(f + f_c)]$
Complex signal frequency translation	$w(t)\,e^{j\omega_c t}$	$W(f - f_c)$
Bandpass signal	$\text{Re}\{g(t)\,e^{j\omega_c t}\}$	$\frac{1}{2}[G(f - f_c) + G^*(-f - f_c)]$
Differentiation	$\frac{d^n w(t)}{dt^n}$	$(j2\pi f)^n W(f)$
Integration	$\int_{-\infty}^{t} w(\lambda)d\lambda$	$(j2\pi f)^{-1}W(f) + \frac{1}{2}W(0)\,\delta(f)$
Convolution	$w_1(t) * w_2(t) = \int_{-\infty}^{\infty} w_1(\lambda) \cdot w_2(t - \lambda)\,d\lambda$	$W_1(f)W_2(f)$
Multiplication[b]	$w_1(t)w_2(t)$	$W_1(f) * W_2(f) = \int_{-\infty}^{\infty} W_1(\lambda)\,W_2(f - \lambda)\,d\lambda$
Multiplication by t^n	$t^n w(t)$	$(-j2\pi)^{-n}\frac{d^n W(f)}{df^n}$

2.2 Fourier transform and spectra

Dirac Delta Function

DEFINATION. The ***Dirac delta function*** $\delta(x)$ is defined by

$$\int_{-\infty}^{\infty} w(x)\delta(x)\,dt = w(0)$$



where *w*(*x*) is any function that is continuous at *x* = 0.
An alternative definition of δ(*x*) is:

$$\delta(x) = \begin{cases} 0 & if \quad x \neq 0 \\ \infty & if \quad x = 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

2.2 Fourier transform and spectra

Dirac Delta Function

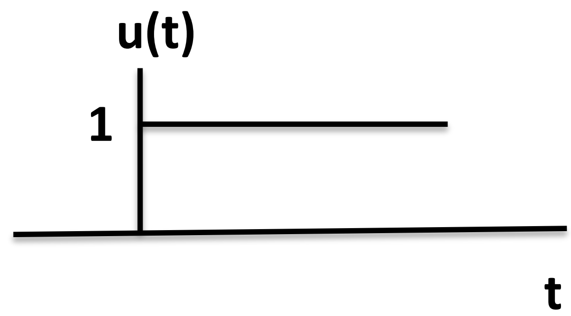
The ***Sifting Property of*** $\delta(x)$ is

$$\int_{-\infty}^{\infty} w(x)\delta(x-x_0)\,dt = w(x_0)$$

2.2 Fourier transform and spectra

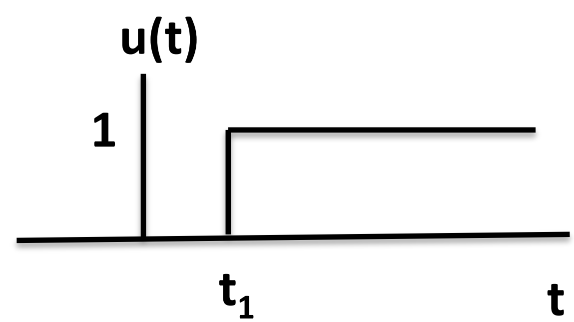
Unit Step Function

$$u(t) = \begin{cases} 1 & if \quad t > 0 \\ 0 & if \quad t < 0 \end{cases}$$



Time shift of the unit-step function

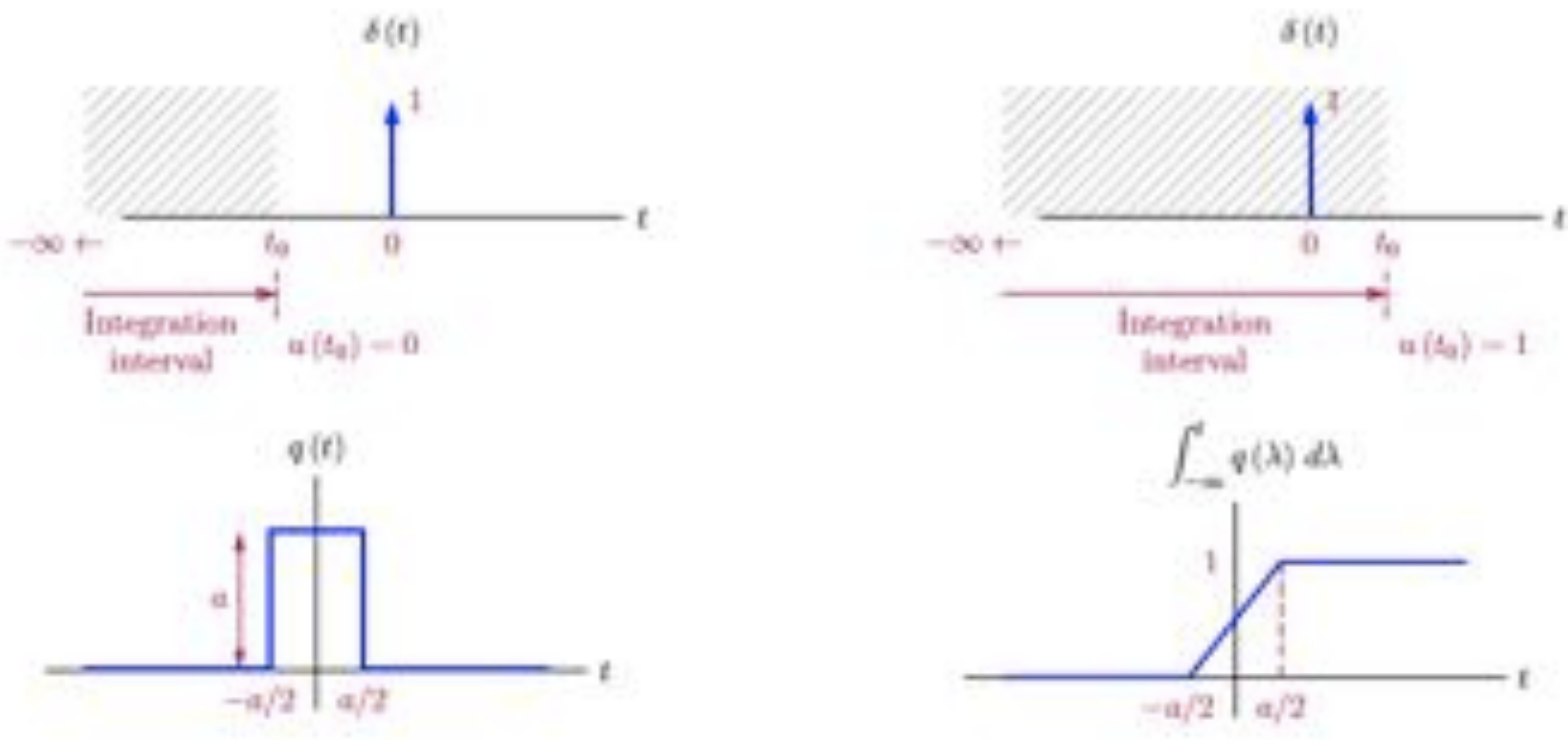
$$u(t - t_1) = \begin{cases} 1 & if \quad t > t_1 \\ 0 & if \quad t < t_1 \end{cases}$$



