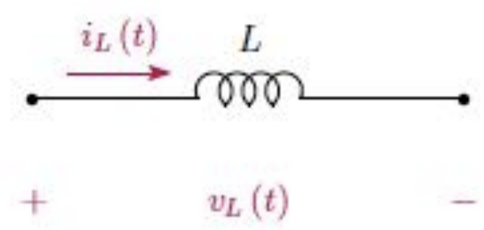
2.3 Differential Equations for Continuous-Time Systems

One method of representing the relationship established by a system between its **input** and **output** signals is a **differential equation** that approximately describes the interplay of the physical quantities within the system.

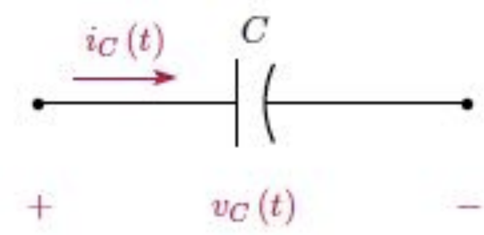
Example: $$\frac{d^2y}{dt^2} + 3x(t)\frac{dy}{dt} + y(t) - 2x(t) = 0$$

Many physical components have mathematical models that involve Integral and differential relationships between signals.



Ideal inductor:

$$v_L(t) = L\frac{di_L(t)}{dt}$$



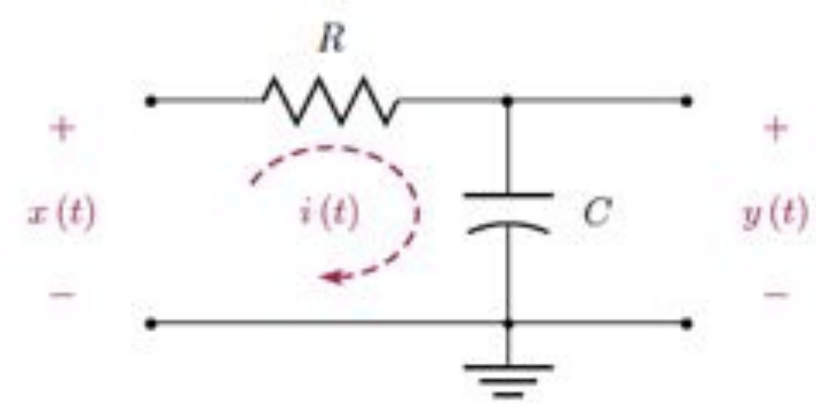
Ideal capacitor:

$$i_C(t) = L\frac{dv_C(t)}{dt}$$

2.3 Differential Equations for Continuous-Time Systems

Example 2.4 Differential equation for simple RC circuit

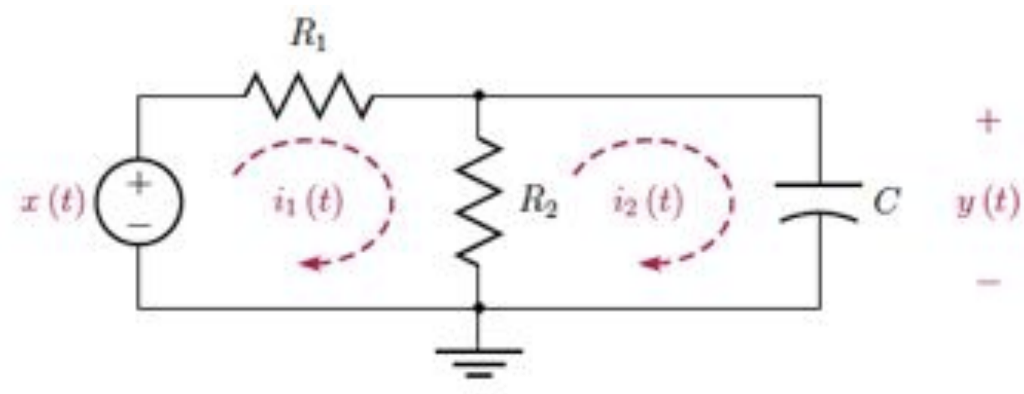
Find a differential equation between the input signal x(t) and the output signal y(t) to serve as a mathematical model for the first-order RC circuit



2.3 Differential Equations for Continuous-Time Systems

Example 2.5 Another RC circuit

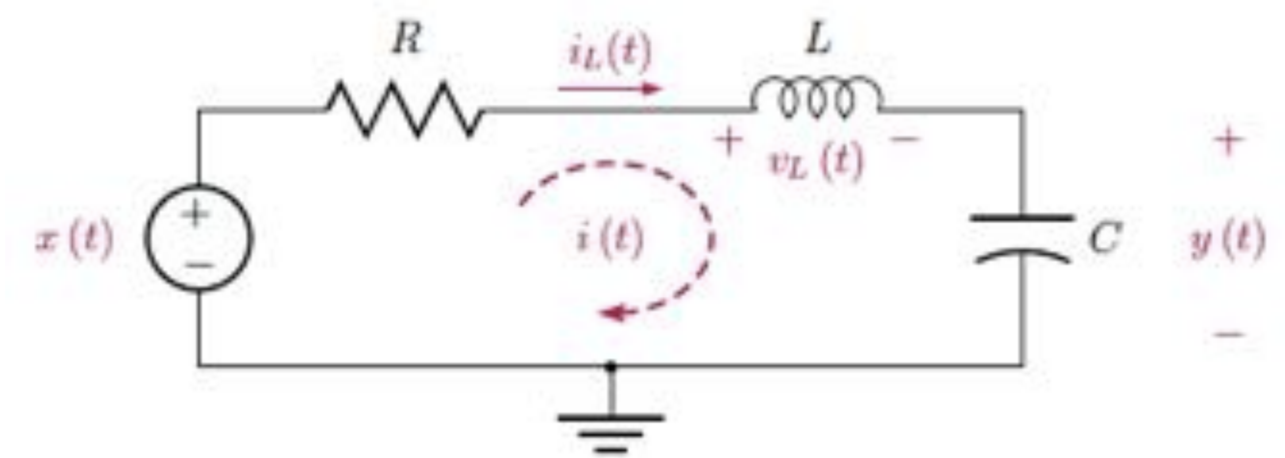
Find a differential equation between the input signal x(t) and the output signal y(t) to serve as a mathematical model for the first-order RC circuit



