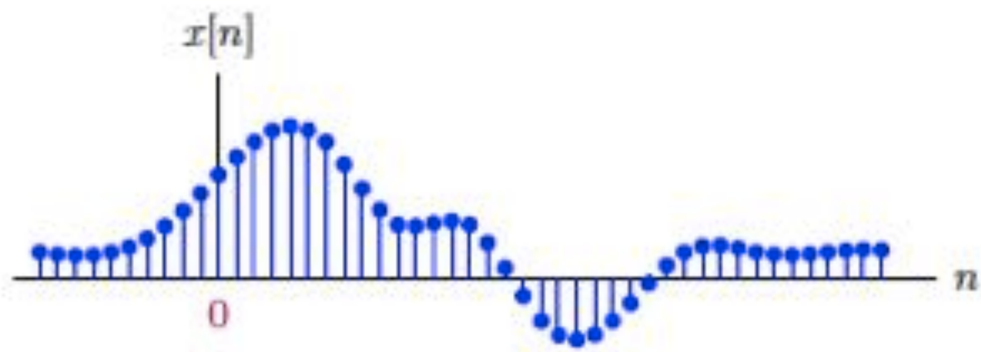
1.4 Discrete-Time Signals

Discrete-time signals are not defined at all time instants. Instead, they are defined only at time instants that are integer multiples of a fixed time increment ***T***, that is, at ***t = nT***. Consequently, the mathematical model for a discrete-time signal is a function ***x[n***] in which independent variable ***n*** is an Integer, and is referred to as the ***sample index***.