2.2 Fourier transform and spectra

The **relationship** between ***unit-step*** and ***Delta*** functions

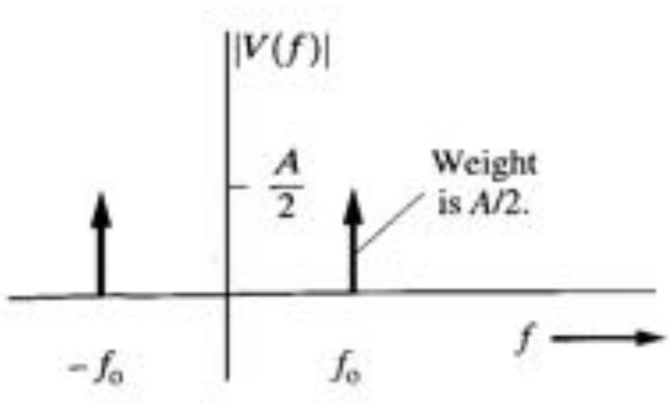
$$u(t) = \int_{-\infty}^{\infty} \delta(\lambda) d\lambda \quad \Longleftrightarrow \quad \delta(t) = \frac{du}{dt}$$



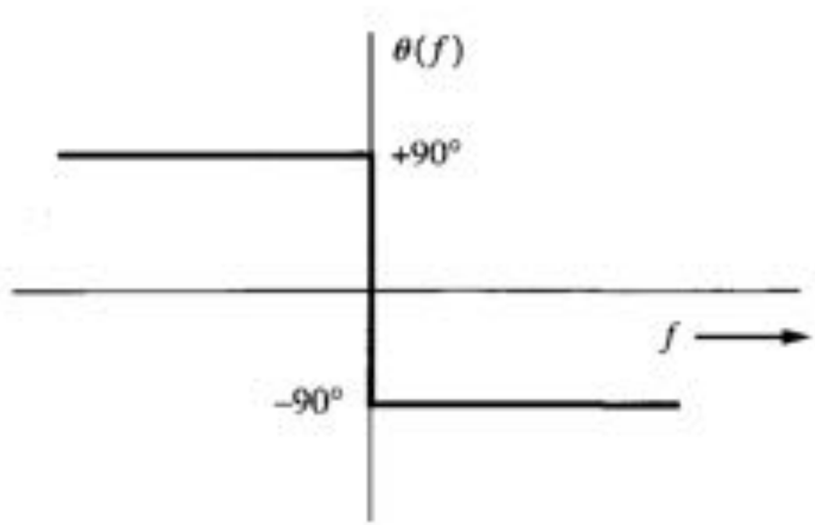
2.2 Fourier transform and spectra

Example 2-5. Spectrum of a sinusoid

Find the spectrum of a sinusoidal voltage waveform that has a frequency f_0 and A peak value of A volts. That is $v(t) = A\sin\omega_0 t$ where $\omega_0 = 2\pi f_0$ find its spectrum?



(a) Magnitude Spectrum



(b) Phase Spectrum ($\theta_0 = 0$)

2.2 Fourier transform and spectra

Rectangular Pulses

DEFINATION. The single ***rectangular pulse*** is denoted as $\prod(\bullet)$

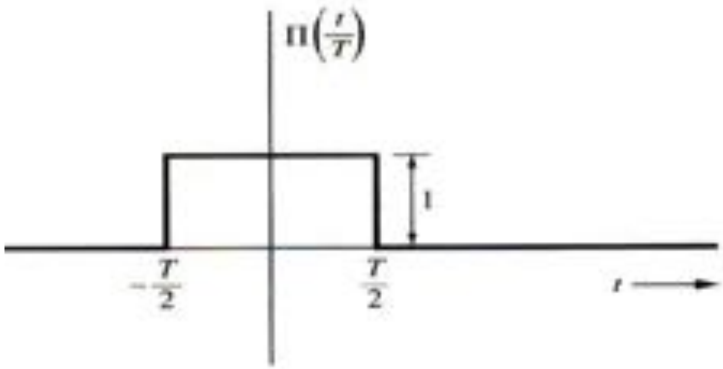
$$\prod\left(\frac{t}{T}\right) \overset{\Delta}{=} \begin{cases} 1, & |t| < T/2 \\ 0, & |t| > T/2 \end{cases}$$

