2.7 Impulse Response and Convolution

$$h(t) = Sys\{\delta(t)\}$$

$$\delta(t) \longrightarrow \boxed{Sys\{..\}} \longrightarrow h(t)$$

For a CTLTI system: the impulse response also constitutes a complete description of the system

Finding the impulse response of a CTLTI system from the differential equation

1. Use a unit-step function for the input signal, and compute the forced response of the system, i.e., the *unit-step response*.
2. Differentiate the unit-step response of the system to obtain the impulse response, i.e.,

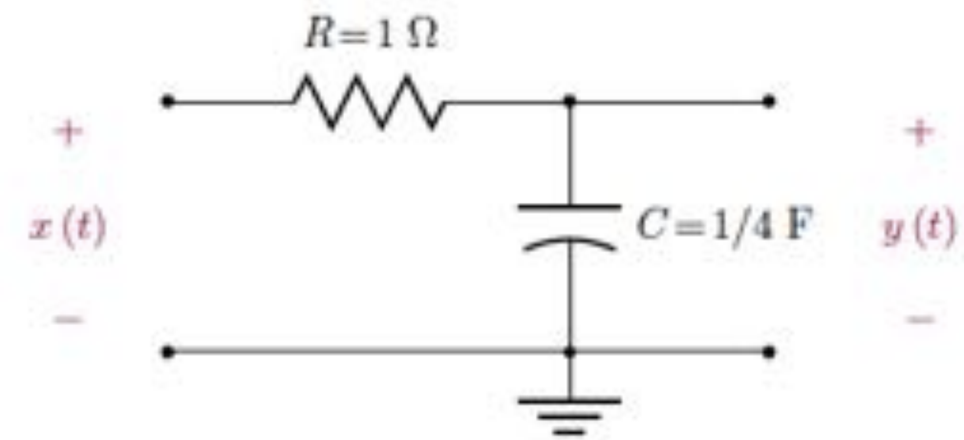
$$h(t) = \frac{dy(t)}{dt}$$

This idea relies on the fact that differentiation is a linear operator

$$Sys\{\delta(t)\} = Sys\{\frac{du(t)}{dt}\} = \frac{d}{dt}[Sys\{u(t)\}]$$

2.7.1 Finding Impulse Response of a CTLTI system

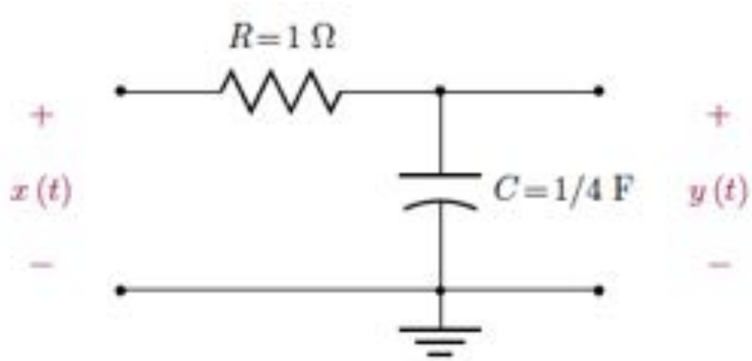
Example 2.18 Unit-step response of the simple RC circuit