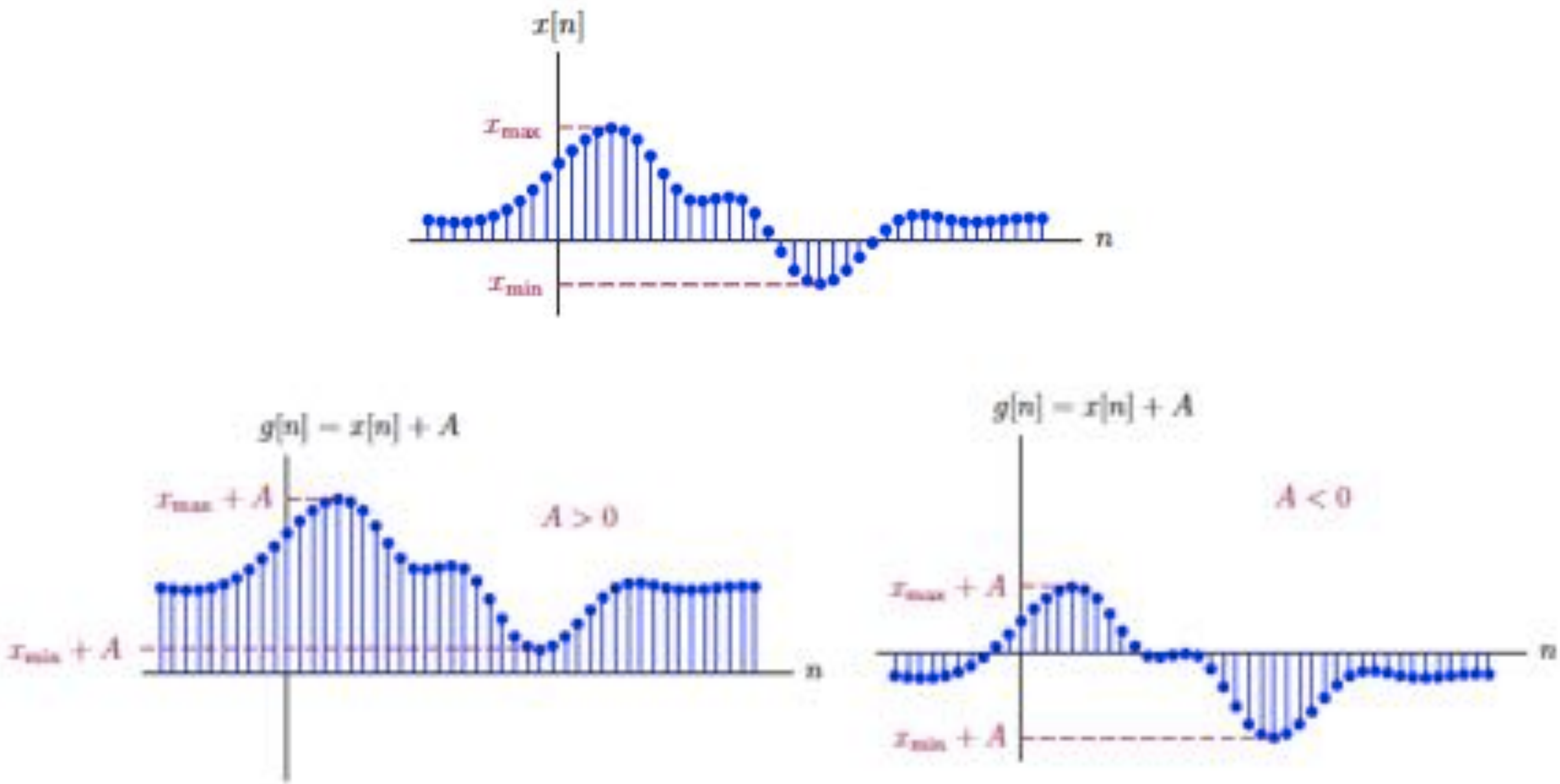


1.4.1 Signal Operations

Arithmetic Operations

Addition of a constant offset ***A*** to the signal ***x[n]***

$$g[n] = x[n] + A$$

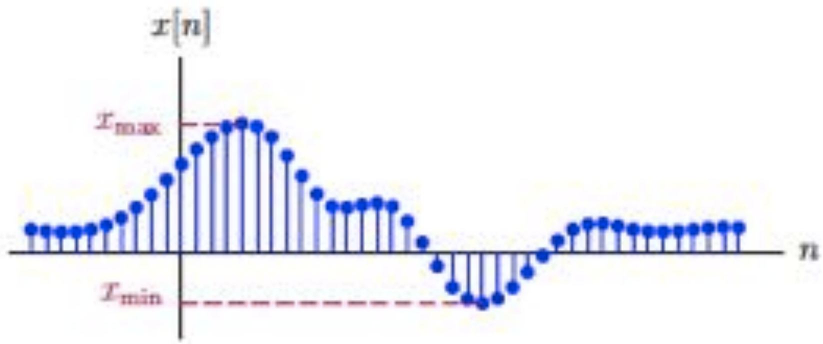


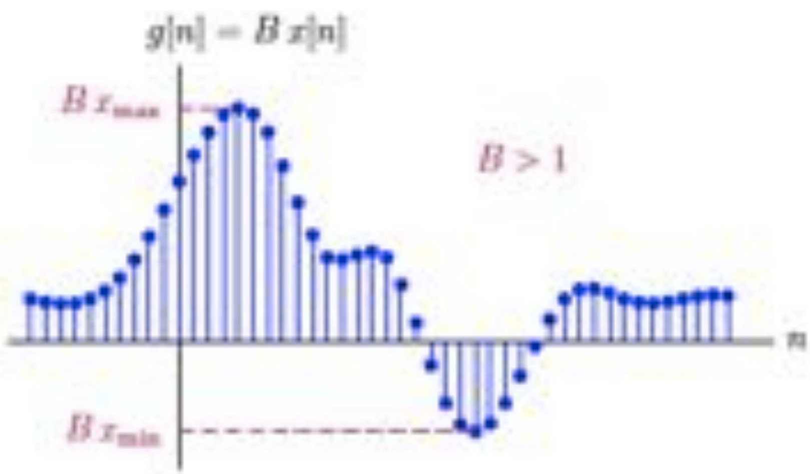
1.4.1 Signal Operations

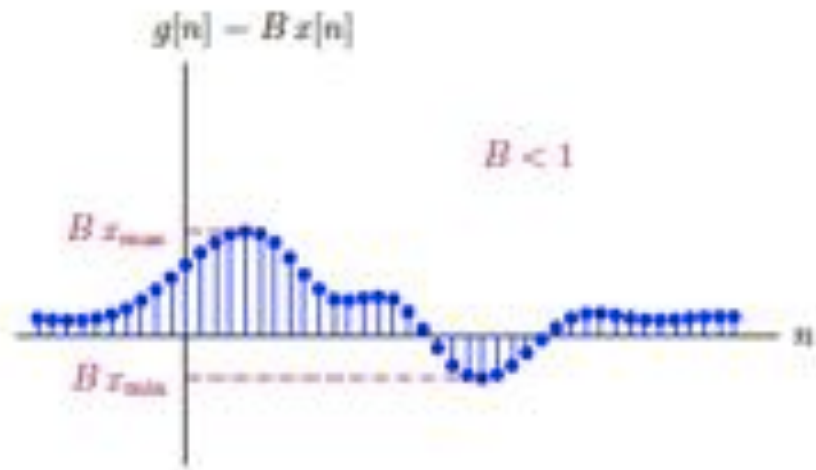
Arithmetic Operations

Multiplication of a constant gain ***B*** to the signal ***x[n]***

$$g[n] = Bx[n]$$





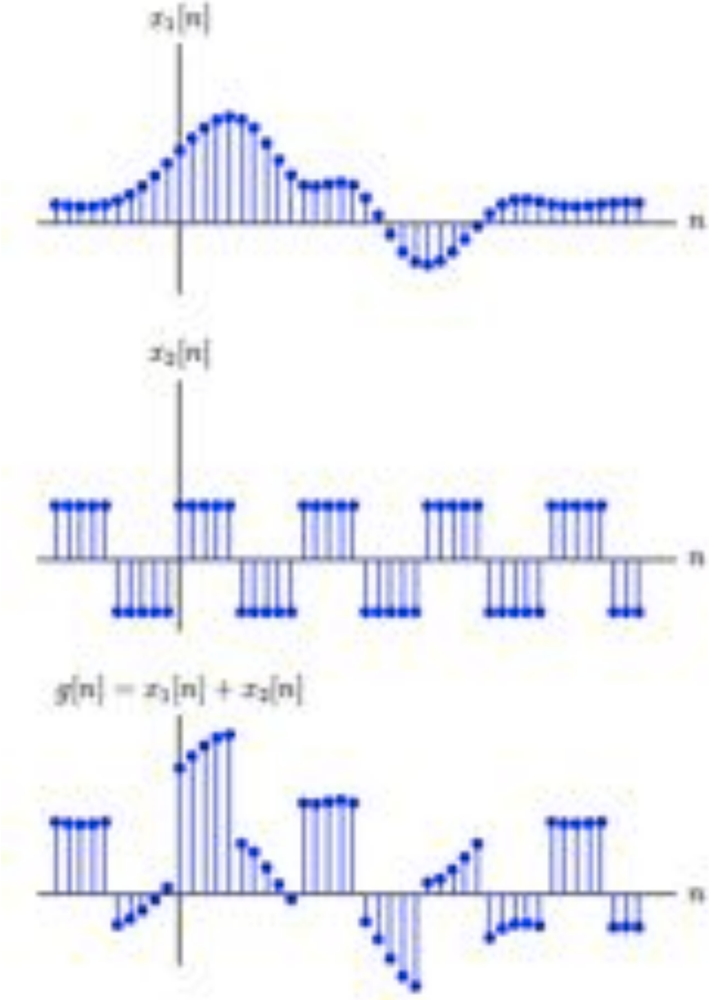


1.4.1 Signal Operations

Arithmetic Operations

Summation of two signals $x_1[n]$ and $x_2[n]$

