Chapter 5 Multiple Random Variables

§ 5.1 Joint Cumulative Distribution Function

The joint CDF $F_{X,Y}(x,y)=P[X<=x, Y<=y]$ is a ***complete probability*** model for any pair of random variables **X** and **Y**.

Definition 5.1 Joint Cumulative Distribution Function (CDF)

$$F_{X,Y}(x,y) = P[X \leq x, Y \leq y]$$

§ 5.1 Joint Cumulative Distribution Function

Theorem 5.1

For any pair of random variables, X, Y,

$a.)\ 0 \le F_{X,Y}(x, y) \le 1$

$b.)\ F_{X,Y}(\infty, \infty) = 1$

$c.)\ F_X(x) = F_{X,Y}(x, \infty)$

$d.)\ F_Y(y) = F_{X,Y}(\infty, y)$

$e.)\ F_{X,Y}(x, -\infty) = 0$

$f.)\ F_{X,Y}(-\infty, y) = 0$

g.) If $x \le x_1$ *and* $y \le y_1$, *then*

$$F_{X,Y}(x, y) \le F_{X,Y}(x_1, y_1)$$

§ 5.1 Joint Cumulative Distribution Function

Example 5.2

X years is the age of children entering first grade in a school. Y is the age of children entering second grade. The joint CDF of X and Y is:

$$F_{X,Y}(x,y) = \begin{cases} 0, & x < 5 \\ 0, & y < 6 \\ (x-5)(y-6), & 5 \le x < 6, 6 \le y < 7 \\ y-6, & x \ge 6, 6 \le y < 7 \\ x-5, & 5 \le x < 6, y \ge 7 \\ 1, & otherwise \end{cases}$$

Fin $F_X(x)$ and $F_Y(y)$

§ 5.1 Joint Cumulative Distribution Function

Theorem 5.2

$$\boldsymbol{P[x_1 < X \le x_2, y_1 < Y \le y_2] = F_{X,Y}(x_2, y_2) - F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1)}$$

Quiz 5.1

Express the following extreme values of the joint CDF $F_{X,Y}(x, y)$ as numbers or in terms of the CDF's $F_X(x)$ and $F_Y(y)$

$a.)\ F_{X,Y}(-\infty, 2)$

$b.)\ F_{X,Y}(-\infty, \infty)$

$c.)\ F_{X,Y}(\infty, y)$

$d.)\ F_{X,Y}(\infty, -\infty)$

§ 5.2 Joint Probability Mass Function (Discrete)

Definition 5.2 Joint Probability Mass Function (PMF)

The **joint probability mass function** of **discrete random** variable X and Y is

$$P_{X,Y}(x, y) = P[X = x, Y = y]$$

Necessary and Sufficient Conditions for a Function to be a discrete Joint Probability Mass Function

1. $P_{X,Y}(x, y) \geq 0$

2. $\sum_{all\ x} \sum_{all\ y} P_{X,Y}(x, y) = 1$

§ 5.2 Joint Probability Mass Function (Discrete)

Example

In an automobile plant two tasks are performed by robots. The first entails welding two joints; the second, tightening three bolts. Let X denote the number of defective welds and Y the number of improperly tightened bolts produced per car. Since X and Y are each discrete, (X,Y) is a two-dimensional discrete random variable. Past data indicate that the joint mass function is as shown in Table.

x/y	1	2	3	4
0	0.840	0.030	0.020	0.010
1	0.060	0.010	0.008	0.002
2	0.010	0.005	0.004	0.001

$$P_{X,Y}(x,y) = \begin{cases} 0.840, & x = 0, y = 0 \\ 0.030, & x = 0, y = 1 \\ 0.020, & x = 0, y = 2 \\ 0.010, & x = 0, y = 3 \\ \cdots, & \cdots\cdot\cdot \\ 0, & otherwise \end{cases}$$

§ 5.2 Joint Probability Mass Function (Discrete)