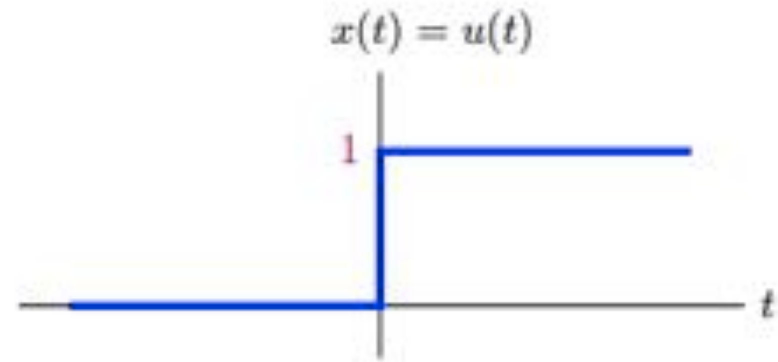


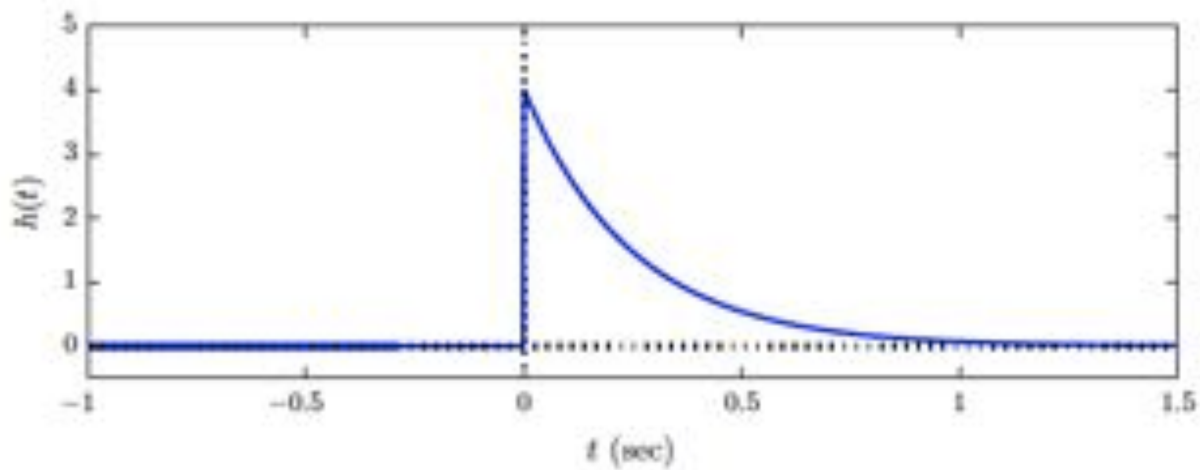
Determine the impulse response of the first-order RC circuit. Assume the system is initially relaxed, that is, there is no initial energy stored in the system.

2.7.1 Finding Impulse Response of a CTLTI system

Example 2.18 Unit-step response of the simple RC circuit







2.7.2 Convolution Operation for CTLTI systems

The output signal ***y(t)*** of a CTLTI system is equal to the ***convolution*** of Its impulse response ***h(t)*** with the input signal ***x(t)***

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda) h(t-\lambda) d\lambda$$

$$= h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda) x(t-\lambda) d\lambda$$

The symbol ***** represents ***convolution operator***

2.7.2 Convolution Operation for CTLTI systems

Steps involved in computing the convolution of two signals

To compute the convolution of $x(t)$ and $h(t)$ at a specific time-instant t:

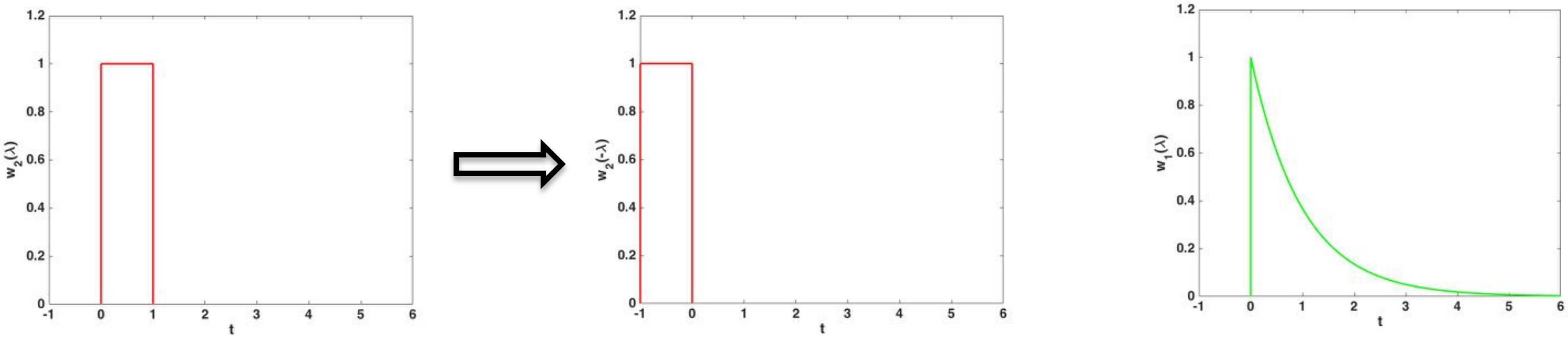
1. Sketch the signal $x(\lambda)$ as a function of the independent variable λ. This corresponds to a simple name change on the independent variable, and the graph of the signal $x(\lambda)$ appears identical to the graph of the signal $x(t)$.
2. For one specific value of t, sketch the signal $h(t-\lambda)$ as a function of the independent variable λ. This task can be broken down into two steps as follows:
   - 2a. Sketch $h(-\lambda)$ as a function of λ. This step amounts to time-reversal of $h(\lambda)$.
   - 2b. In $h(\lambda)$ substitute $\lambda \to \lambda - t$. This step yields

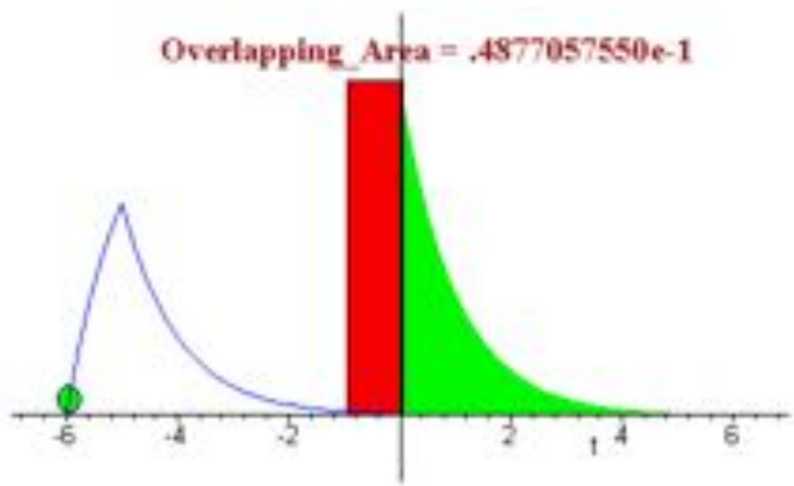
     $$h(-\lambda)\Big|_{\lambda \to \lambda - t} = h(t-\lambda)$$

     and amounts to time-shifting $h(-\lambda)$ by t.
3. Multiply the two signals in 1 and 2 to obtain $f(\lambda) = x(\lambda)\, h(t-\lambda)$.
4. Compute the area under the product $f(\lambda) = x(\lambda)\, h(t-\lambda)$ by integrating it over the independent variable λ. The result is the value of the output signal at the specific time instant t.
5. Repeat steps 1 through 4 for all values of t that are of interest.