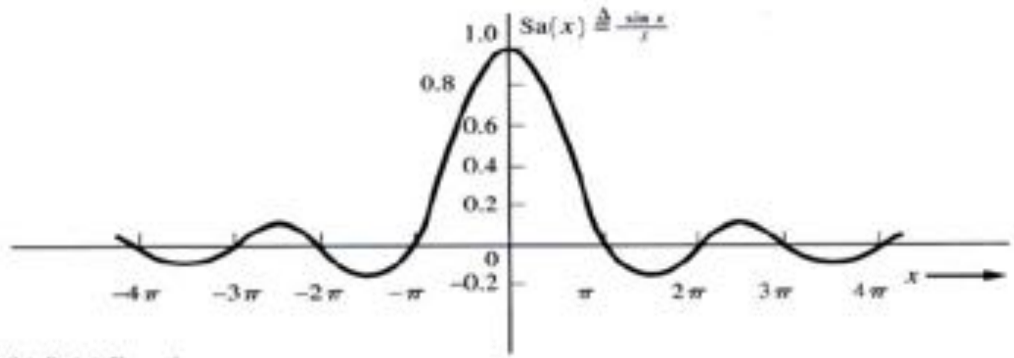
DEFINATION. $Sa(\bullet)$ Denoted the function $Sa(x) = \dfrac{\sin x}{x}$

2.2 Fourier transform and spectra

Example 2-6. Spectrum of a rectangular pulse

Find the spectrum of a rectangular pulse $w(t) = \prod(t/T)$





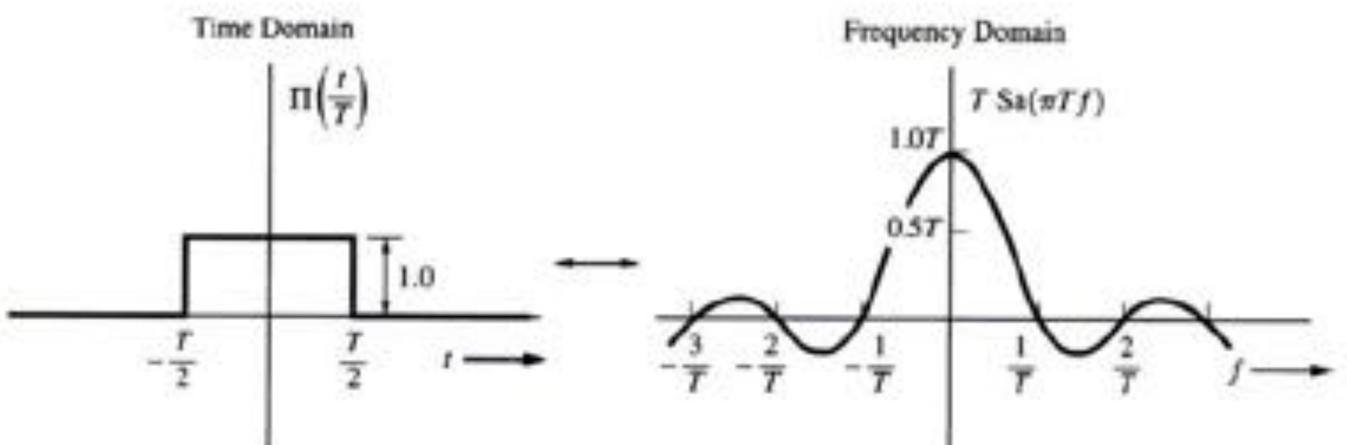
(b) Sa(x) Function

2.2 Fourier transform and spectra

Spectrum of a Rectangular Pulse

$$w(t) = \Pi\left(\frac{t}{T}\right) \Leftrightarrow W(f) = T \cdot Sa\left(fT\right)$$

- Rectangular pulse is a time window.
- FT is a sinc function, infinite frequency content.
- Shrinking time axis causes stretching of frequency axis.
- Signals cannot be both time-limited and bandwidth-limited.



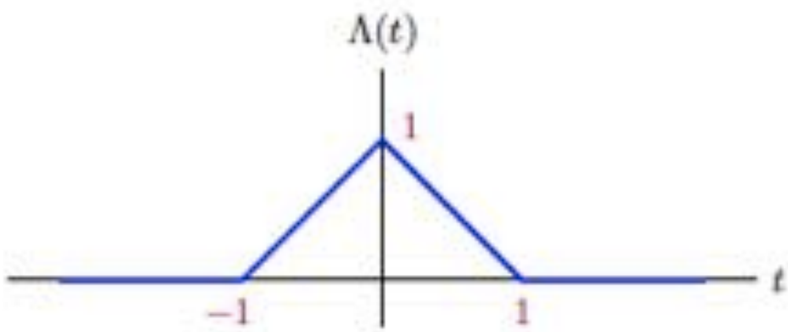
Note the inverse relationship between the pulse width T and the zero crossing 1/T

2.2 Fourier transform and spectra

Triangular Pulses

DEFINATION. The single ***triangular function*** is denoted as $\Lambda(\bullet)$