Theorem 5.3

For discrete random variable X and Y and any set B in the X, Y plane, the probability of the event $\{(X, Y) \in B\}$ is

$$P[B] = \sum_{(x,y)\in B} P_{X,Y}(x, y)$$

§ 5.3 Marginal Probability Mass Function (Discrete)

For discrete random variable, the marginal PMFs PX(x) and PY(y) are probability models for the individual random variables X and Y

Theorem 5.4

For discrete random variables X and Y with joint PMF $P_{X,Y}(x,y)$

$$P_X[x] = \sum_{y \in S_Y} P_{X,Y}(x,y)$$

Y choose all the *r. v.* from S_Y

$$P_Y[y] = \sum_{x \in S_X} P_{X,Y}(x,y)$$

X choose all the *r. v.* from S_X

§ 5.4 Joint Probability Density Function (Continuous)

Definition 5.3 Joint Probability Density Function (PDF)

The joint PDF of the continuous random variables X and Y is a function $f_{X,Y}(x,y)$ with the property

$$F_{X,Y}(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(u,v)dudv$$

Theorem 5.5

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

§ 5.6 Independent Random Variable

Definition 5.4 Independent Random Variable

Random variables ***X*** and ***Y*** are **independent** **if and only if**

Discrete: $P_{X,Y}(x,y) = P_X(x)P_Y(y)$

Continuous: $f_{X,Y}(x,y) = f_X(x)f_Y(y)$

Example 5.12

$$f_{X,Y} = \begin{cases} 4xy, & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & otherwise \end{cases}$$

Are X and Y independent?

§ 5.6 Independent Random Variable

Quiz 5.6

(A) Random variables ***X*** and ***Y***, and Random variables Q and G have joint PMFs:

$P_{X,Y}(x,y)$	0	1	2
0	0.01	0	0
1	0.09	0.09	0
2	0	0	0.81

Are X and Y independent

$P_{Q,G}(q,g)$	0	1	2
0	0.06	0.18	0.12
1	0.04	0.12	0.8

Are Q and G independent

(B) Random variables X1 and X2 are independent and identically distributed with probability density function

$$f_X(x) = \begin{cases} x/2, & 0 \le x \le 2 \\ 0, & otherwise \end{cases}$$

What is the joint PDF $f_{X_1,X_2}(x_1, x_2)$?

§ 5.7 Expected Value of a Function of Two Random Variables

Theorem 5.9

For Random variables ***X*** and ***Y***, the expected value of $W = g(X, Y)$ is

Discrete: $$E[W] = \sum_{x \in S_X} \sum_{y \in S_Y} g(x, y) P_{X,Y}(x, y)$$

Continuous: $$E[W] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y)\, dx\, dy$$

Theorem 5.10

$$E[a_1 g_1(X, Y) + \cdots + a_n g_n(X, Y)] = a_1 E[g_1(X, Y)] + \cdots + a_n E[g_n(X, Y)]$$

§ 5.7 Expected Value of a Function of Two Random Variables

Theorem 5.11

For any two Random variables ***X*** and ***Y***

$$E[X+Y] = E[X] + E[Y]$$

Theorem 5.12

$$Var[X+Y] = Var[X] + Var[Y] + 2E[(X-\mu_X)(Y-\mu_Y)]$$

§ 5.7 Expected Value of a Function of Two Random Variables

Example

Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 5x^2/2, & -1 \le x \le 1, 0 \le y \le x^2 \\ 0, & otherwise \end{cases}$$

(a) What are E[X] and Var[X]?
(b) What are E[Y] and Var[Y]?
(c) What is Cov[X, Y]?
(d) What is E[X+Y]?
(e) What is Var[X+Y]?

