2.7 Impulse Response and Convolution

\[ h(t) = \text{Sys}\{\delta(t)\} \]

For a CTTLTI system: the impulse response also constitutes a complete description of the system.

Finding the impulse response of a CTTLTI system from the differential equation

1. Use a unit-step function for the input signal, and compute the forced response of the system, i.e., the unit-step response.
2. Differentiate the unit-step response of the system to obtain the impulse response, i.e.,

\[ h(t) = \frac{dy(t)}{dt} \]

This idea relies on the fact that differentiation is a linear operator

\[ \text{Sys}\{\delta(t)\} = \text{Sys}\{\frac{du(t)}{dt}\} = \frac{d}{dt}\left[\text{Sys}\{u(t)\}\right] \]
2.7.1 Finding Impulse Response of a CTLTI system

Example 2.18 Unit-step response of the simple RC circuit

Determine the impulse response of the first-order RC circuit. Assume the system is initially relaxed, that is, there is no initial energy stored in the system.
2.7.1 Finding Impulse Response of a CTLTI system

Example 2.18 Unit-step response of the simple RC circuit
2.7.2 Convolution Operation for CTLTI systems

The output signal $y(t)$ of a CTLTI system is equal to the *convolution* of its impulse response $h(t)$ with the input signal $x(t)$

\[ y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda)h(t - \lambda)d\lambda \]

\[ = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)x(t - \lambda)d\lambda \]

The symbol * represents *convolution operator*
2.7.2 Convolution Operation for CTLTI systems

Steps involved in computing the convolution of two signals

To compute the convolution of \( x(t) \) and \( h(t) \) at a specific time-instant \( t \):

1. Sketch the signal \( x(\lambda) \) as a function of the independent variable \( \lambda \). This corresponds to a simple name change on the independent variable, and the graph of the signal \( x(\lambda) \) appears identical to the graph of the signal \( x(t) \).

2. For one specific value of \( t \), sketch the signal \( h(t-\lambda) \) as a function of the independent variable \( \lambda \). This task can be broken down into two steps as follows:
   2a. Sketch \( h(-\lambda) \) as a function of \( \lambda \). This step amounts to time-reversal of \( h(\lambda) \).
   2b. In \( h(\lambda) \) substitute \( \lambda \rightarrow \lambda - t \). This step yields

\[
\left. h(-\lambda) \right|_{\lambda \rightarrow \lambda - t} = h(t - \lambda)
\]

and amounts to time-shifting \( h(-\lambda) \) by \( t \).

3. Multiply the two signals in 1 and 2 to obtain \( f(\lambda) = x(\lambda) h(t - \lambda) \).

4. Compute the area under the product \( f(\lambda) = x(\lambda) h(t - \lambda) \) by integrating it over the independent variable \( \lambda \). The result is the value of the output signal at the specific time instant \( t \).

5. Repeat steps 1 through 4 for all values of \( t \) that are of interest.
2.7.2 Convolution Operation for CTLTI systems

\[ \omega_1(t) = e^{-t}u(t) \quad \text{and} \quad \omega_2(t) = \prod(t-1) \]
2.7.1 Finding Impulse Response of a CTLTI system

Example 2.20 Unit-step response of the simple RC circuit

Determine the impulse response of the first-order RC circuit. Assume the system is initially relaxed, that is, there is no initial energy stored in the system. Solve the same problem using the convolution operation.
2.7.1 Finding Impulse Response of a CTDLTI system

Example 2.20 Unit-step response of the simple RC circuit
2.8 Causality in Continuous-time Systems

A system is said to be causal if the current value of the output signal depends only on current and past values of the input signal, but not on its future values.

Causality in Continuous-time Linear Time-Invariant (CTLTI) systems

\[ y(t) = h(t) \ast x(t) = \int_{-\infty}^{\infty} h(\lambda)x(t - \lambda) \, d\lambda \]

For \( \lambda < 0 \), the term \( x(t - \lambda) \) refers to future values of the input signal.

For a CTLTI system to be causal, the impulse response of the system must be equal to zero for all negative values of its argument.

\[ h(t) = 0 \text{ for all } t < 0 \]
2.9 Stability in Continuous-time Systems

A system is said to be \textit{stable} in the \textit{bounded-input bounded-output (BIBO)} sense if any bounded input signal is guaranteed to produce a bounded output signal

\textbf{CTLTI system}

\[ y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)x(t - \lambda)d\lambda \]

For \( \lambda < 0 \), the term \( x(t-\lambda) \) refers to future values of the input signal

\textbf{Causality in CTLTI systems}

For a CTLTI system to be stable, the impulse response of the system must be \textit{absolute integrable}.

\[ \int_{-\infty}^{\infty} |h(\lambda)| d\lambda < \infty \]
2.10 Approximate numerical solution of a differential equation

First-order linear differential equation:
\[
\frac{dy(t)}{dt} + \frac{1}{RC} y(t) = \frac{1}{RC} x(t)
\]

Rearrange terms:
\[
\frac{dy(t)}{dt} = -\frac{1}{RC} y(t) + \frac{1}{RC} x(t)
\]

General form:
\[
\frac{dy(t)}{dt} = g[t, y(t)] \quad \text{where} \quad g[t, y(t)] = -\frac{1}{RC} y(t) + \frac{1}{RC} x(t)
\]

\[
\left. \frac{dy(t)}{dt} \right|_{t=t_0} \approx \frac{y(t_0 + T) - y(t_0)}{T}
\]

\(T\): small step size
2.10 Approximate numerical solution of a differential equation

\[
\frac{y(t_0 + T) - y(t_0)}{T} \approx g[t_0, y(t_0)] \quad y(t_0 + T) \approx y(t_0) + Tg[t_0, y(t_0)]
\]

For the RC circuit, using \( t_0 = 0 \):

\[
y(T) \approx y(0) + Tg[0, y(t_0)] \\
= y(0) + T[-\frac{1}{RC}y(0) + \frac{1}{RC}x(0)]
\]

\[
y(2T) \approx y(0) + Tg[T, y(T)] \\
= y(T) + T[-\frac{1}{RC}y(T) + \frac{1}{RC}x(T)]
\]

This is known as the Euler method. More sophisticated methods exist with better accuracy.