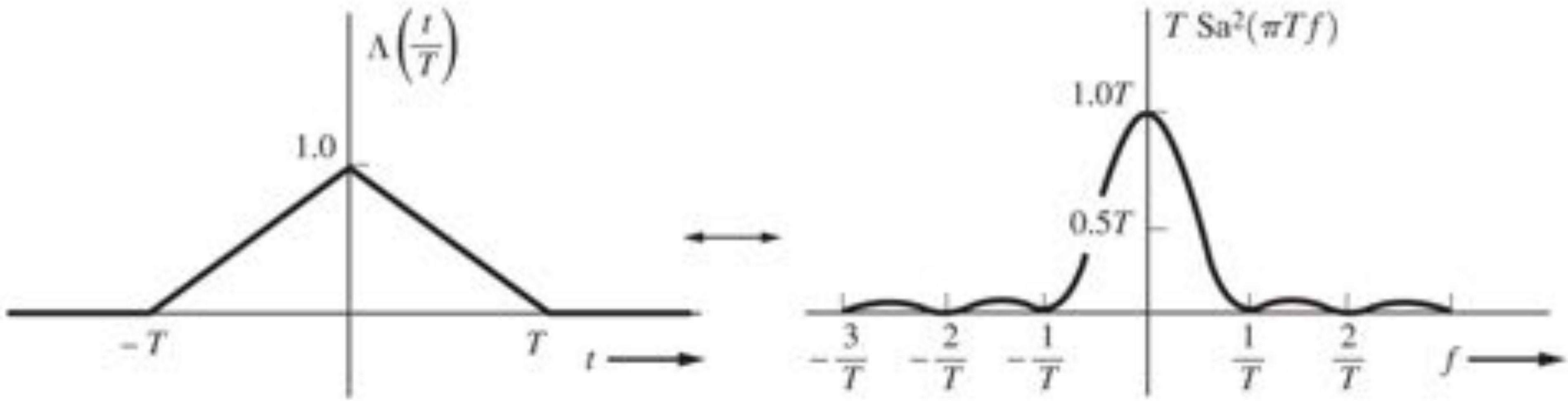
$$\Lambda(\frac{t}{T}) \stackrel{\Delta}{=} \begin{cases} 1-\frac{|t|}{T}, & |t| \leq T \\ 0, & |t| > T \end{cases}$$



2.2 Fourier transform and spectra

Example 2-7. Spectrum of a triangular pulse

Find the spectrum of a triangular pulse $w(t) = \Lambda(t/T)$



$$w(t) = \Lambda\left(\frac{t}{T}\right) \Leftrightarrow W(f) = T \cdot Sa^2(\pi f T)$$

2.2 Fourier transform and spectra

Convolution

DEFNITION. The ***convolution*** of a waveform ***$w_1(t)$*** with a wave ***$w_2(t)$*** to produce a third waveform ***$w_3(t)$*** is

$$\omega_3(t) = \omega_1(t) * \omega_2(t) = \int_{-\infty}^{\infty} \omega_1(\lambda)\omega_2(t-\lambda)d\lambda$$

$$= \int_{-\infty}^{\infty} \omega_1(\lambda)\omega_2(-(\lambda-t))d\lambda$$

Where $\omega_1(t) * \omega_2(t)$ is a shorthand notation for this integration operation and * is read "**convolved with**."

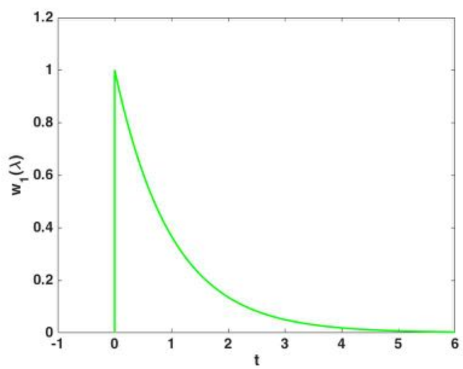
The ***convolution*** can be obtained through three steps:

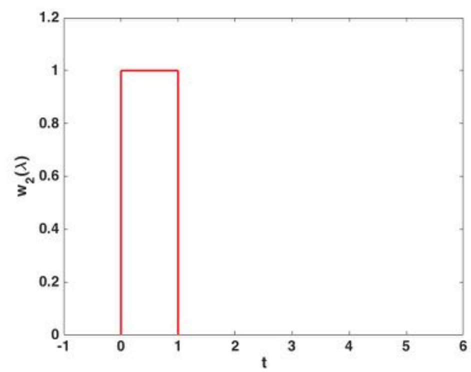
1. **Time reversal** of $\omega_2(t)$ to obtain $\omega_2(-\lambda)$
2. **Time shifting** of ω_2 by *t* seconds to obtain $\omega_2(-(\lambda - t))$
3. **Multiplying** this result by ω_1 to form the integrand $\omega_1(\lambda)\omega_2(-(\lambda - t))$