2.3 Differential Equations for Continuous-Time Systems

<u>Example 2.6 Differential equation for RLC circuit</u>

Find a differential equation between the input signal x(t) and the output signal y(t) to serve as a mathematical model for the first-order RLC circuit



2.4 Constant-coefficient ordinary differential equations

General constant-coefficient differential equation for a CTLTI system

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \ldots + a_1 \frac{dy(t)}{dt} + a_0 y(t) =$$
$$b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \ldots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

Constant-coefficient ordinary differential equation in closed summation form

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

We need to know $y(t_0)$, ..., $\left.\frac{d^{N-1} y(t)}{dt^{N-1}}\right|_{t=t_0}$ to find the solution for $t > t_0$

2.4 Constant-coefficient ordinary differential equations

Example 2.7 Checking linearity and time invariance of a differential equation

Determine whether the first-order constant-coefficient differential equation

$$\frac{dy(t)}{dt} + a_0 y(t) = b_0 x(t)$$

Represents a CTLTI system.

2.4 Constant-coefficient ordinary differential equations

The differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Represents a CTLTI system provided that all initial conditions are equal to zeros

$$y(t_0) = 0, \quad \left.\frac{dy(t)}{dt}\right|_{t=t_0} = 0, \quad \ldots.., \quad \left.\frac{d^{N-1} y(t)}{dt^{N-1}}\right|_{t=t_0} = 0$$

It is typical, but not required, to have t0 = 0.

2.5 Solving Differential Equations

2.5.1 Solution of the first-order differential equation

The first-order differential equation (represents a ***first-order CTLTI*** system)

$$\frac{dy(t)}{dt} + \alpha y(t) = r(t) \qquad y(t_0): \text{ specified}$$

Is solved as

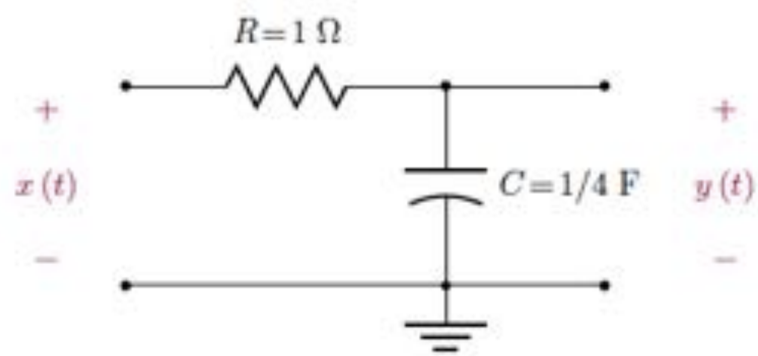
$$y(t) = e^{-\alpha(t-t_0)} y(t_0) + \int_{t_0}^{t} e^{-\alpha(t-\tau)} r(\tau) d\tau$$

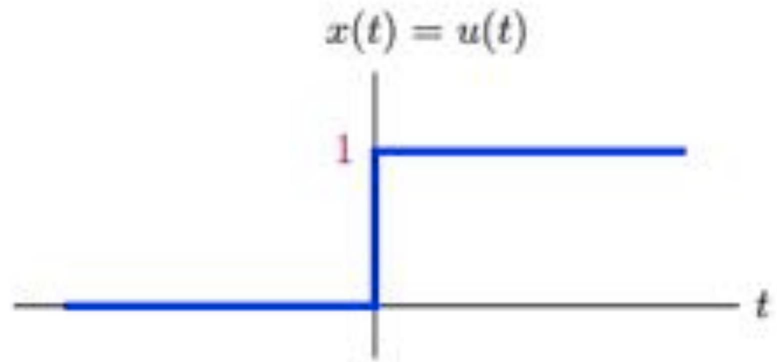
Note:

- ✧ This result is only applicable to a first-order differential equation.
- ✧ It is also useful for working with higher order system through the use of ***state-space*** models.

