3.3 Difference Equations for Discrete-Time Systems

A **Discrete-time** system can be modeled with ***difference equations*** involving **current**, **past**, or **future** samples of input and output signals

Example 3.3 Moving-average filter

A **length-N** ***moving average filter*** is a simple system that produces an output equal to the arithmetic average of the **most recent** *N* samples of the input signal.

Compute $y[100]$:

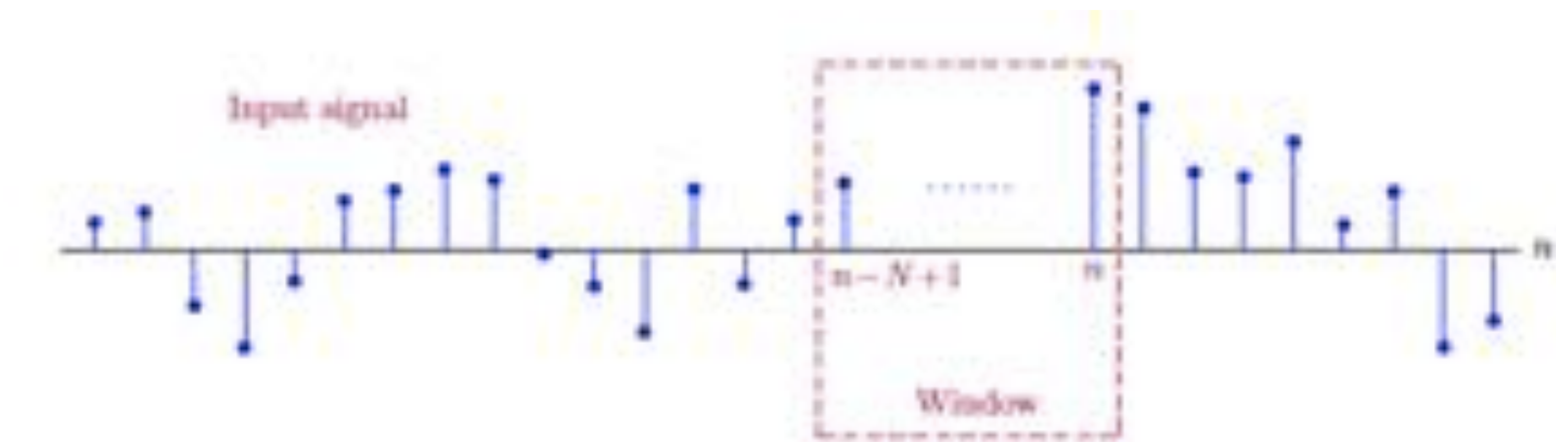
$$y[100] = \frac{x[100] + x[99] + \ldots + x[100-(N-1)]}{N} = \frac{1}{N}\sum_{k=0}^{N-1} x[100-k]$$

The general expression for the length-N moving average filter:

$$y[n] = \frac{x[n] + x[n-1] + \ldots + x[n-(N-1)]}{N} = \frac{1}{N}\sum_{k=0}^{N-1} x[n-k]$$