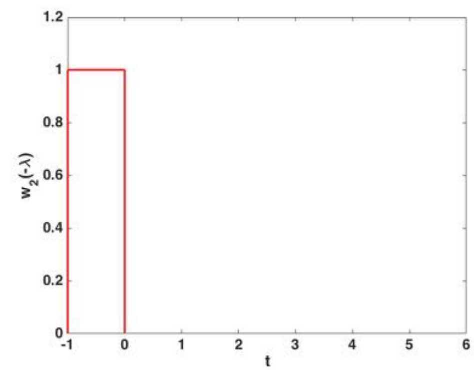
2.2 Fourier transform and spectra

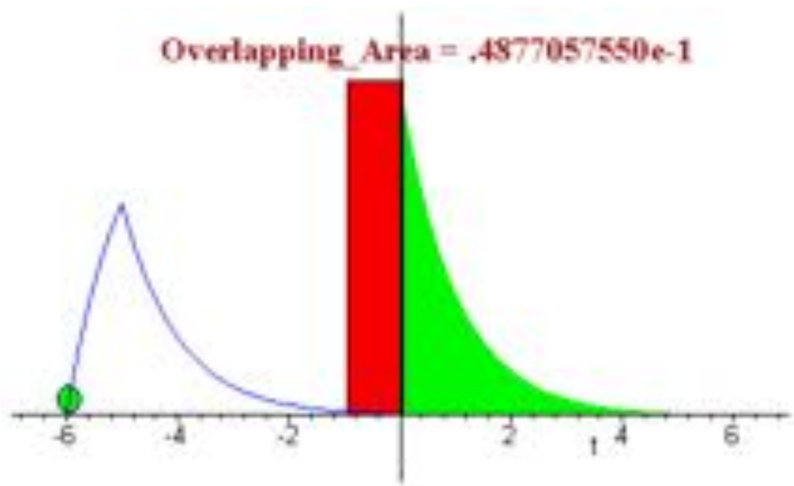
Convolution of a rectangle with and exponential

$\omega_1(t) = e^{-t}u(t)$ and $\omega_2(t) = \prod(t-1)$









2.2 Fourier transform and spectra

Example 2-8. Convolution of a rectangle with an exponential

let $$w_1(t) = \Pi\left(\frac{t - \frac{1}{2}T}{T}\right)$$

and

$$w_2(t) = e^{-t/T} u(t)$$

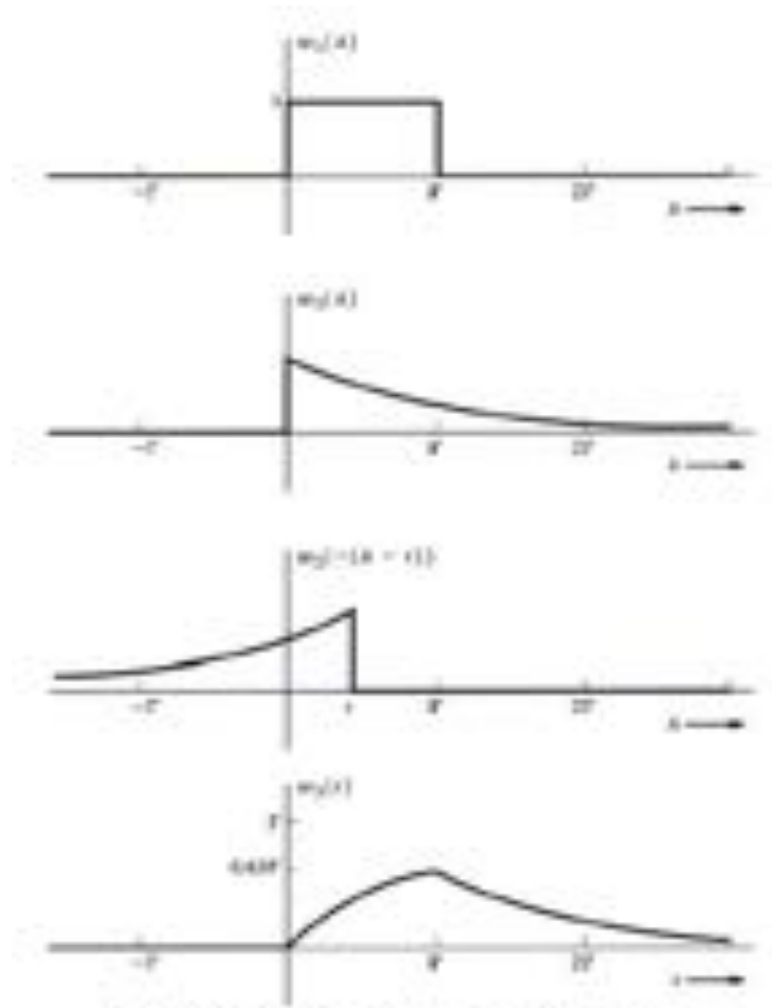


Figure 2-7 Convolution of a rectangle and an exponential.

2.3 Power spectral density and autocorrelation function

Power spectral density

DEFNITION. The ***power spectral density*** (***PSD***) for a deterministic power waveform is

$$p_w(f) = \lim_{T \to \infty} \left(\frac{\left| W_r(f) \right|^2}{T} \right)$$

Where $w_T(t) \leftrightarrow W_T(f)$ and $p_w(f)$ has units of *watts per hertz*.

Note:

1.) The **PSD** represents the ***normalized power*** of a waveform in its frequency domain

2.) The **PSD** is always a ***real nonnegative*** function of frequency.

3.) The **PSD** is ***not sensitive*** to the phase spectrum of *w*(*t*).