$$g[n] = x_1[n] + x_2[n]$$

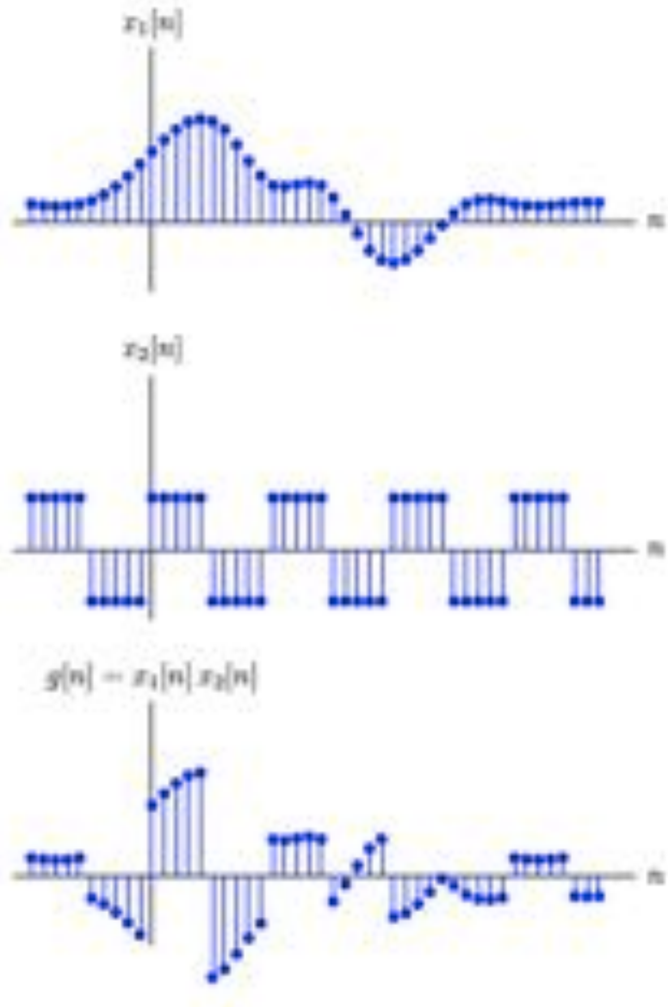


1.4.1 Signal Operations

Arithmetic Operations

Multiplication of two signals $x_1[n]$ and $x_2[n]$

$$g[n] = x_1[n]\, x_2[n]$$



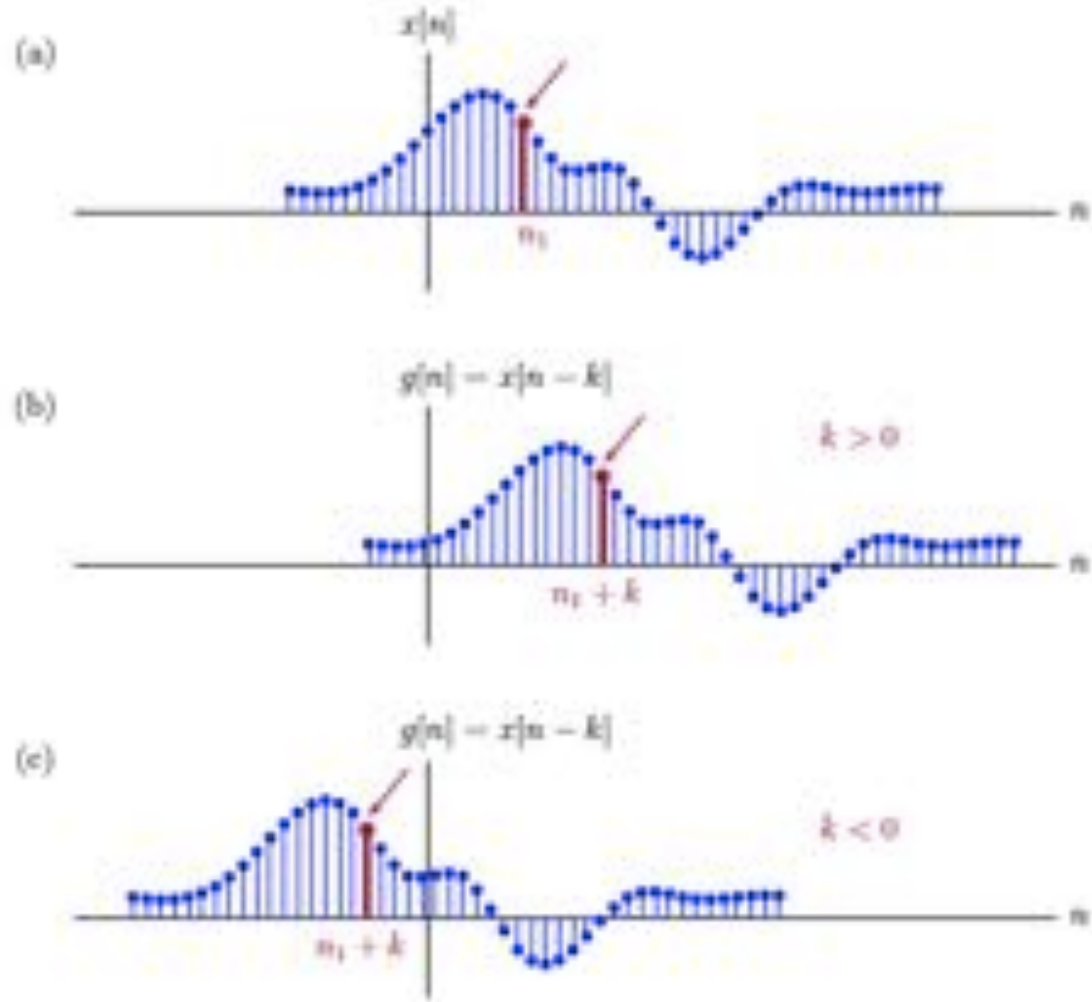
1.4.1 Signal Operations

Time shifting

A ***time shifted*** version of the signal ***x[n]*** can be obtained through

g[n] = x[n-k]

k: **integer**



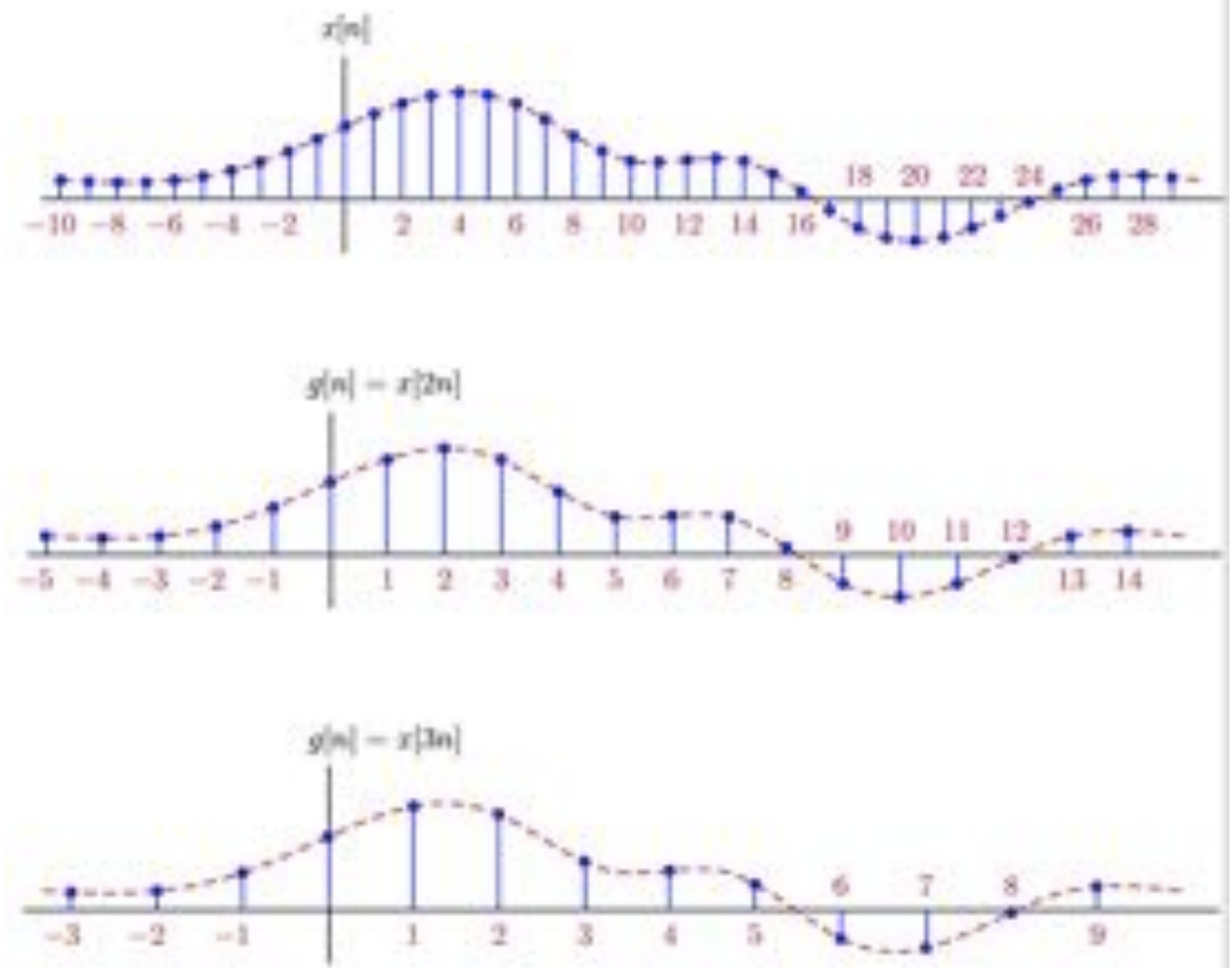
1.4.1 Signal Operations

Time scaling

A ***time scaling*** version of the signal x[n] can be obtained through

g[n] = x[kn]