3.3 Difference Equations for Discrete-Time Systems

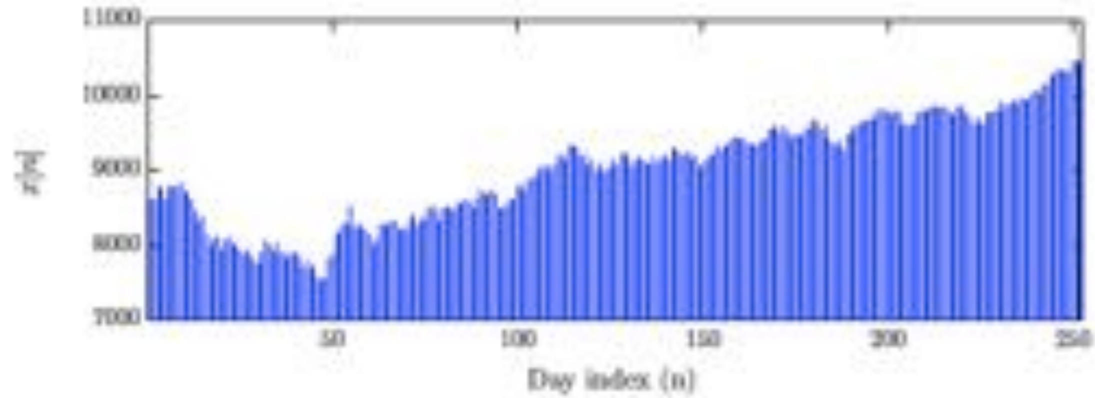
Example 3.3 Moving-average filter



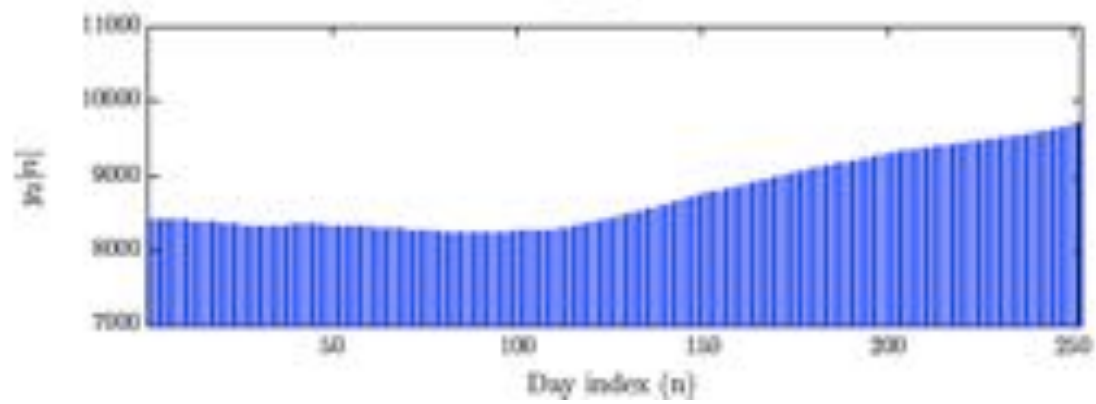
3.3 Difference Equations for Discrete-Time Systems

Example 3.3 Moving-average filter

Example: 100-Day moving average of the Dow Jones index



$$y_2[n] = \frac{1}{100} \sum_{k=0}^{99} x[n-k]$$



3.3 Difference Equations for Discrete-Time Systems

Example 3.4 Length-2 moving-average filter

Solution:

A length-2 moving average filter produces an output by averaging the current input sample and the previous input sample.

$$y[n] = \frac{1}{2}\,x[n] + \frac{1}{2}\,x[n-1]$$

For a few values of the index we can write the following set of equations:

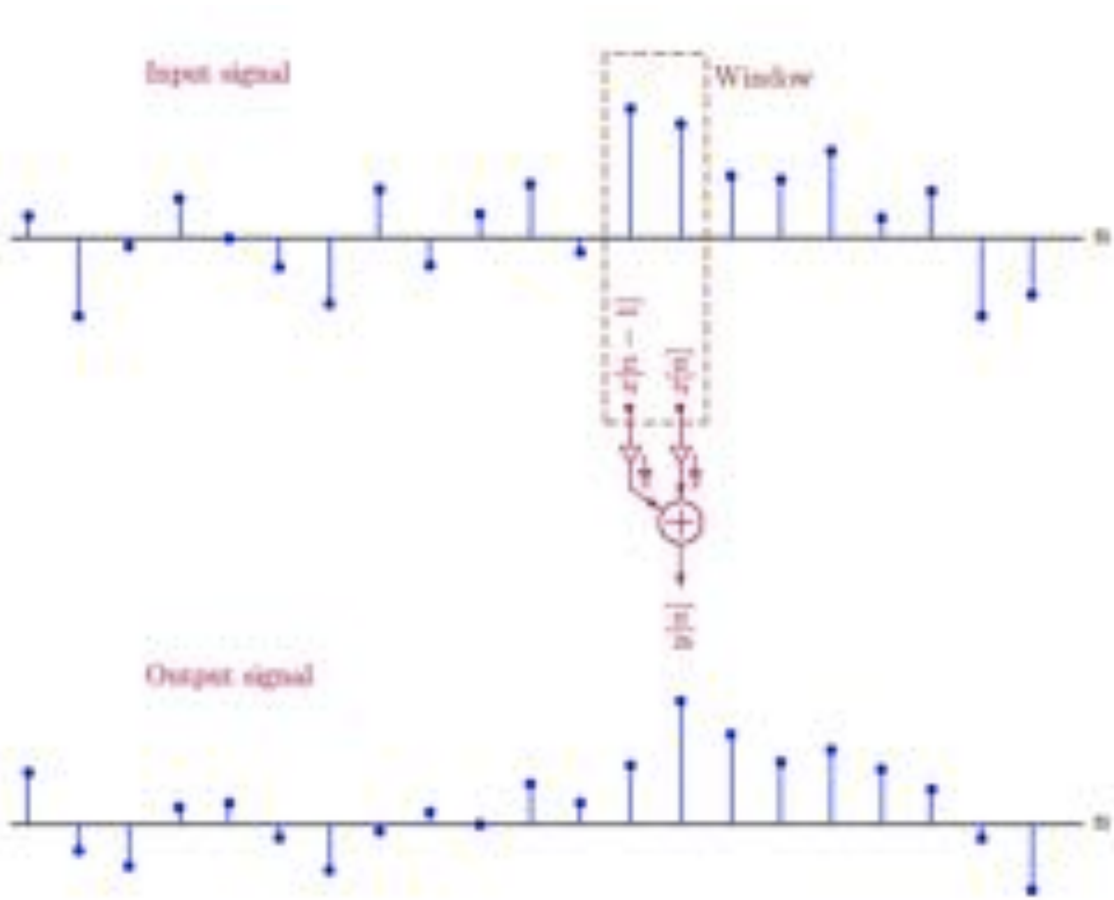
$$n = 0: \qquad y[0] = \tfrac{1}{2}\,x[0] + \tfrac{1}{2}\,x[-1]$$

$$n = 1: \qquad y[1] = \tfrac{1}{2}\,x[1] + \tfrac{1}{2}\,x[0]$$

$$n = 2: \qquad y[2] = \tfrac{1}{2}\,x[2] + \tfrac{1}{2}\,x[1]$$

3.3 Difference Equations for Discrete-Time Systems

Example 3.4 Length-2 moving-average filter



3.3 Difference Equations for Discrete-Time Systems

Example 3.6 Exponential smoother

Solution:

The current output sample is computed as a mix of the current input sample and the previous output sample through the equation

$$y[n] = (1 - \alpha)\; y[n-1] + \alpha\, x[n]$$

The parameter α is a constant in the range $0 < \alpha < 1$, and it controls the degree of smoothing.