2.7.2 Convolution Operation for CTLTI systems

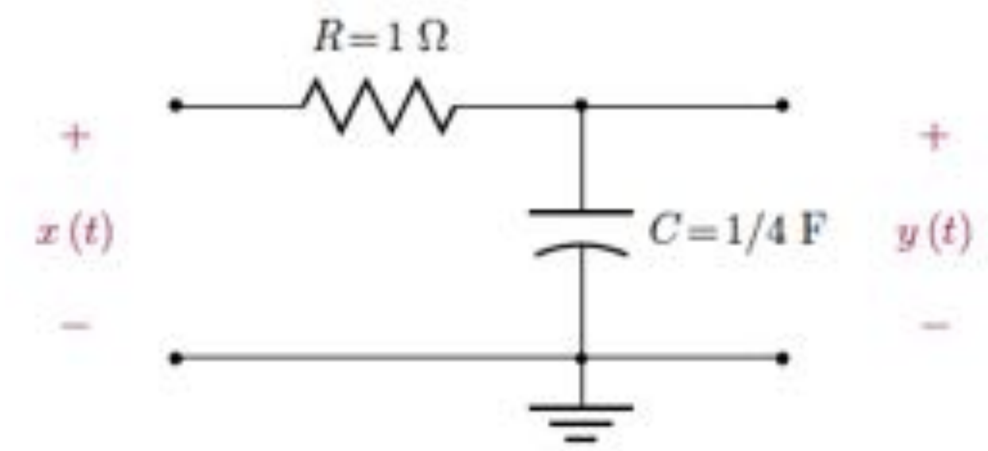
$$\omega_1(t) = e^{-t}u(t) \quad \text{and} \quad \omega_2(t) = \prod(t-1)$$





2.7.1 Finding Impulse Response of a CTLTI system

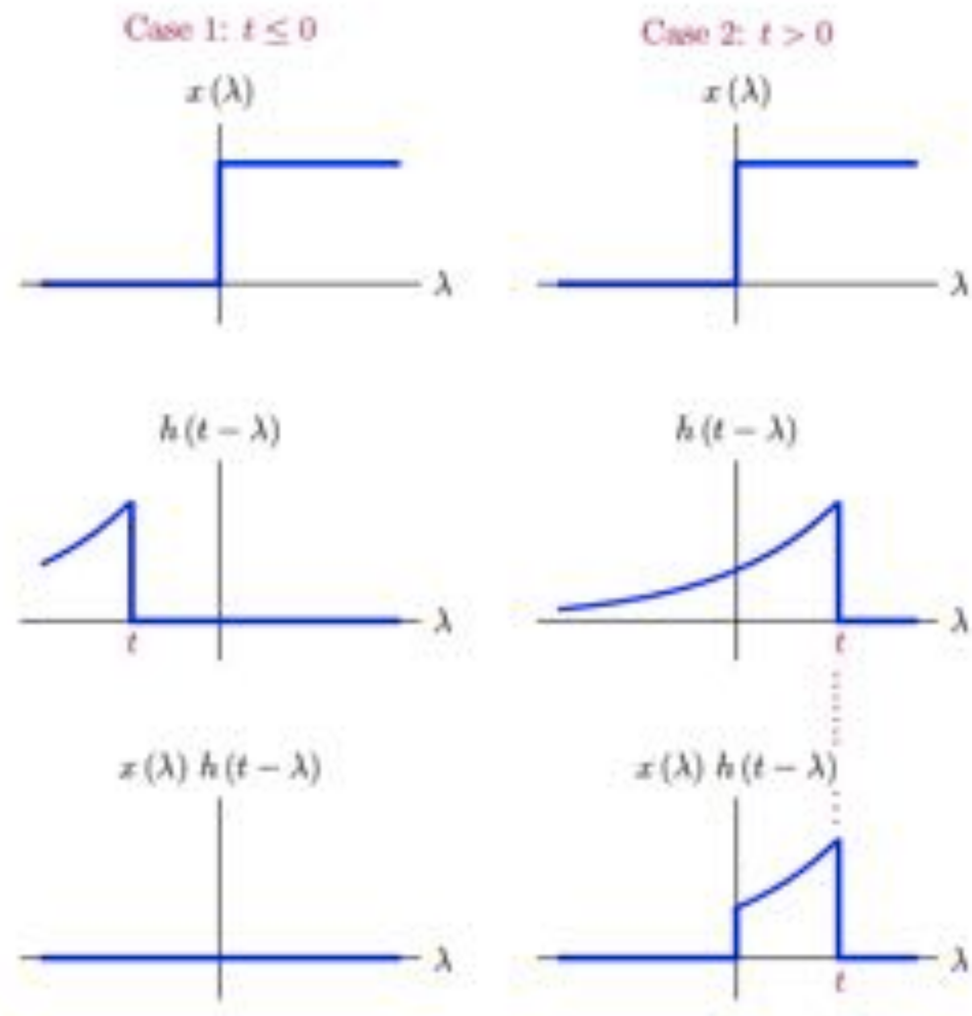
Example 2.20 Unit-step response of the simple RC circuit



Determine the impulse response of the first-order RC circuit. Assume the system is initially relaxed, that is, there is no initial energy stored in the system. Solve the same problem using the **convolution** operation.