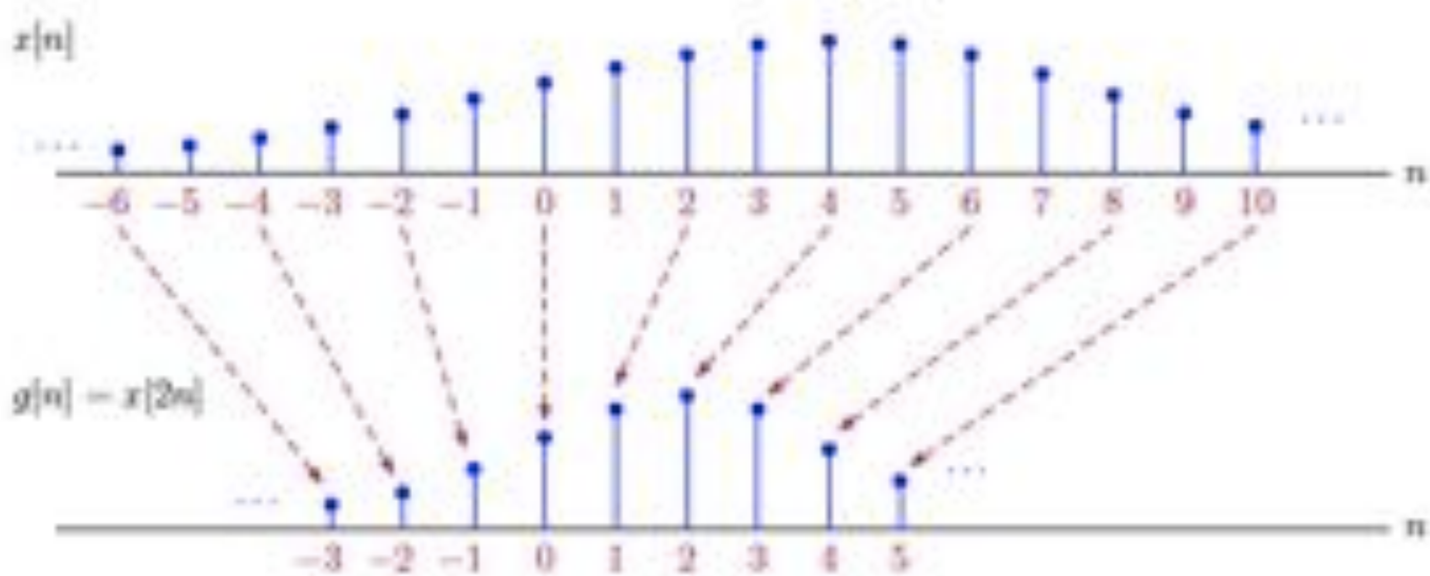
k: **integer**



1.4.1 Signal Operations

Time scaling (downsampling)

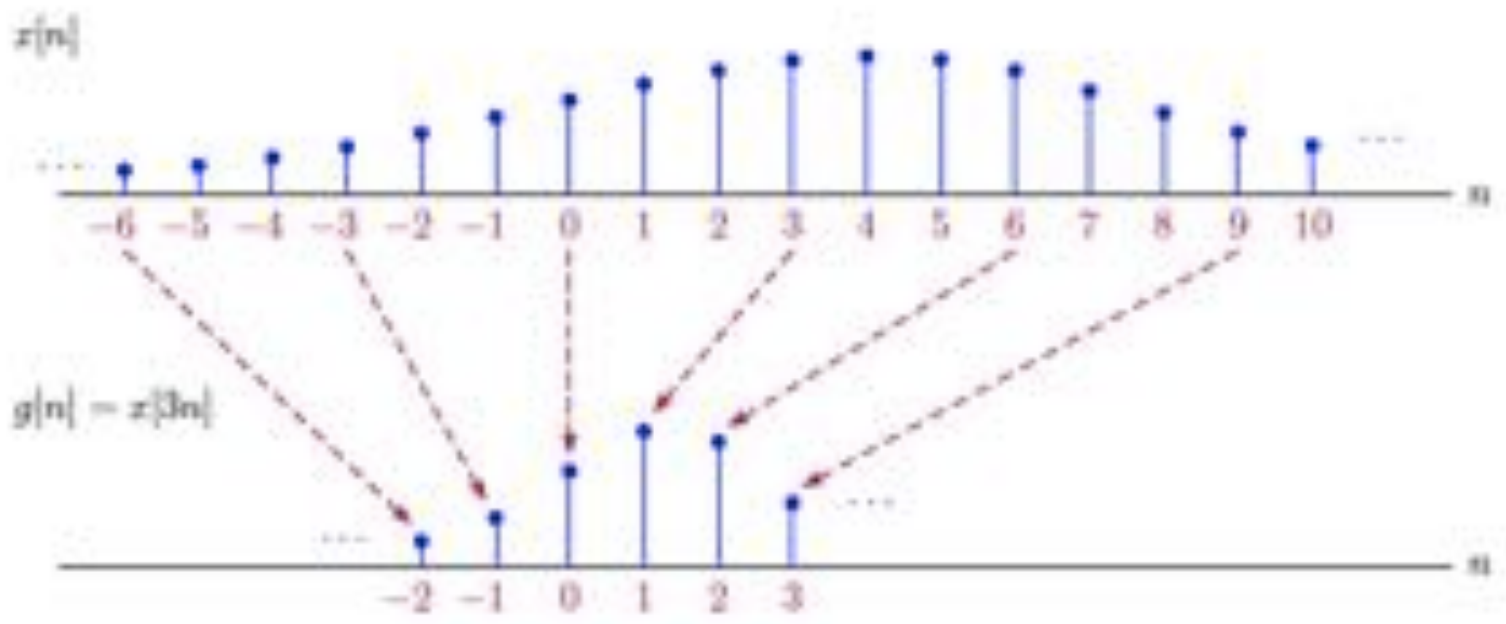
g[n] = x[2n]



1.4.1 Signal Operations

Time scaling (downsampling)

g[n] = x[3n]