2.5.1 Solution of the first-order differential equation

Example 2.8 Unit-step response of the simple RC circuit

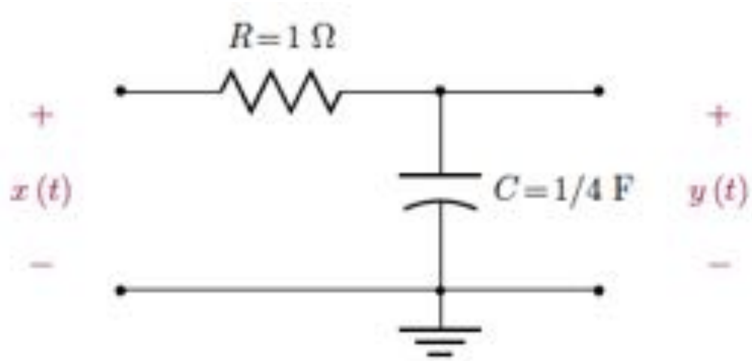


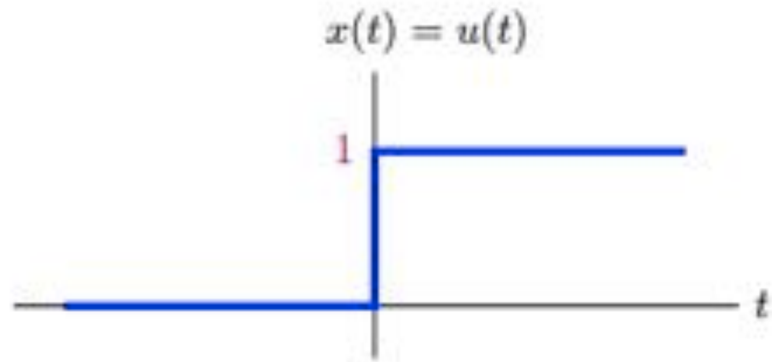


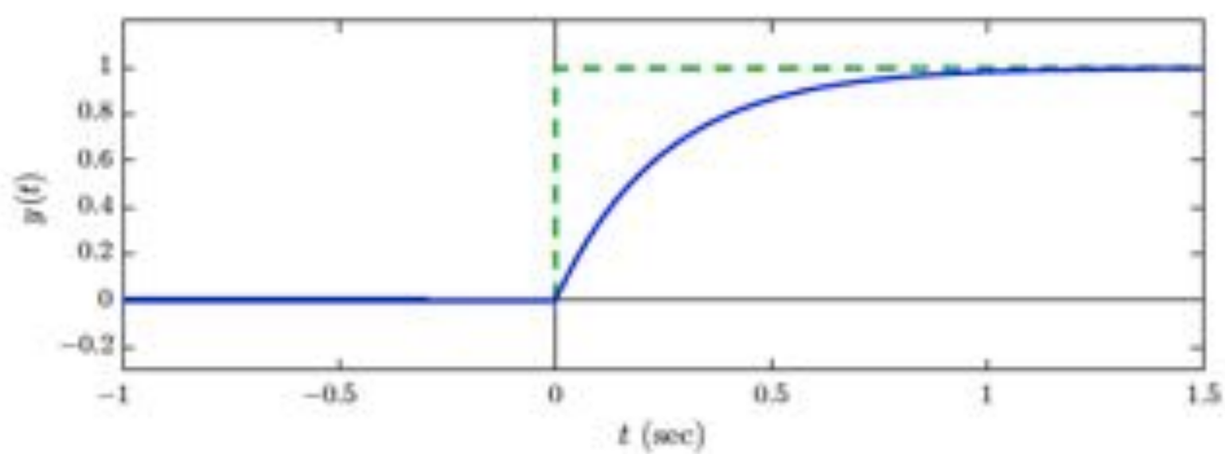
Consider the simple RC circuit as shown above. Let the element value be R = 1Ω, and C = 1/4F. Assume the initial value of the output at time = 0 is $y(0) = 0$. Determine the response of the system to an input signal in the form of a unit-step function, i.e. $x(t) = u(t)$.