§ 5.8 Covariance, Correlation and Independence

Definition 5.5 Covariance

The **covariance** of two random variables ***X*** and ***Y*** is

$$Cov[X,Y] = E[(X-\mu_X)(Y-\mu_Y)]$$

Definition 5.6 Correlation Coefficient

The **correlation coefficient** of two random variables ***X*** and ***Y*** is

$$\rho_{X,Y} = \frac{Cov[X,Y]}{\sqrt{Var[X]Var[Y]}} = \frac{Cov[X,Y]}{\sigma_X \sigma_Y}$$

§ 5.8 Covariance, Correlation and Independence

Theorem 5.13

If $\hat{X} = aX + b$ and $\hat{Y} = cY + d$, then

(a) $\rho_{\hat{X},\hat{Y}} = \rho_{X,Y}$

(b) $Cov[\hat{X},\hat{Y}] = acCOV[X,Y]$

Theorem 5.14

$$-1 \leq \rho_{X,Y} \leq 1$$

Theorem 5.15

If X and Y are random variables such that ***Y = aX + b***

$$\rho_{X,Y} = \begin{cases} -1 & a < 0 \\ 0 & a = 0 \\ 1 & a > 0 \end{cases}$$

§ 5.8 Covariance, Correlation and Independence

Definition 5.7

The correlation of ***X*** and ***Y*** is $r_{X,Y} = E[XY]$

Theorem 5.16

(a) $Cov[X,Y] = r_{X,Y} - \mu_X \mu_Y$ **The relation between covariance and correlation**

(b) Var$[X+Y] = Var[X] + Var[Y] + 2Cov[X,Y]$ **The relation between variance and covariance**

(c) If X = Y, Cov$[X,Y] = Var[X] = Var[Y]$, and $r_{X,Y} = E[X^2] = E[Y^2]$

§ 5.8 Covariance, Correlation and Independence

Definition 5.8 Orthogonal Random Variables

Random variables ***X*** and ***Y*** are **orthogonal** if $r_{X,Y} = 0$

Definition 5.9 Uncorrelated Random Variables

Random variables ***X*** and ***Y*** are **uncorrelated** if $Cov[X, Y] = 0$

§ 5.8 Covariance, Correlation and Independence

Theorem 5.17

For **independent** random variables X and Y,

(a) $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$

(b) $r_{X,Y} = E[XY] = E[X]E[Y]$

(c) $\text{Cov}[X, Y] = \rho_{X,Y} = 0$

(c) $\text{Var}[X, Y] = Var[X] + Var[Y]$

§ 5.9 Bivariate Gaussian Random Variables

Definition 5.10 Bivariate Gaussian Random Variables

Random variable ***X*** and ***Y*** have a bivariate Gaussian PDF with parameters μ_X , μ_Y , $\sigma_X > 0$ $\sigma_Y > 0$ and $\rho_{X,Y}$ satisfying $-1 < \rho_{X,Y} < 1$ if

$$f_{X,Y}(x,y) = \frac{\exp\left[-\dfrac{\left(\dfrac{x-\mu_X}{\sigma_X}\right)^2 - \dfrac{2\rho_{X,Y}(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \left(\dfrac{y-\mu_Y}{\sigma_Y}\right)^2}{2\left(1-\rho_{X,Y}^2\right)} \right]}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{X,Y}^2}}$$

§ 5.9 Bivariate Gaussian Random Variables

Theorem 5.18

If ***X*** and ***Y*** are the bivariate Gaussian random variable in definition 5.10, X is the Gaussian (μ_X, σ_X) random variable and Y is the Gaussian (μ_Y , σ_Y) random variable:

$$f_X(x) = \frac{1}{\sigma_X\sqrt{2\pi}} \exp\left(\frac{-(x-\mu_X)^2}{2\sigma_X^2}\right) \qquad f_Y(y) = \frac{1}{\sigma_Y\sqrt{2\pi}} \exp\left(\frac{-(y-\mu_Y)^2}{2\sigma_Y^2}\right)$$

Theorem 5.19

Bivariate Gaussian random variable ***X*** and ***Y*** in definition 5.10, have correlation coefficient $\rho_{X,Y}$