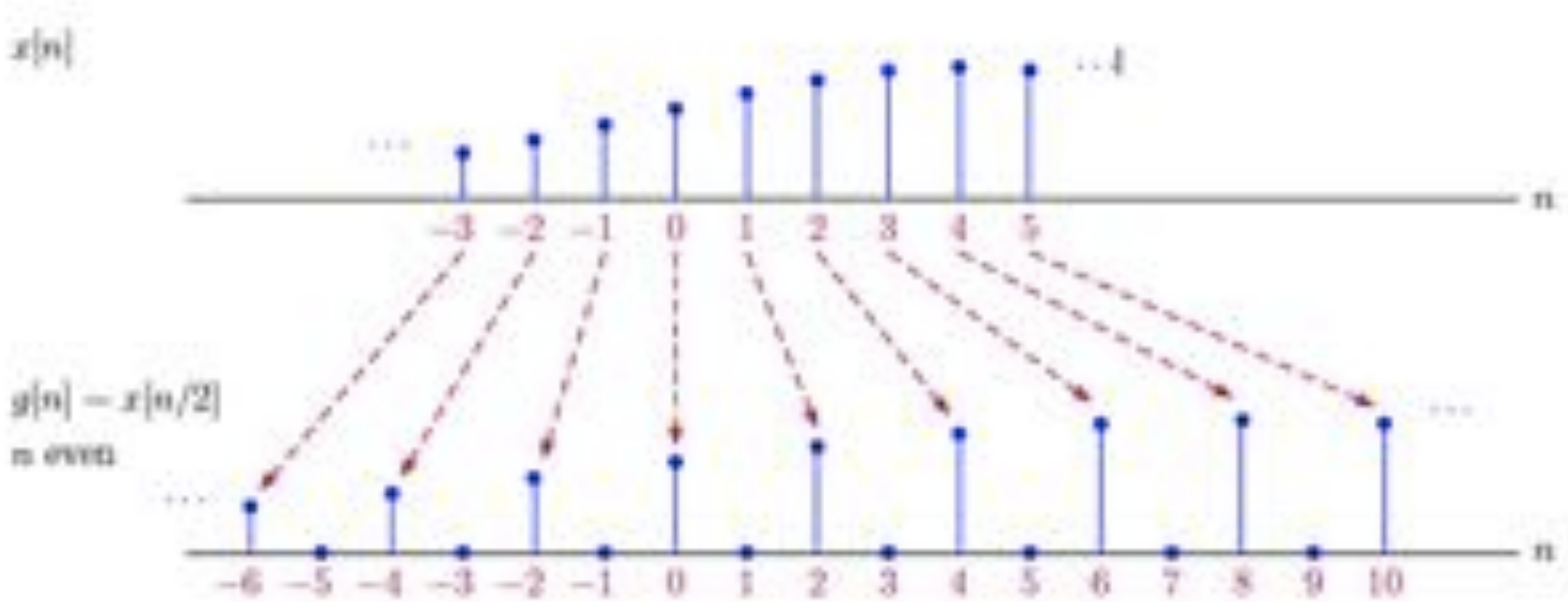


1.4.1 Signal Operations

Time scaling (upsampling)

g[n] = x[n/2] (*how do we handle odd value of n?*)

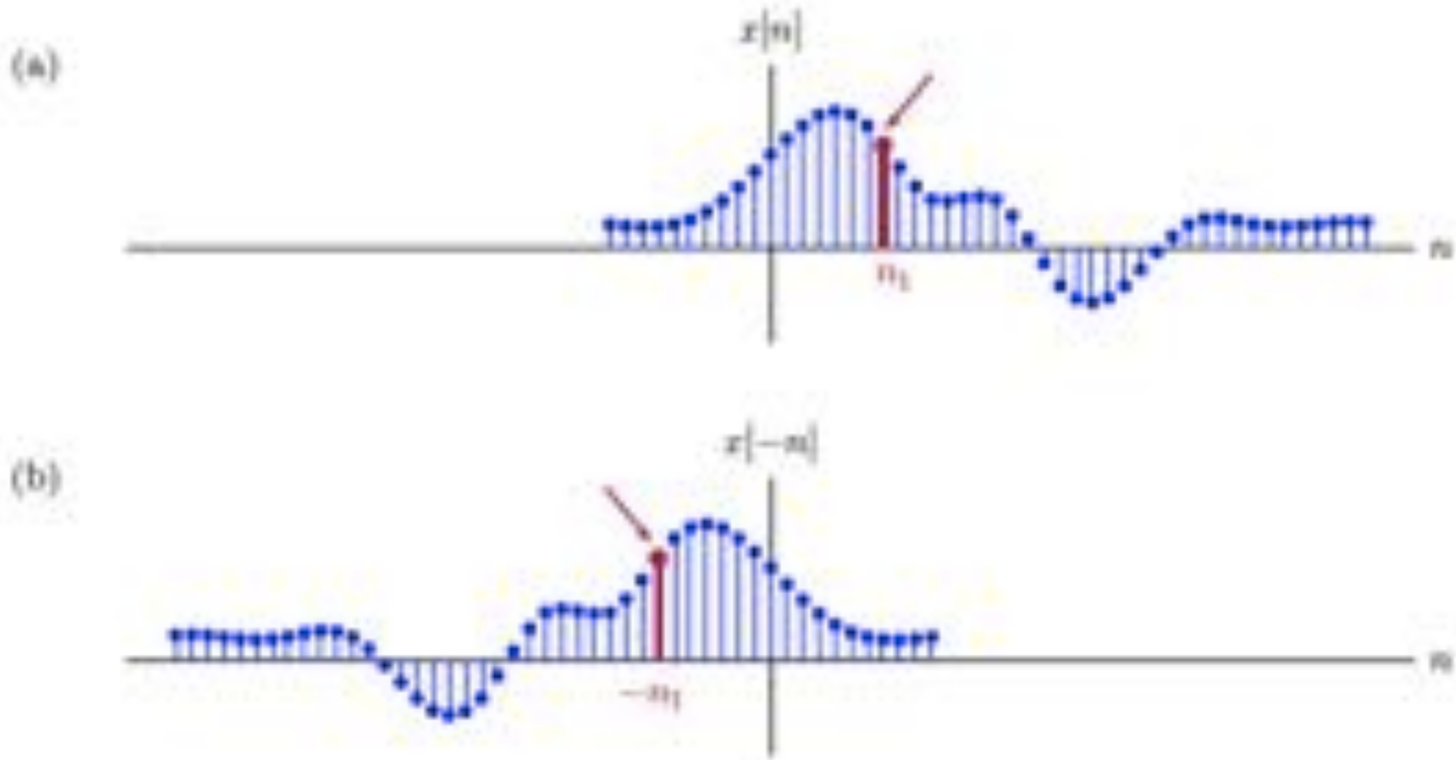
$$g[n] = \begin{cases} x[n/2], & \text{if n/2 is integer} \\ 0, & \text{otherwise} \end{cases}$$