2.5.1 Solution of the first-order differential equation

<u>Example 2.8 Unit-step response of the simple RC circuit</u>

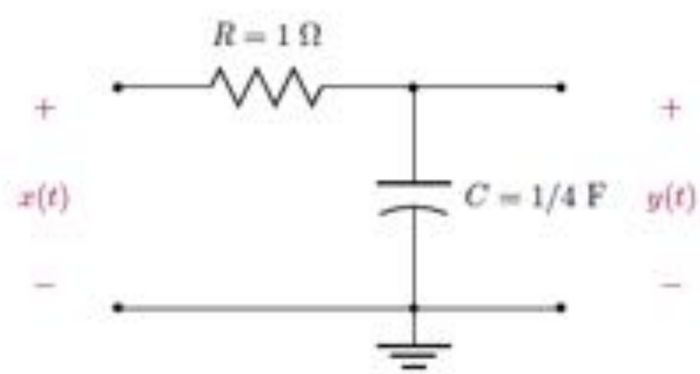


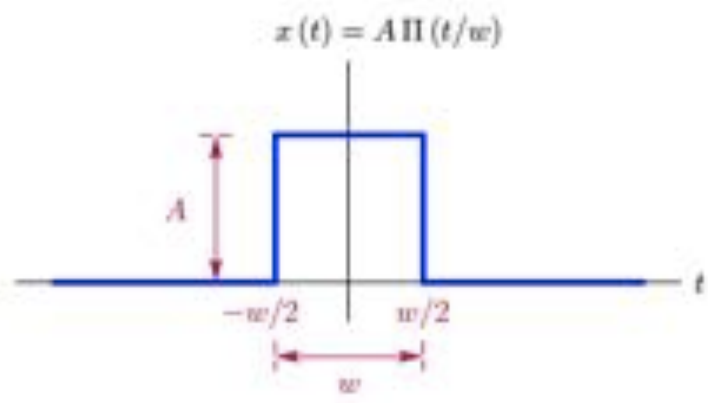




2.5.1 Solution of the first-order differential equation

Example 2.9 Pulse response of the simple RC circuit





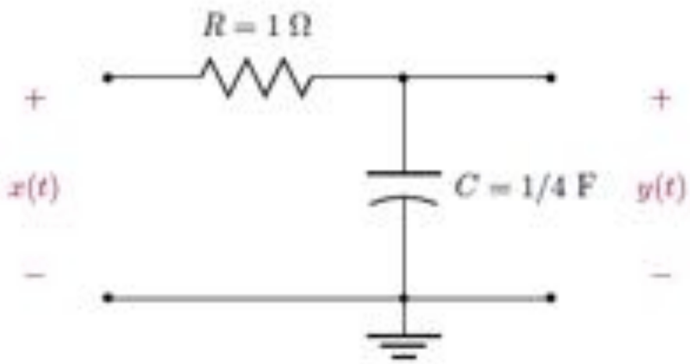
Determine the response of the RC circuit to a rectangular pulse signal

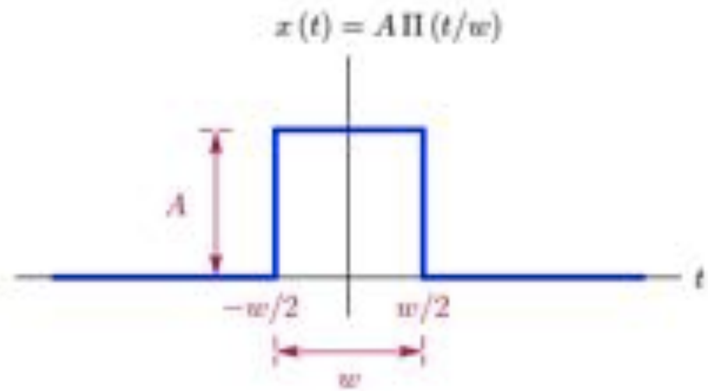
$$x(t) = A\Pi(t/w)$$

Element values for the circuit are $R = 1\Omega$, and $C = 1/4$F. The initial value of the output signal at time $t = -w/2$ is $y(-w/2) = 0$.

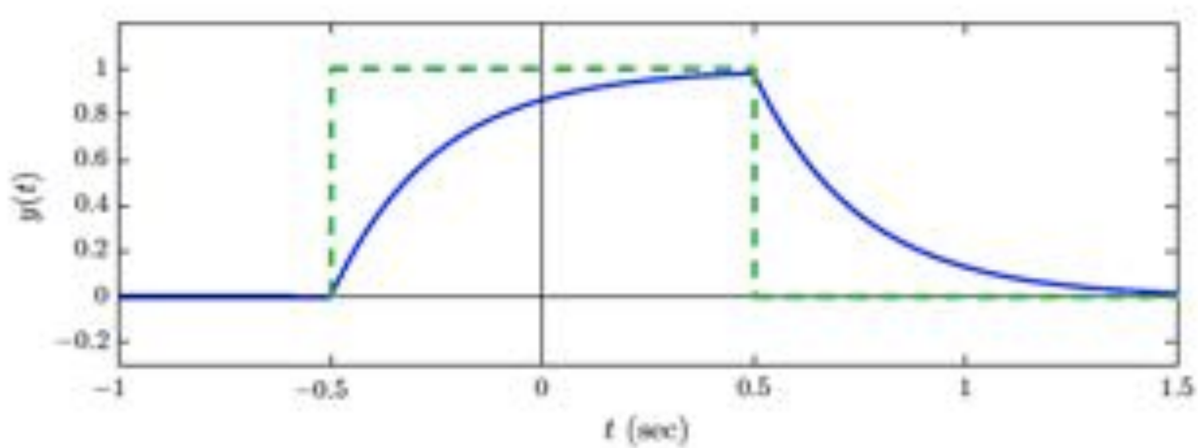
2.5.1 Solution of the first-order differential equation