Theorem 5.20

Bivariate Gaussian random variable ***X*** and ***Y*** are uncorrelated if and only if they are independent

§ 5.9 Bivariate Gaussian Random Variables

Theorem 5.21

If ***X*** and ***Y*** are the bivariate Gaussian random variable in definition 5.10, and W_1 and W_2 are given by the linearly independent equations

$$W_1 = a_1 X + b_1 Y \qquad W_2 = a_2 X + b_2 Y$$

Then W_1 and W_2 are bivariate Gaussian random variable such that

$$E[W_i] = a_i \mu_X + b_i \mu_Y$$

$$Var[W_i] = a_i^2 \sigma_X^2 + b_i^2 \sigma_Y^2 + 2 a_i b_i \rho_{X,Y} \sigma_X \sigma_Y$$

$$Cov[W_1, W_2] = a_1 a_2 \sigma_X^2 + b_1 b_2 \sigma_Y^2 + (a_1 b_2 + a_2 b_1) \rho_{X,Y} \sigma_X \sigma_Y$$

§ 5.10 Multivariate Probability Models

Definition 5.11 Multivariate Joint CDF

The ***joint CDF*** of $X_1, X_2, \ldots X_n$ is

$$F_{X_1,\ldots,X_n}(x_1,x_2,\ldots,x_n) = P\left[X_1 \le x_1, X_2 \le x_2, \ldots, X_N \le x_n\right]$$

Definition 5.12 Multivariate Joint PMF

The ***joint PMF*** of the discrete random variable $X_1, X_2, \ldots X_n$ is

$$P_{X_1,\ldots,X_n}(x_1,x_2,\ldots,x_n) = P\left[X_1 = x_1, X_2 = x_2, \ldots, X_N = x_n\right]$$

Definition 5.12 Multivariate Joint PDF

The ***joint PDF*** of the continuous random variable $X_1, X_2, \ldots X_n$ is

$$f_{X_1,\ldots,X_n}(x_1,x_2,\ldots,x_n) = \frac{\partial^n F_{X_1\ldots X_n}(x_1,x_2,\ldots,x_n)}{\partial x_1 \partial x_2 \ldots \partial x_n}$$

§ 5.10 Multivariate Probability Models

Theorem 5.22

If $X_1, X_2, \ldots X_n$ are discrete random variables with joint PMF $P_{X_1,\ldots,X_n}(x_1, x_2, \ldots, x_n)$

$$P_{X_1,\ldots,X_n}(x_1, x_2, \ldots, x_n) \geq 0$$

$$\sum_{x_1 \in S_{X1}} \cdots \sum_{x_n \in S_{Xn}} P_{X_1,\ldots,X_n}(x_1, x_2, \ldots, x_n) = 1$$

Theorem 5.23

If $X_1, X_2, \ldots X_n$ are discrete random variables with joint PMF $f_{X_1,\ldots,X_n}(x_1, x_2, \ldots, x_n)$

$$f_{X_1,\ldots,X_n}(x_1, x_2, \ldots, x_n) \geq 0$$

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1,\ldots,X_n}(x_1, x_2, \ldots, x_n)\, dx_1 \ldots dx_n = 1$$

$$F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1,\ldots,X_n}(u_1, u_2, \ldots, u_n)\, du_1 \ldots du_n$$

§ 5.10 Multivariate Probability Models

Theorem 5.24

The probability of an event A expressed in terms of the random variables $X_1, X_2, \ldots X_n$

Discrete

$$P[A] = \sum_{(x_1,\ldots,x_n)\in A} P_{X_1,\ldots,X_n}(x_1,x_2,\ldots,x_n)$$

Continuous

$$P[A] = \int \cdots \iint_A f_{X_1,\ldots,X_n}(x_1,x_2,\ldots,x_n)\,dx_1 \ldots dx_n$$