1.4.1 Signal Operations

Time reversal

A ***time reversal*** version of the signal *x*[n] can be obtained through



1.4.2 Basic Building Blocks for discrete-time Signals

Basic building blocks

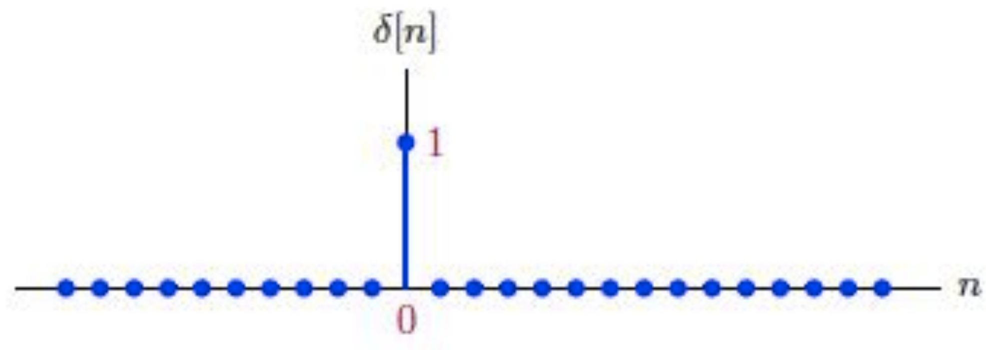
- ✧ Unit-impulse function
- ✧ Unit-step function
- ✧ Unit-ramp function
- ✧ Sinusoidal signals

1.4.2 Basic Building Blocks for discrete-time Signals

unit-impulse function

Mathematical definition.

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

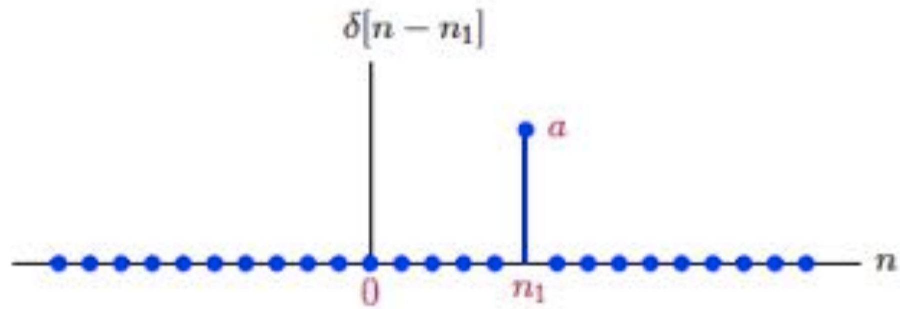


1.4.2 Basic Building Blocks for discrete-time Signals

unit-impulse function

Scaling and time shifting

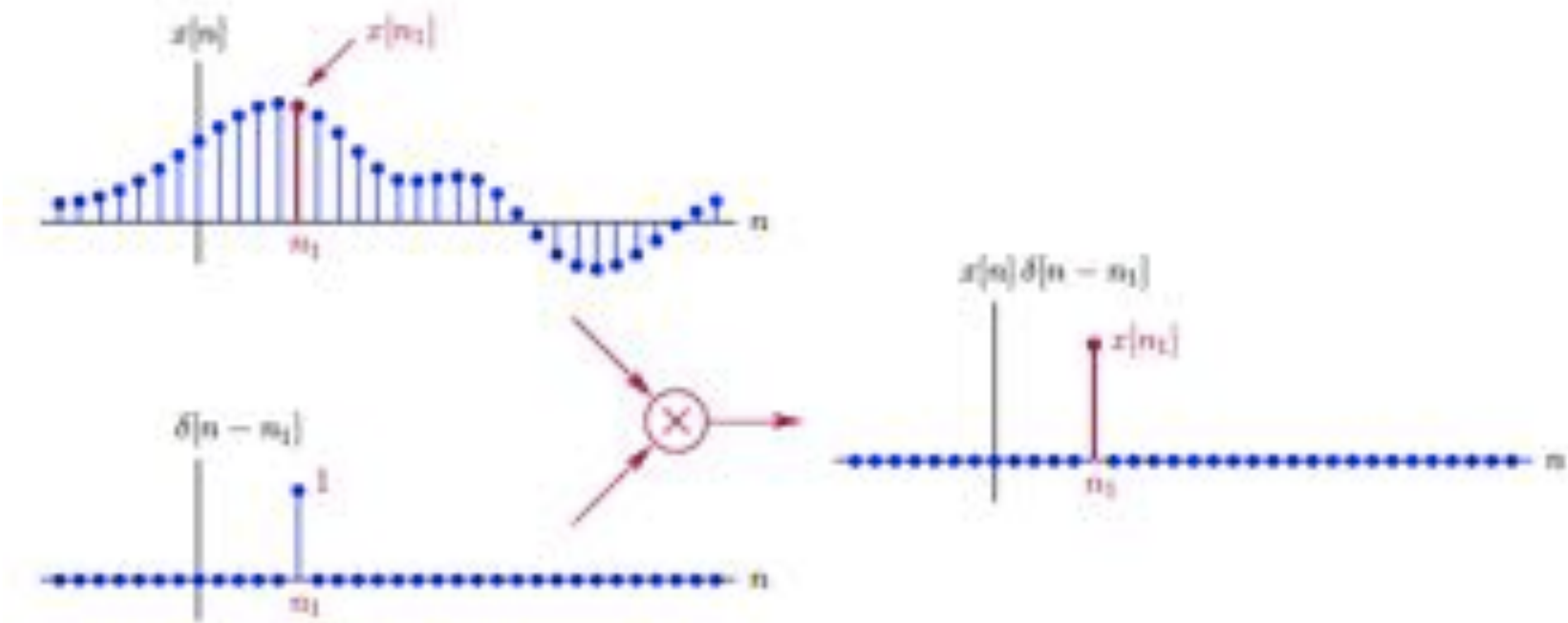
$$a\delta[n-n_1] = \begin{cases} a, & n = n_1 \\ 0, & n \neq n_1 \end{cases}$$



1.4.2 Basic Building Blocks for discrete-time Signals

Sampling property of the unit-impulse function

$$x[n]\,\delta[n-n_1] = x[n_1]\,\delta[n-n_1]$$



$$x[n]\,\delta[n-n_1] = \begin{cases} x[n_1], & n = n_1 \\ 0, & n \neq n_1 \end{cases}$$