Example 2.9 Pulse response of the simple RC circuit





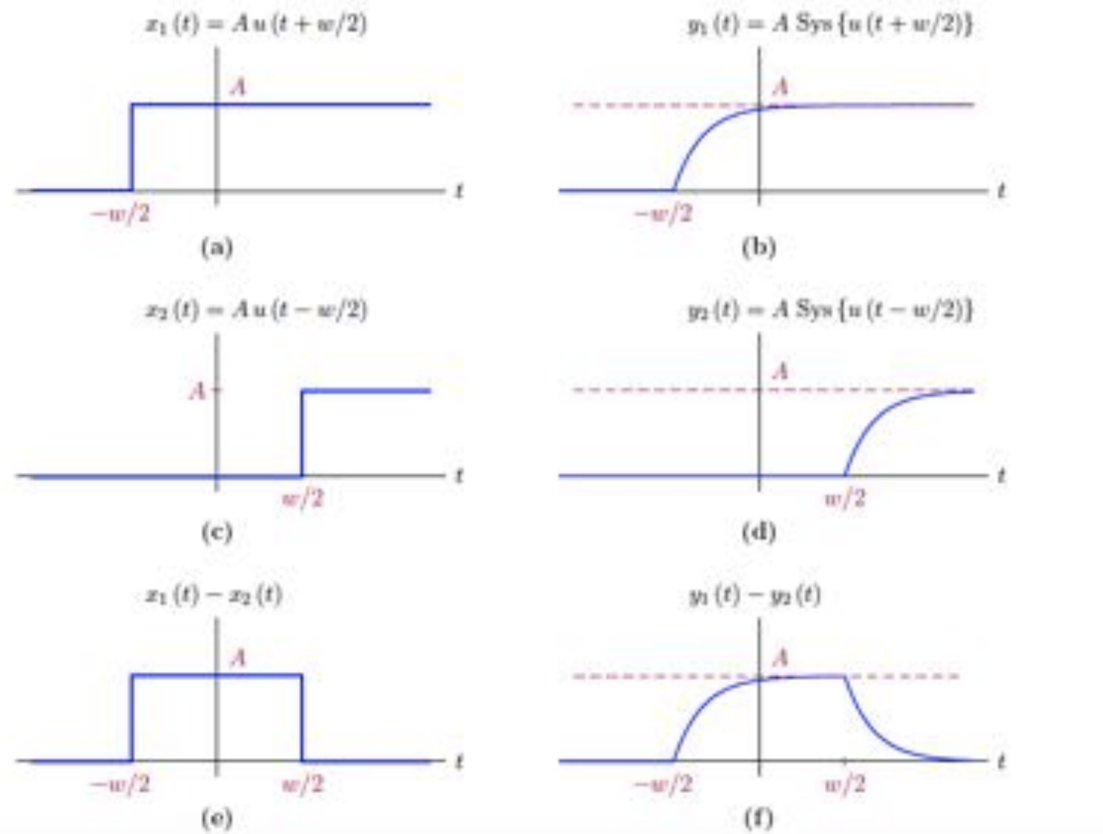
$A = 1$ and $w = 1$



2.5.1 Solution of the first-order differential equation

Example 2.10 Pulse response of the simple RC circuit

Rework the problem in Ex. 2.9 by making use of the unit-step response found in Ex. 2.8 along with the linearity and time-invariance properties of the RC circuit.



2.5.2 Solution of the general differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Initial conditions:

$$y(t_0), \quad \left.\frac{dy(t)}{dt}\right|_{t=t_0}, \ \ldots\ldots, \quad \left.\frac{d^{N-1}y(t)}{dt^{N-1}}\right|_{t=t_0}$$

General solution:

$$y(t) = y_h(t) + y_p(h)$$

- ✧ ***$y_h(t)$:*** is the ***homogeneous*** solution of the differential equation
  natural response
- ✧ ***$y_p(t)$:*** is the ***particular solution*** of the differential equation
- ✧ ***$y(t) = y_h(t) + y_p(t)$*** is the ***forced solution*** of the differential equation
  forced response

2.5.3 Finding the natural response of a continuous-time system

Homogeneous differential equation:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0$$

First-order homogeneous differential equation:

$$\frac{dy(t)}{dt} + \alpha y(t) = 0$$

Where α is any arbitrary constant

solution:

$$y(t) = ce^{-\alpha t}$$

The constant *c* must be determined based on the desired initial value $y(t)$ at $t = t_0$

2.5.3 Finding the natural response of a continuous-time system

Justification:

$$\frac{dy(t)}{dt} + \alpha y(t) = 0 \quad \Longrightarrow \quad \frac{dy(t)}{y} = -\alpha dt$$

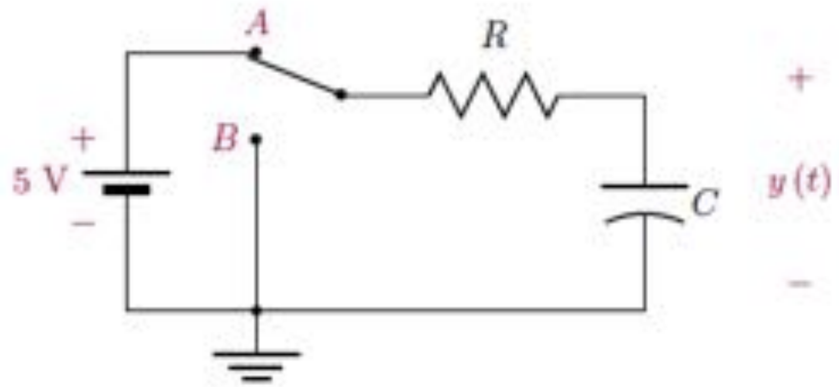
Integrating both sides leads to

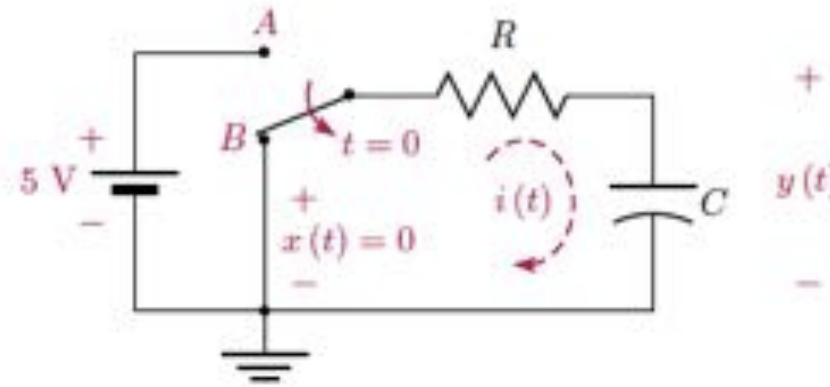
$$\int \frac{dy(t)}{y} = -\int \alpha \, dt \quad \Longrightarrow \quad \ln(y) = -\alpha t + K$$