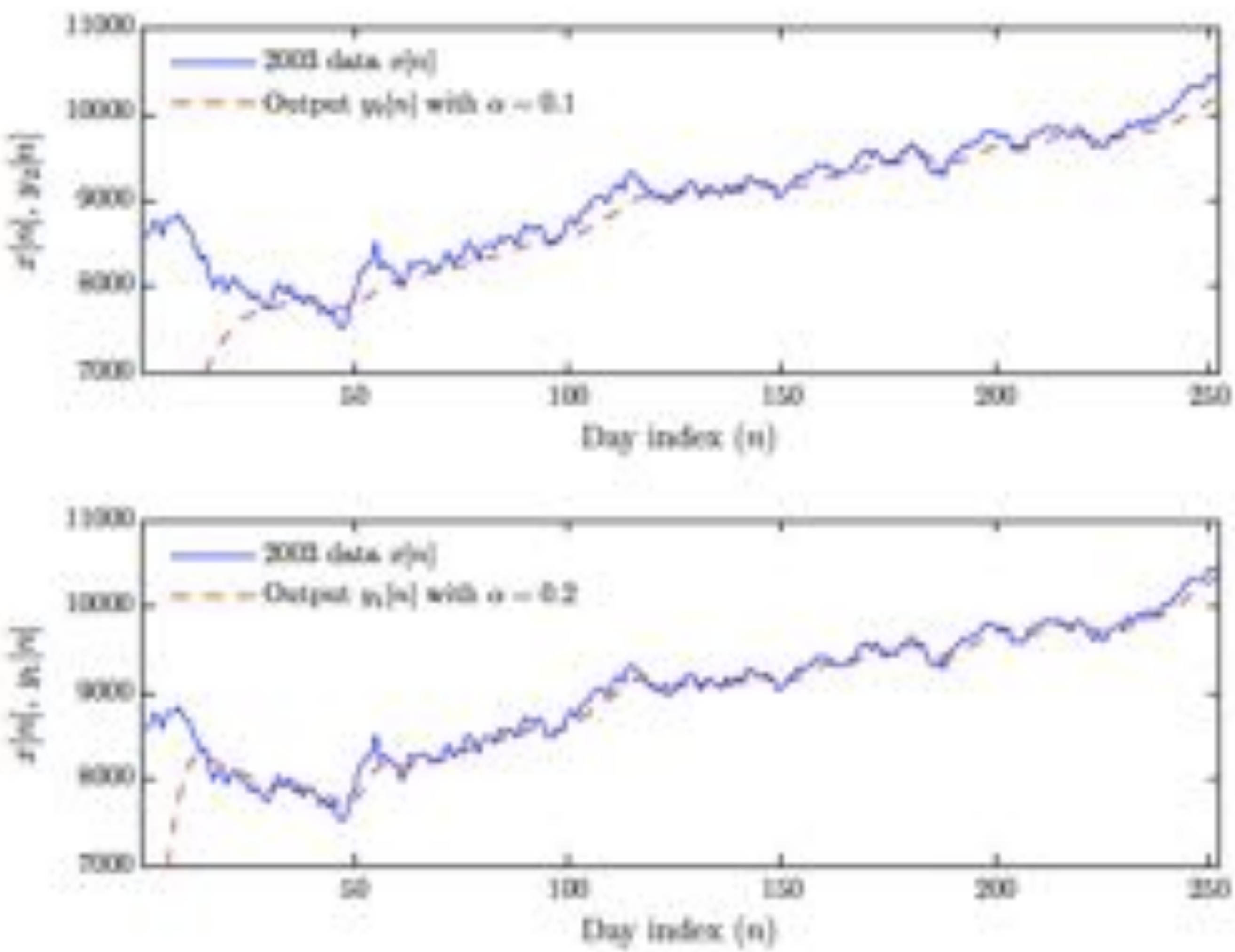
$$n = 0: \quad y[0] = (1 - \alpha)\; y[-1] + \alpha\, x[0]$$

$$n = 1: \quad y[1] = (1 - \alpha)\; y[0] + \alpha\, x[1]$$

$$n = 2: \quad y[2] = (1 - \alpha)\; y[1] + \alpha\, x[2]$$

3.3 Difference Equations for Discrete-Time Systems

Example 3.6 Exponential smoother

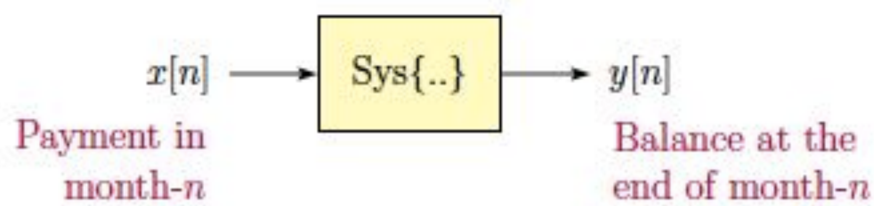


3.3 Difference Equations for Discrete-Time Systems

Example 3.7 Loan payments

Express the act of taking a bank loan an paying it back over time as a discrete-time system problem with monthly payments representing the Input signal and monthly balance representing the output signal.

$$y[n] = (1+c)y[n-1] + x[n]$$



"c" is the monthly interest rate

3.3 Difference Equations for Discrete-Time Systems

Example 3.7 Loan payments

Express the act of taking a bank loan an paying it back over time as a discrete-time system problem with monthly payments representing the Input signal and monthly balance representing the output signal.