1.4.2 Basic Building Blocks for Discrete-time Signals

Sifting property of the unit-impulse function

$$\sum_{n=-\infty}^{\infty} x[n]\,\delta[n-n_1] = x[n_1]$$

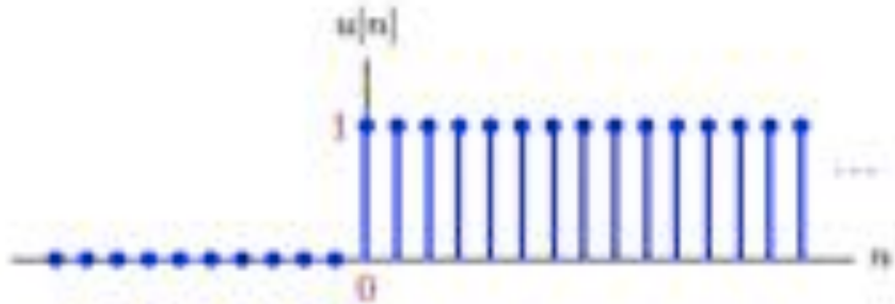
Using the sampling property:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x[n]\,\delta[n-n_1] &= \sum_{n=-\infty}^{\infty} x[n_1]\,\delta[n-n_1]\\ &= x[n_1] \sum_{n=-\infty}^{\infty} \delta[n-n_1]\\ &= x[n_1]\end{aligned}$$

1.4.2 Basic Building Blocks for Discrete-time Signals

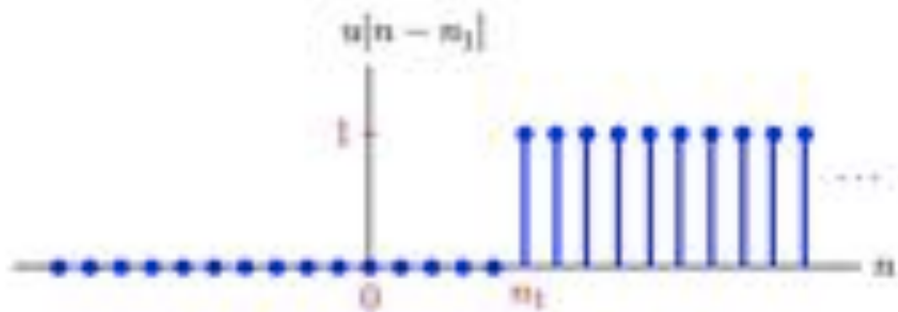
unit-step function

$$u[n] = \begin{cases} 1 & if \quad n > 0 \\ 0 & if \quad n < 0 \end{cases}$$



Time shift of the unit-step function

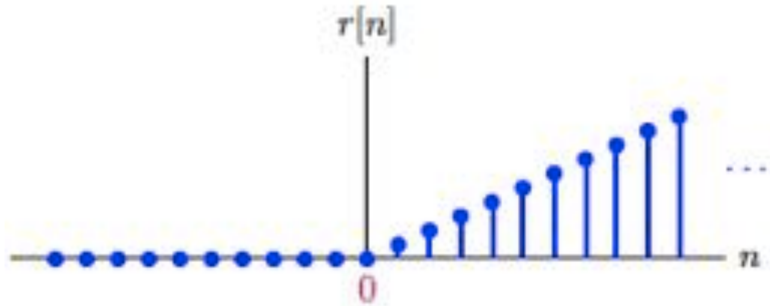
$$u[n-n_1] = \begin{cases} 1 & if \quad n > n_1 \\ 0 & if \quad n < n_1 \end{cases}$$