2.7.1 Finding Impulse Response of a CTLTI system

Example 2.20 Unit-step response of the simple RC circuit



2.8 Causality in Continuous-time Systems

A system is said to be ***causal*** if the current value of the output signal depends only on current and past values of the input signal, but not on its future values

CTLTI system

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda$$

For $\lambda < 0$, the term $x(t-\lambda)$ refers to future values of the input signal

Causality in CTLTI systems

For a CTLTI system to be causal, the impulse response of the system must be equal to zero for all negative values of its argument.

$h(t) = 0$ for all $t < 0$

2.9 Stability in Continuous-time Systems

A system is said to be ***stable*** in the ***bounded-input bounded-output (BIBO)*** sense if any bounded input signal is guaranteed to produce a bounded output signal

CTLTI system

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda) x(t-\lambda) d\lambda$$

For $\lambda < 0$, the term $x(t-\lambda)$ refers to future values of the input signal

Causality in CTLTI systems

For a CTLTI system to be stable, the impulse response of the system must be ***absolute integrable.***

$$\int_{-\infty}^{\infty} |h(\lambda)| d\lambda < \infty$$

2.10 Approximate numerical solution of a differential equation

First-order linear differential equation:

$$\frac{dy(t)}{dt} + \frac{1}{RC}\,y(t) = \frac{1}{RC}\,x(t)$$

Rearrange terms:

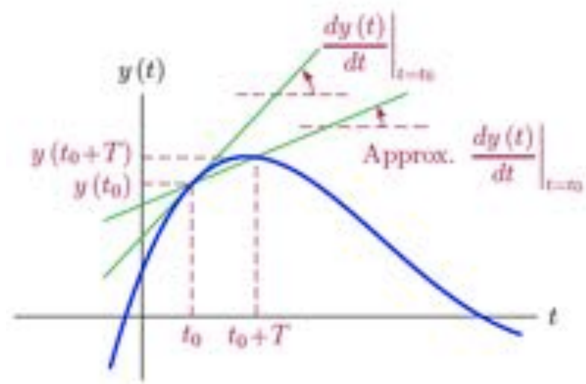
$$\frac{dy(t)}{dt} = -\frac{1}{RC}\,y(t) + \frac{1}{RC}\,x(t)$$

General form:

$$\frac{dy(t)}{dt} = g[t, y(t)] \qquad \text{where} \quad g[t, y(t)] = -\frac{1}{RC}\,y(t) + \frac{1}{RC}\,x(t)$$

$$\left.\frac{dy(t)}{dt}\right|_{t=t_0} \approx \frac{y(t_0+T) - y(t_0)}{T}$$

T: small step size



2.10 Approximate numerical solution of a differential equation

$$\frac{y(t_0+T)-y(t_0)}{T} \approx g[t_0, y(t_0)] \qquad y(t_0+T) \approx y(t_0) + Tg[t_0, y(t_0)]$$

For the RC circuit, using $t_0 = 0$:

$$\begin{aligned} y(T) &\approx y(0) + Tg[0, y(t_0)] \\ &= y(0) + T[-\frac{1}{RC}y(0) + \frac{1}{RC}x(0)] \end{aligned}$$

$$\begin{aligned} y(2T) &\approx y(0) + Tg[T, y(T)] \\ &= y(T) + T[-\frac{1}{RC}y(T) + \frac{1}{RC}x(T)] \end{aligned}$$

This is known as the Euler method. More sophisticated methods
Exist with better accuracy.