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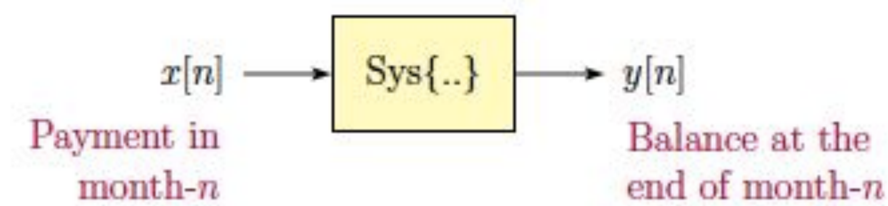


"c" is the monthly interest rate

We can find out the monthly payment x[n] by solving the difference equation

3.4 Constant-Coefficient Linear Difference Equations

Constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Initial conditions:

$y[n_0\text{-}1]$, $y[n_0\text{-}2]$, ... $y[n_0\text{-}N]$

It is typical, but not required, to have $n_0 = 0$.

3.4 Constant-Coefficient Linear Difference Equations

Iteratively solving difference equations.

Consider the difference equation for the exponential smoother of Ex.3.6

$$y[n] = (1-\alpha)y[n-1] + \alpha\, x[n]$$

✧ Given the initial value $y[-1]$ of the output signal, $y[0]$ is found by

$$y[0] = (1-\alpha)y[-1] + \alpha\, x[0]$$

✧ Knowing $y[0]$, the next output sample $y[1]$ is found by

$$y[1] = (1-\alpha)y[0] + \alpha\, x[1]$$

✧ Knowing $y[1]$, the next output sample $y[2]$ is found by

$$y[2] = (1-\alpha)y[1] + \alpha\, x[2]$$

3.5 Solving Difference Equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Initial conditions:

$$y[n_0-1],\ y[n_0-2],\ ,\ \ldots..,\quad y[n_0-N]$$

General solution:

$$y[n] = y_h[n] + y_p[n]$$

- ✧ ***$y_n([n]$:*** is the ***homogeneous*** solution of the differential equation
  natural response
- ✧ ***$y_p[n]$:*** is the ***particular solution*** of the differential equation
- ✧ ***$y[n] = y_h[n] + y_p[n]$*** is the ***forced solution*** of the differential equation
  forced response