$$\Longrightarrow \quad y(t) = e^{K} e^{-\alpha t} = c e^{-\alpha t}$$

2.5.3 Finding the natural response of a continuous-time system

Example 2.11 Natural response of the simple RC circuit

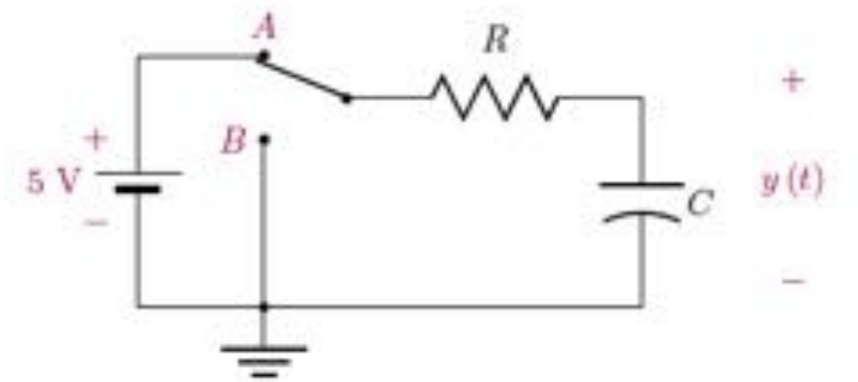


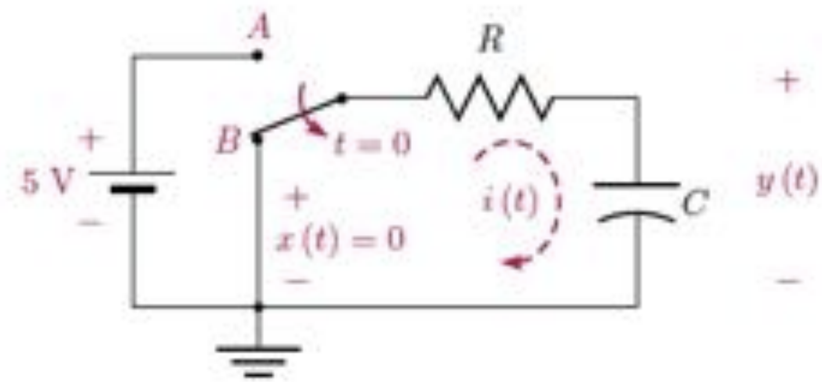


Consider the simple RC circuit as shown above. Let the element value be R = 1Ω, and C = 1/4F. Let the input terminals of the circuit be connected to a battery that supplies the circuit with an input voltage of 5 V up to the time instant t = 0. Assuming the battery has been connected to the circuit for a lot time before t = 0, the capacitor voltage has remained at a steady-state value of 5 V. Let the switch be moved from position A to position B at t = 0 ensuring that x(t) = 0 for t >= 0. The initial value of capacitor voltage at time t = 0 is y(0) = 5 V. Find the output signal as a function of time.

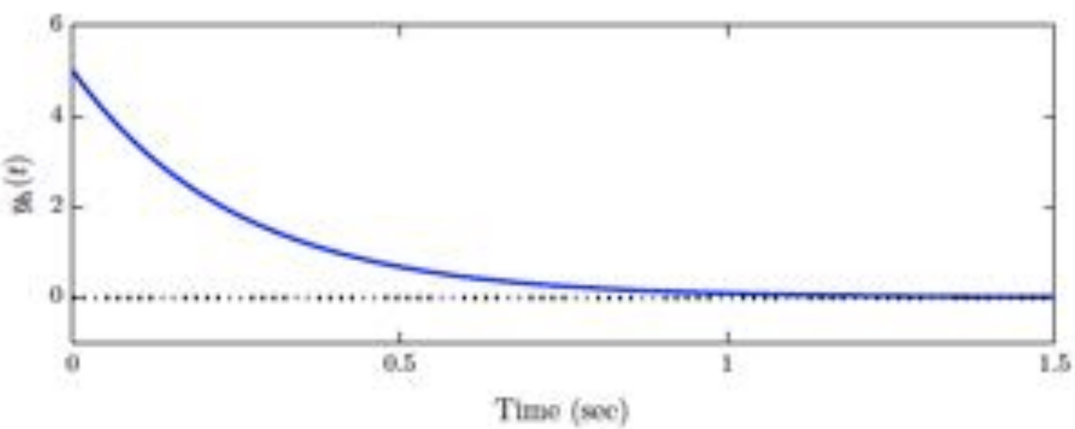
2.5.3 Finding the natural response of a continuous-time system