2.3 Power spectral density and autocorrelation function

The ***Normalized Average Power***

$$P = \langle w^2(t) \rangle = \int_{-\infty}^{\infty} P_w(f)$$

This means the **area** under the **PSD function** is the **normalized average power**

2.3 Power spectral density and autocorrelation function

Autocorrelation Function

DEFNITION. The ***Autocorrelation*** of a real (physical) waveform is

$$R_w(\tau) = \langle \omega(t)\omega(t+\tau) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \omega(t)\omega(t+\tau)\,dt$$

Wiener-Khintchine Theorem: The ***PSD*** and the ***autocorrelation*** function are Fourier transform pairs:

$$R_w(\tau) \leftrightarrow P_w(f)$$

The **PSD** can be evaluated by either of the following two methods:

- **Direct method:** by using the definition.
- **Indirect method:** by first evaluating the autocorrelation function and then taking the FT.

$$P_w(f) = \Im[R_w(\tau)]$$

2.3 Power spectral density and autocorrelation function

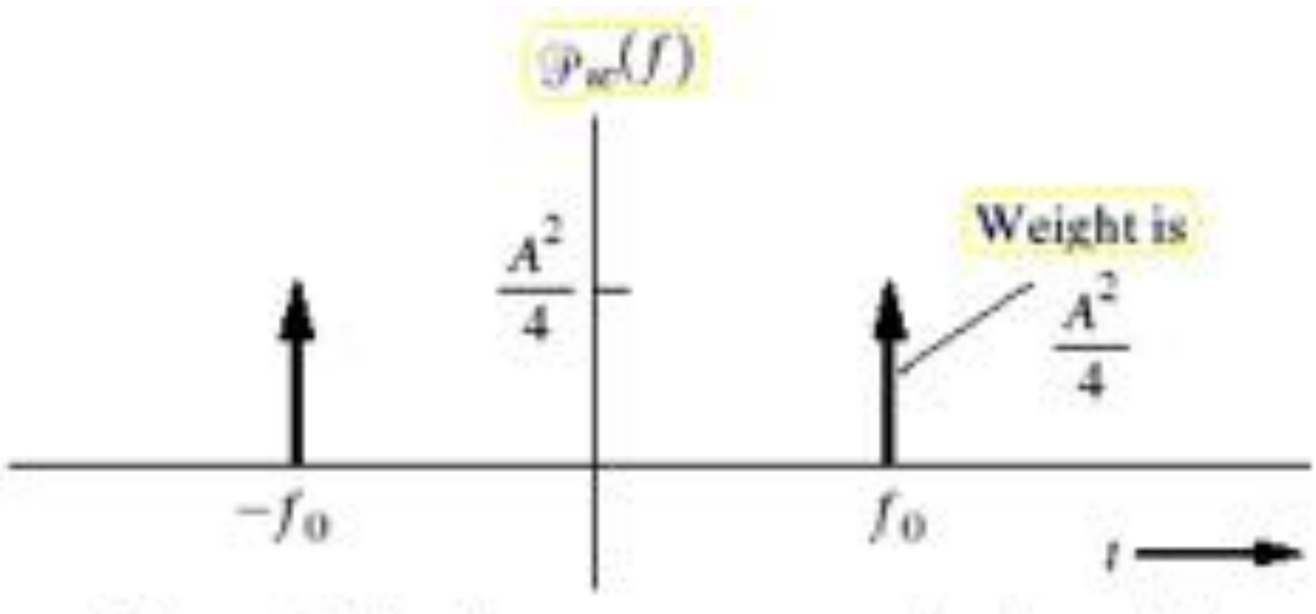
The average power can be obtained by any of the four techniques

$$P = \langle w^2(t) \rangle = W_{rms}^2 = \int_{-\infty}^{\infty} P_w(f)\,df = R_W(0)$$

2.3 Power spectral density and autocorrelation function

Example 2-9. PSD of a sinusoid

let $w(t) = \sin \omega_0 t$



2.4 Orthogonal Series Representation of Signal and Noise

Orthogonal Function

DEFNITION. Functions *$\varphi_n(t)$* and *$\varphi_m(t)$* are said to be ***orthogonal*** with respect to each other over the interval $a < t < b$ if they satisfy the condition

$$\int_a^b \varphi_n(t)\varphi_m^*(t)\,dt = \begin{cases} 0 & n \neq m \\ K_n & n = m \end{cases} = K_n \delta_{nm}$$

$$\delta_{nm} \equiv \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

- ✧ $\boldsymbol{\delta_{nm}}$ is called the ***Kronecker delta function***
- ✧ If the constants $\boldsymbol{K_n}$ are all equal to 1 then the *$\varphi_n(t)$* are said to be ***orthonormal functions.***

2.4 Orthogonal Series Representation of Signal and Noise

Orthogonal Series

Assume that ***w*(*t*)** represents some practical waveform (signal, noise, or signal-noise combination) that we wish to represent over the interval ***a* < *t* < *b***. Then we can obtain an equivalent orthogonal series representation by using the following theorem.

THEOREM. ***w(t)*** can be represented over the interval (***a*,*b***) by the series

$$w(t) = \sum_n a_n \varphi_n(t)$$

where the ***orthogonal coefficients*** are given by

$$a_n = \frac{1}{K_n} \int_a^b w(t) \varphi_n^*(t)\, dt$$

And the range of ***n*** is over the integer values that correspond to the subscripts that were used to denote the orthogonal function in the complete orthogonal set

2.4 Orthogonal Series Representation of Signal and Noise

Application of Orthogonal Series

- It is also possible to generate $\boldsymbol{w(t)}$ from the $\boldsymbol{\varphi_j(t)}$ functions and the coefficients $\boldsymbol{a_j}$.
- In this case, $\boldsymbol{w(t)}$ is approximated by using a reasonable number of the $\phi_j(t)$ functions.