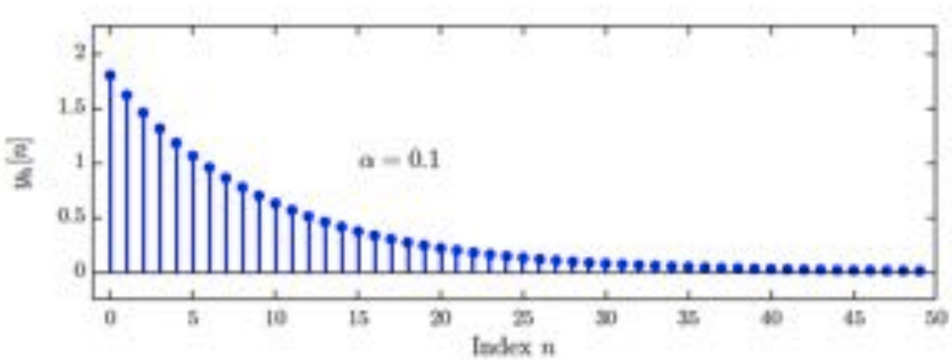
3.5 Solving Difference Equations

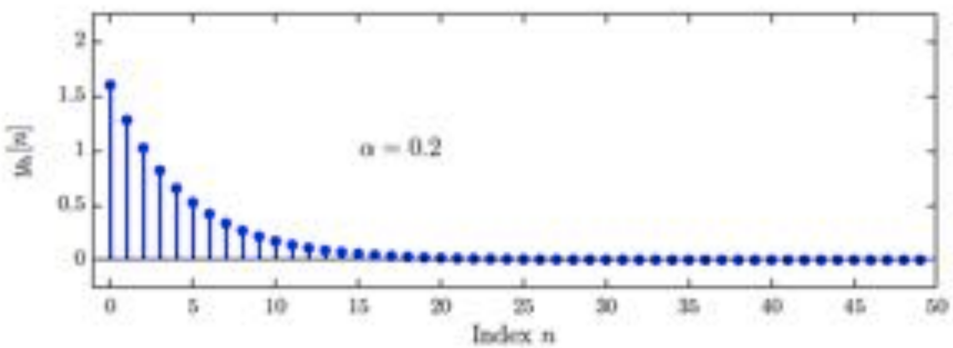
Example 3.11 Natural response of exponential smoother

Determine the natural response of the exponential smoother defined as:

$$y[n] = (1-\alpha)\,y[n-1] + \alpha x$$

If y[-1] = 2





3.5.1 Finding the natural response of a discrete-time system

General homogeneous differential equation:

$$\sum_{k=0}^{N} a_k y[n-k] = 0$$

Characteristic equation:

$$\sum_{k=0}^{N} a_k z^{-k} = 0$$

To obtain the characteristic equation, substitute:

$$y[n-k] \rightarrow z^{-k}$$

3.5.1 Finding the natural response of a discrete-time system

Write the characteristic equation in open form:

$$a_0 + a_1 z^{-1} + ... + a_{N-1} z^{-N+1} + a_N z^{-N} = 0$$

Multiply both sides by z^N to obtain

$$a_0 z^N + a_1 z^{N-1} + ... + a_{N-1} z^1 + a_N = 0$$

In factored form:

$$a_0 (a - z_1)(a - z_2)...(a - z_N) = 0$$

Homogeneous solution (assuming roots are distinct):

$$y_h[n] = c_1 z_1^n + c_2 z_2^n + ... + c_N z_N^n = \sum_{k=1}^{N} c_k z_k^n$$

Where unknown coefficients c_1, c_2, …., c_N are determined by the initial condition, the terms z_i^n are called the ***modes of the system***

3.5.1 Finding the natural response of a discrete-time system

Example 3.12 Natural response of second-order system

A second-order system is described by the difference equation:

$$y[n]-\frac{5}{6}y[n-1]+\frac{1}{6}y[n-2]=0$$

Determine the natural response of this system for n >= 0 subject to initial condition:
y[-1] = 19, and *y*[-2] = 53