Example 2.11 Natural response of the simple RC circuit



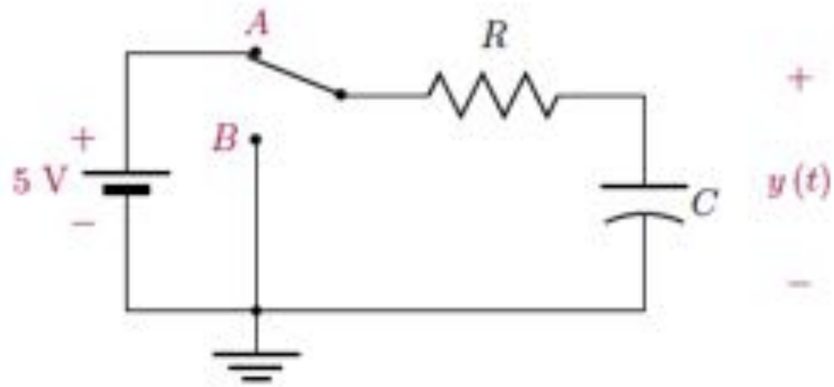


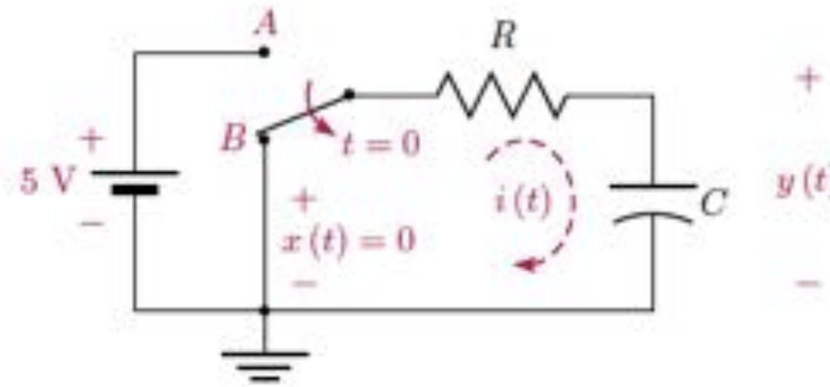
$$y_h(t) = 5\,e^{-4t}\,u(t)$$



2.5.3 Finding the natural response of a continuous-time system

Example 2.12 Natural response of the simple RC circuit





Rework Ex. 2.11 with one minor change: The initial value of the output signal is specified at the time instant t = -0.5 seconds instead of at t = 0, and its value is *y*(-0.5) = 10.

2.5.3 Finding the natural response of a continuous-time system

General homogeneous differential equation:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0$$

Characteristic equation:

$$\sum_{k=0}^{N} a_k s^k = 0$$

To obtain the characteristic equation, substitute:

$$\frac{d^k y(t)}{dt^k} \rightarrow s^k$$

2.5.3 Finding the natural response of a continuous-time system

Write the characteristic equation in open form:

$$a_N s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0 = 0$$

In factored form:

$$a_N (s - s_1)(s - s_2)\ldots(s - s_N) = 0$$

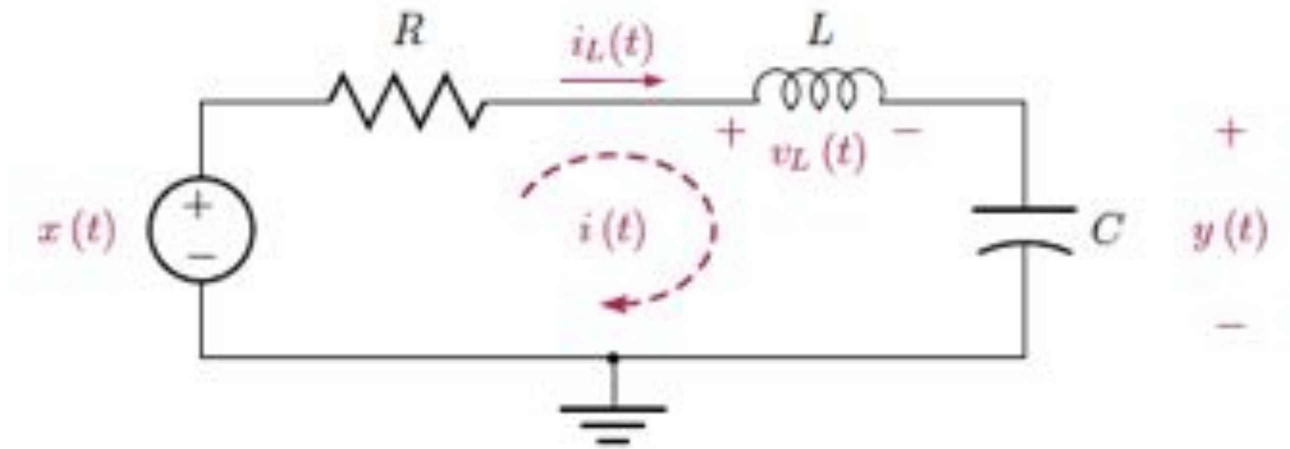
Homogeneous solution:

$$y_h(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} + \cdots + c_N e^{s_N t} = \sum_{k=1}^{N} c_k e^{s_k t}$$

Unknown coefficients $c_1, c_2, \ldots, c_N$ are determined from the ***initial conditions***. Terms $e^{s_k t}$ are called the ***modes of the system***.

2.5.3 Finding the natural response of a continuous-time system

Example 2.14 Natural response of second-order system



Let the element values be R = 5Ω, L = 1 H, and C = 1/6 F. At time = 0, the initial conductor current is i(0) = 2A and the initial capacitor voltage is y(0) = 1.5V. No external input signal is applied to the circuit. Therefore x(t)=0. Determine the output voltage y(t).