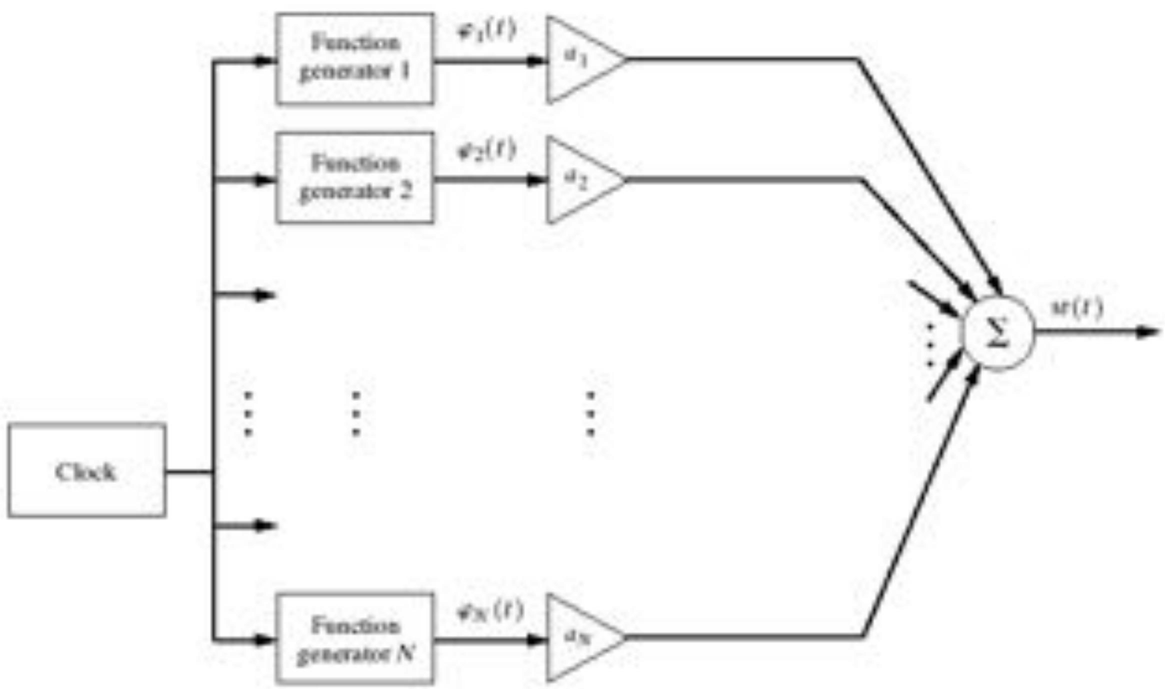


Figure 2–10 Waveform synthesis using orthogonal functions.

$w(t)$ is realized by adding weighted versions of orthogonal functions

2.5 Fourier Series

Complex Fourier Series

The complex Fourier series uses the orthogonal exponential function

THEOREM. A ***physical waveform*** (i.e. finite energy) may be represented over the interval ***$a < t < a+T_0$*** by the complex exponential Fourier series

$$w(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

where the complex (phasor) ***Fourier coefficient*** are

$$c_n = \frac{1}{T_0}\int_a^{a+T_0} w(t)e^{-jnw_0 t}\,dt$$

and where $\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$

2.5 Fourier Series

Complex Fourier Series

$$w(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \longleftrightarrow c_n = \frac{1}{T_0}\int_a^{a+T_0} w(t) e^{-jnw_0 t}\, dt$$

- ***C_n*** is the Fourier Series. In general, it is a complex number. The Fourier coefficient ***C_0*** is equivalent to the **DC** value of the waveform ***w(t)***.

- If the waveform ***w*(*t*)** is periodic with period ***T_0***, this Fourier series representation Is valid over all time.

- For this case of periodic waveforms, the choice of a is arbitrary and is usually taken to be a = 0 or a = -T0/2 for mathematical convenience.

- The frequency $f_0 = 1/T_0$ is said to be the *fundamental frequency* and the frequency *nf_0* is said to be the nth harmonic frequency, when n>1.

2.5 Fourier Series

Some Properties of the Complex Fourier Series

1. If $w(t)$ is real,

$$c_n = c_{-n}^*$$

2. If $w(t)$ is real and even [i.e., $w(t) = w(-t)$],

$$\mathrm{Im}[c_n] = 0$$

3. If $w(t)$ is real and odd [i.e., $w(t) = -w(-t)$],

$$\mathrm{Re}[c_n] = 0$$

4. Parseval's theorem is

$$\frac{1}{T_0}\int_a^{a+T_0} |w(t)|^2 dt = \sum_{n=-\infty}^{n=\infty} |c_n|^2$$

2.5 Fourier Series

Some Properties of the Complex Fourier Series

5. The complex Fourier series coefficients of a real waveform are related to the quadrature Fourier series coefficients by

$$c_n = \begin{cases} \frac{1}{2}a_n - j\frac{1}{2}b_n, & n > 0 \\ a_0, & n = 0 \\ \frac{1}{2}a_{-n} + j\frac{1}{2}b_{-n}, & n < 0 \end{cases}$$

6. The complex Fourier series coefficients of a real waveform are related to the polar Fourier series coefficients by

$$c_n = \begin{cases} \frac{1}{2}D \underline{/\varphi_n}, & n > 0 \\ D_0, & n = 0 \\ \frac{1}{2}D_{-n} \underline{/\varphi_{-n}}, & n < 0 \end{cases}$$

Note that these properties for the complex Fourier series coefficients are similar to those of Fourier transform as given Sec. 2-2

2.5 Fourier Series

Quadrature Fourier Series

The **Quadrature Form** of the Fourier series representing any physical waveform ***w(t)*** over the interval $a < t < a+T_0$ is,

$$w(t) = \sum_{n=0}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=0}^{\infty} b_n \sin(n\omega_0 t)$$