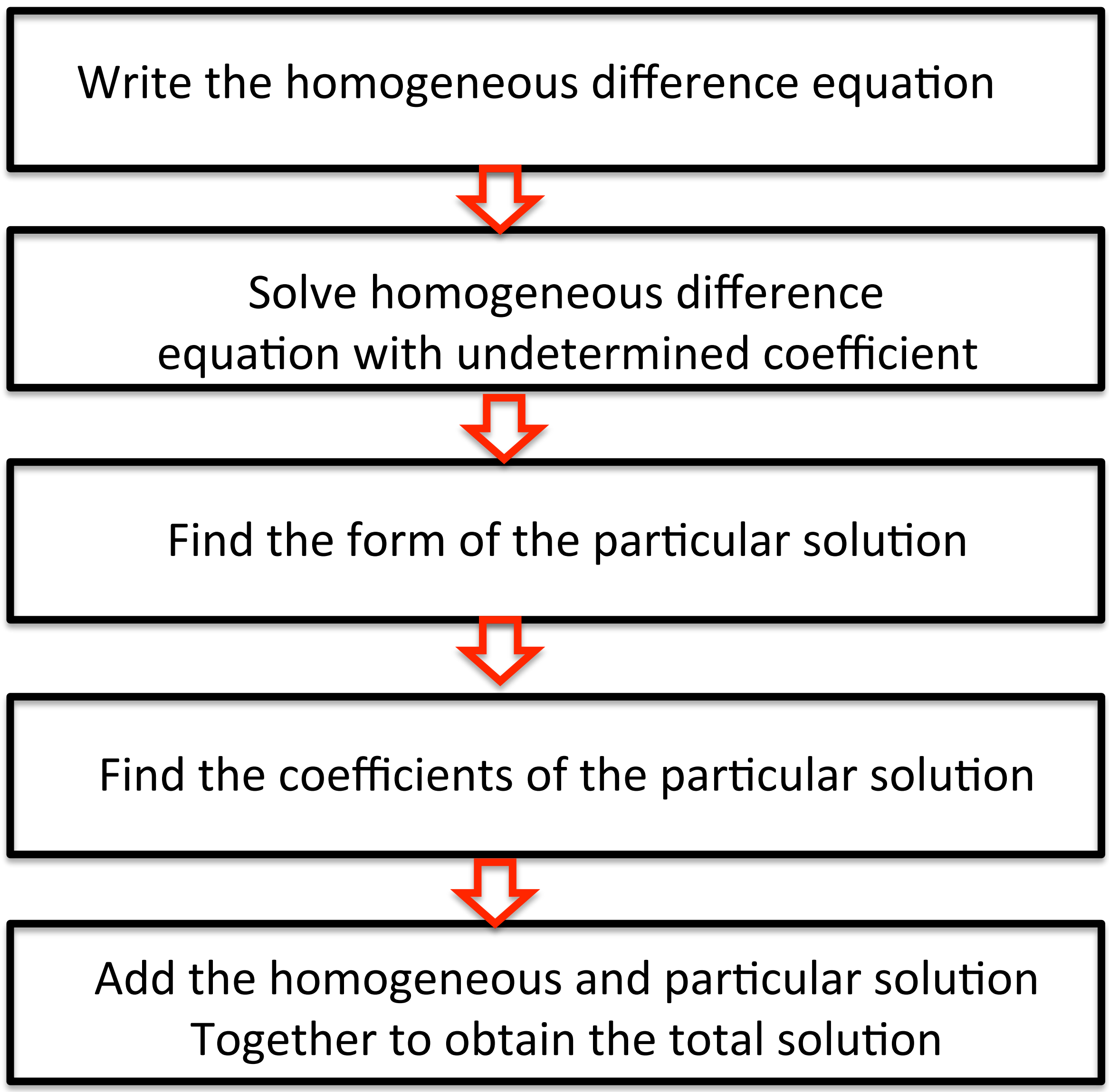
3.5.2 Finding the forced response of a discrete-time system

Choosing a ***particular solution*** for various discrete-time input signals

Input signal	Particular solution
K (constant)	k_1
$K\, e^{an}$	$k_1\, e^{an}$
$K\, \cos(\Omega_0 n)$	$k_1\, \cos(\Omega_0 n) + k_2\, \sin(\Omega_0 n)$
$K\, \sin(\Omega_0 n)$	$k_1\, \cos(\Omega_0 n) + k_2\, \sin(\Omega_0 n)$
$K\, n^m$	$k_m n^m + k_{m-1} n^{m-1} + \ldots + k_1 n + k_0$

The coefficient of the ***particular solution*** are determined from the Difference equation by assuming all initial conditions are equal to zero.

3.5.2 Finding the forced response of a discrete-time system



3.5.2 Finding the forced response of a discrete-time system

Example 3.14 Forced response of exponential smoother

Determine the forced response of the exponential smoother defined as:

$$y[n] = (1-\alpha)\, y[n-1] + \alpha x[n]$$

The input signal is a unit-step function, and y[-1] = 2.5