2.5.4 Finding the FORCED response of a continuous-time system

Choosing a ***particular solution*** for various input signals

Input signal	Particular solution
K (constant)	k_1
$K\, e^{at}$	$k_1\, e^{at}$
$K \cos(at)$	$k_1 \cos(at) + k_2 \sin(at)$
$K \sin(at)$	$k_1 \cos(at) + k_2 \sin(at)$
$K\, t^n$	$k_n t^n + k_{n-1} t^{n-1} + \ldots + k_1 t + k_0$

The coefficient of the ***particular solution*** are determined from the Differential equation by assuming all initial conditions are equal to zero.

2.5.4 Finding the FORCED response of a continuous-time system

Write the homogeneous differential equation

Solve homogeneous differential
Equation with undetermined coefficient

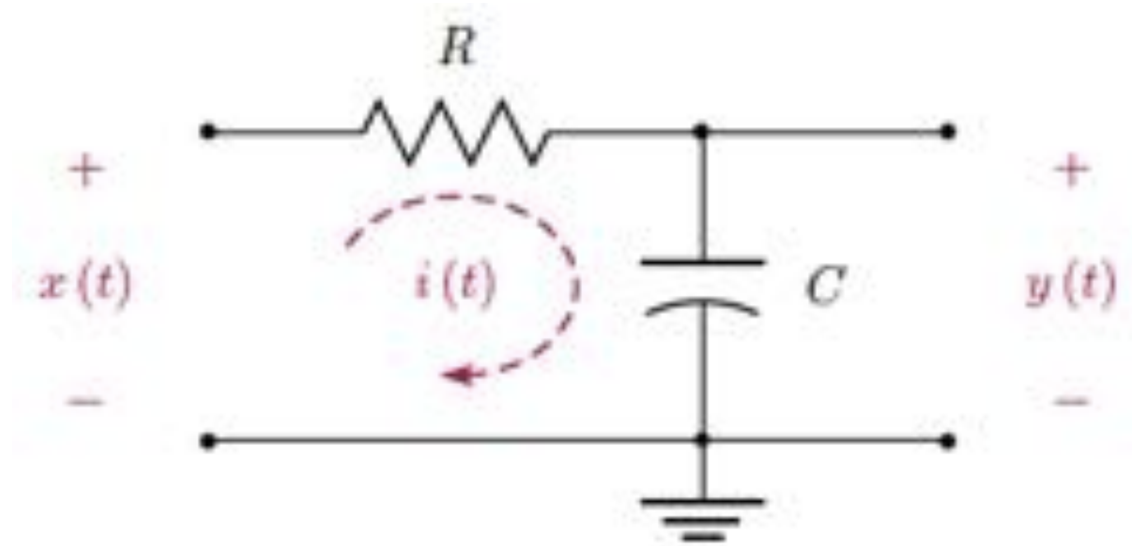
Find the form of the particular solution

Find the coefficients of the particular solution

Add the homogeneous and particular solution
Together to obtain the total solution

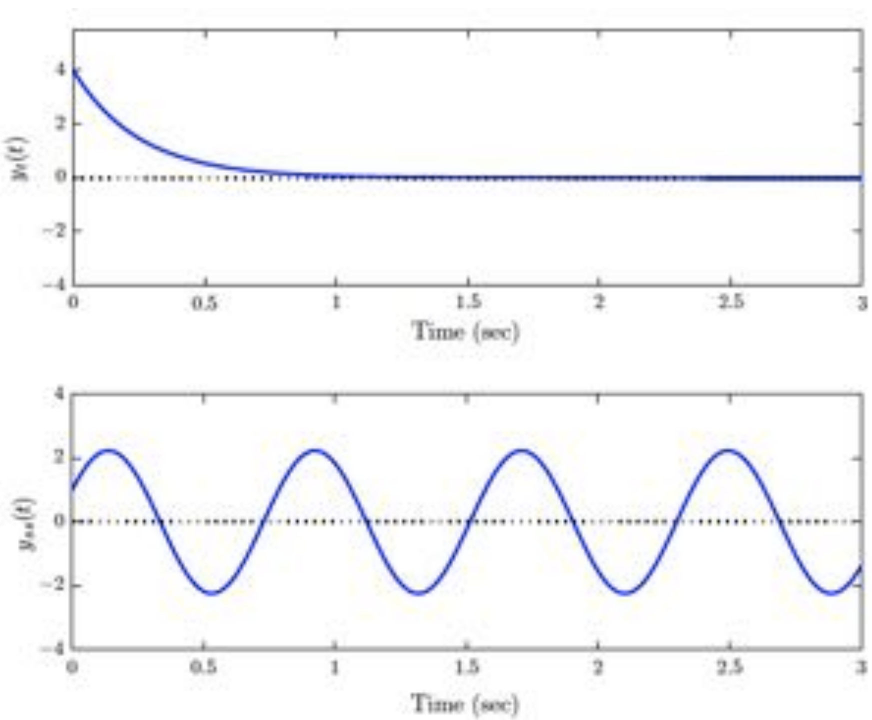
2.5.4 Finding the FORCED response of a continuous-time system

Example 2.16 Forced response of the first-order system for sinusoidal input



2.5.4 Finding the FORCED response of a continuous-time system

Example 2.16 Forced response of the first-order system for sinusoidal input



Complete solution:

$$y(t) = 4\, e^{-4t} + \cos(8t) + 2\,\sin(8t) \quad \text{for } t \geq 0$$