Where the **orthogonal functions** are $\cos(nw_0 t)$ and $\sin(nw_0 t)$. we find that these ***Fourier coefficients*** are given by

$$a_n = \begin{cases} \dfrac{1}{T_0}\displaystyle\int_a^{a+T_0} w(t)\,dt, & n = 0 \\ \dfrac{2}{T_0}\displaystyle\int_a^{a+T_0} w(t)\cos n\omega_0 t\,dt, & n \geq 1 \end{cases}$$

$$b_n = \frac{2}{T_0}\int_a^{a+T_0} w(t)\sin n\omega_0 t\,dt, \quad n > 0$$

2.5 Fourier Series

Polar Fourier Series

The **Polar Form** of the Fourier series representing any physical waveform is,

$$w(t) = D_0 + \sum_{n=1}^{\infty} D_n \cos(n\omega_0 t + \varphi_n)$$

Where w(t) is real and

$$a_n = \begin{cases} D_0, & n = 0 \\ D_n \cos\varphi_n, & n \geq 1 \end{cases} \qquad b_n = -D_n \sin\varphi_n \qquad n \geq 1$$

These two equations may be inverted, we got

$$D_n = \begin{cases} a_0, & n = 0 \\ \sqrt{a_n^2 + b_n^2}, & n \geq 1 \end{cases} = \begin{cases} c_0, & n = 0 \\ 2\,|\,c_n\,|, & n \geq 1 \end{cases} \qquad \varphi_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right) = \angle c_n, n \geq 1$$

2.5 Fourier Series

What is the best form to use?

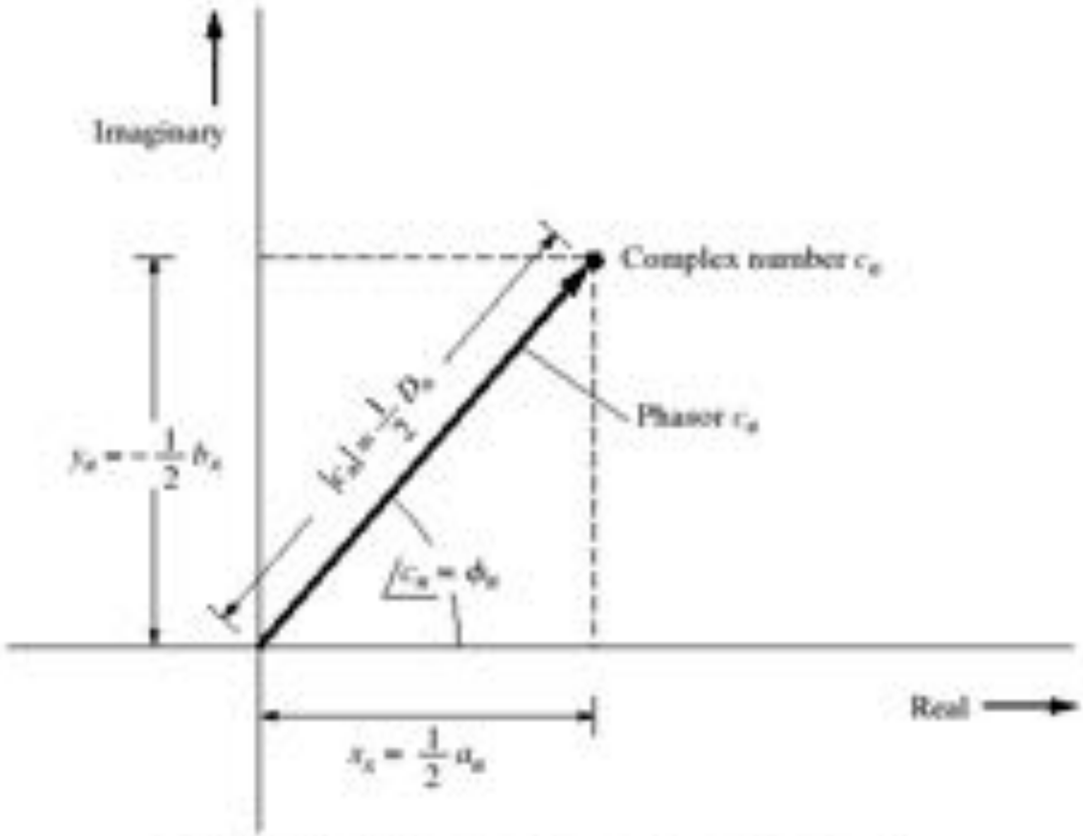


Figure 2-11 Fourier series coefficients, $n \geq 1$.

2.5 Fourier Series

Line Spectra for Periodic Waveforms

THEOREM. If ***w*(*t*)** is periodic with period T_0 and is represented by

$$w(t) = \sum_{n=-\infty}^{\infty} h(t - nT_0) = \sum_{n=-\infty}^{\infty} c_n e^{jnw_0 t}$$

where

$$h(t) = \begin{cases} w(t), & |t| < \frac{T_0}{2} \\ 0, & elsewhere \end{cases}$$

Then the *Fourier coefficients* are given by

$$c_n = f_0 H(nf_0)$$

and where

$H(f) = \Im[h(t)]$ and $f_0 = 1/T_0$

2.5 Fourier Series

THEOREM. For a periodic waveform ***w(t)***, the ***normalized power*** is

$$P_w = \left\langle w^2(t) \right\rangle = \sum_{n=-\infty}^{\infty} \left| c_n \right|^2$$

Where the $\{c_n\}$ are the complex Fourier coefficients for the waveform

2.5 Fourier Series

THEOREM. For a periodic waveform ***w(t)***, the ***power spectral density (PSD)*** is

$$P(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \, \delta(f - nf_0)$$

Where $T_0 = 1/f_0$ is the period of the waveform, and the $\{c_n\}$ are the complex Fourier coefficients for the waveform