1.4.2 Basic Building Blocks for Discrete-time Signals

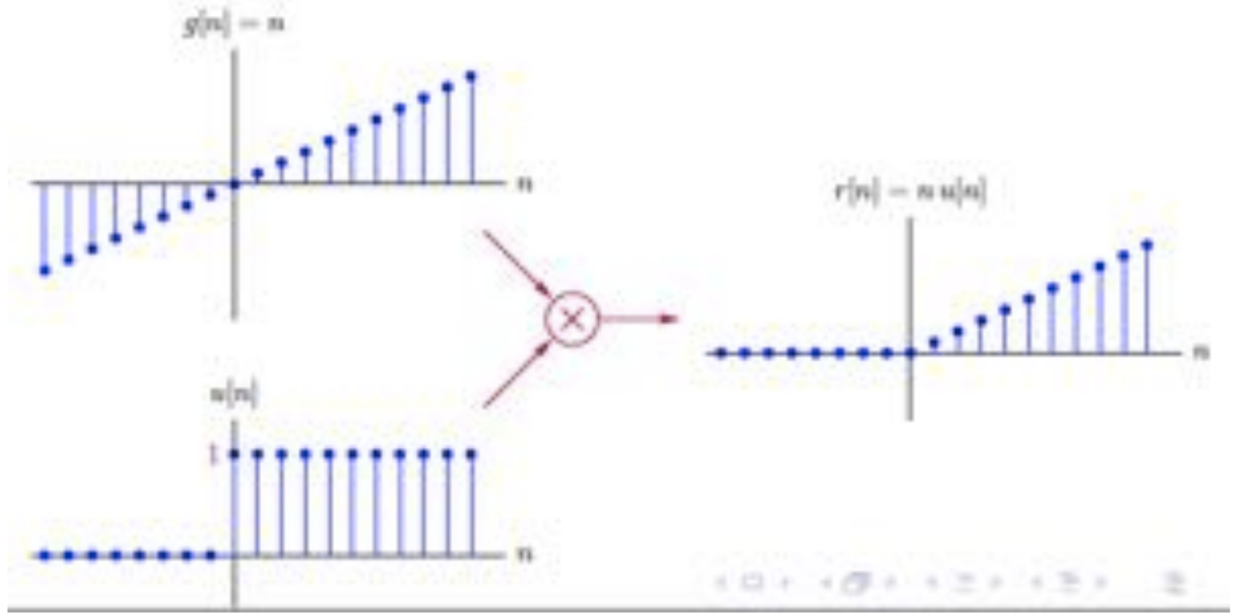
unit-ramp function

$$r[n] = \begin{cases} n, & n \ge 0 \\ 0, & n < 0 \end{cases}$$



or, equivalently

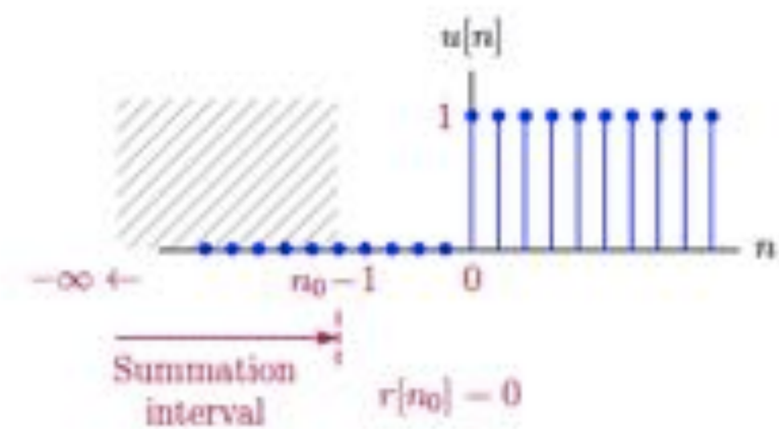
$$r[n] = nu[n]$$

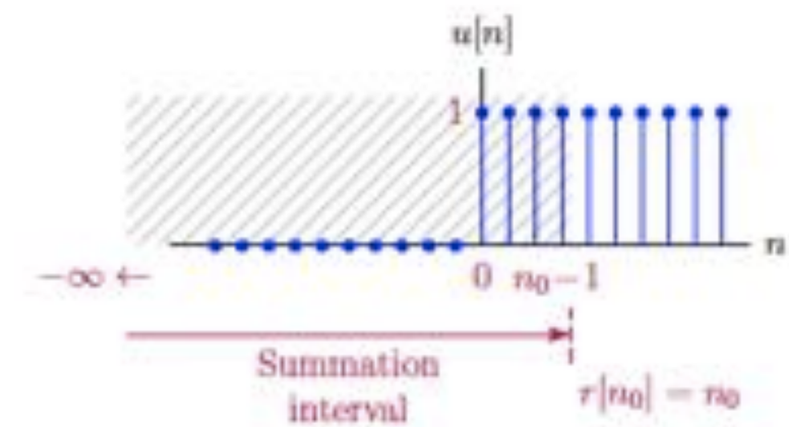


1.4.2 Basic Building Blocks for Discrete-time Signals

Constructing a unit-ramp function from a unit-step

$$r[n] = \sum_{k=-\infty}^{\infty} u[k]$$



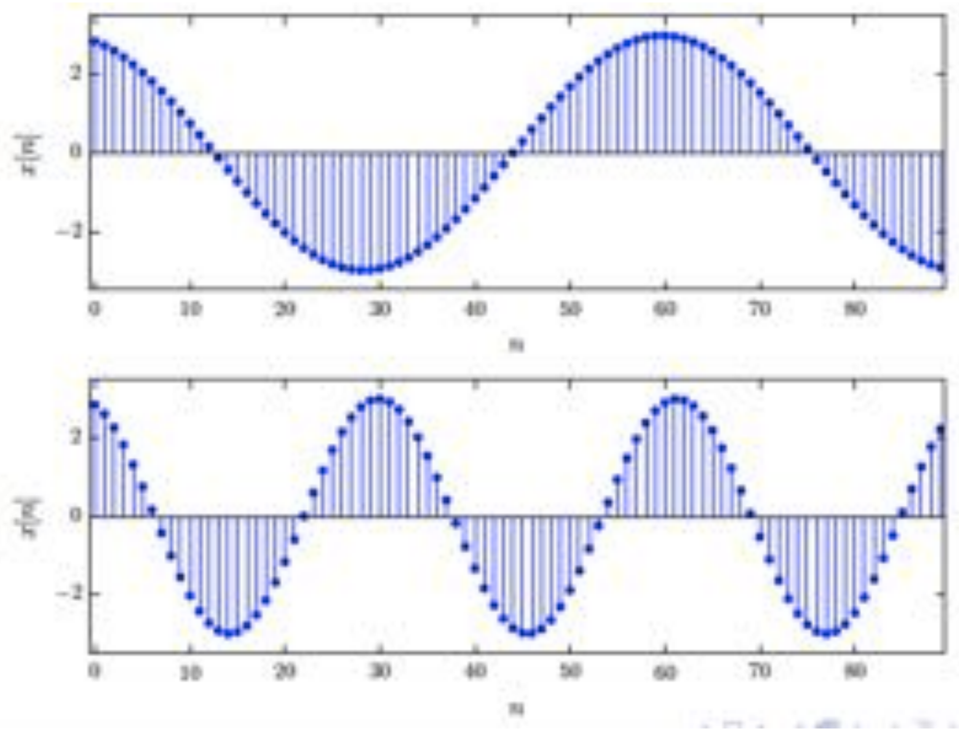


1.4.2 Basic Building Blocks for Discrete-time Signals

Sinusoidal function

$$x[n] = A\cos(\Omega_0 n + \theta)$$

Where ***A*** is the ***amplitude*** of the signal, and Ω_0 is the ***radian frequency*** which has the unit of *radians*. The parameter θ is the ***initial phase angle*** in *radians*.



The ***A*** controls the **peak value** of the signal, and the θ affects the **peaks locations**

1.4.2 Basic Building Blocks for Discrete-time Signals

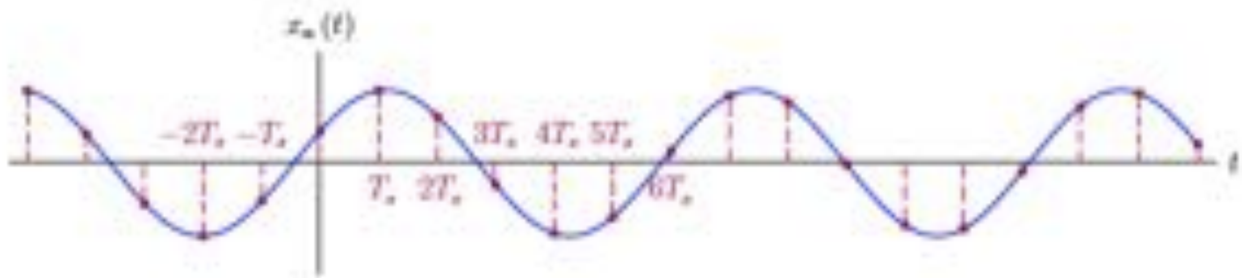
Characteristics of discrete-time sinusoids

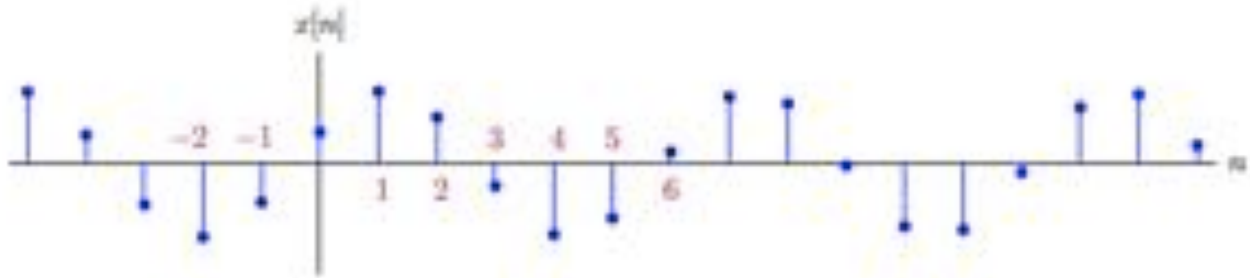
- For continuous-time sinusoidal signal $x_a(t) = A\cos(\omega_0 t)$: ω_0 is in rad/s.
- For discrete-time sinusoidal signal $x[n] = A\cos(\Omega_0 n)$: Ω_0 is in radians.

$$x_a(t) = A\cos(\omega_0 t + \theta)$$

$$\begin{aligned} x[n] &= x_a(nT_s) \\ &= A\cos(\omega_0 T_s n + \theta) \\ &= A\cos(2\pi f_0 T_s n + \theta) \end{aligned}$$

$$\begin{aligned} \Omega_0 &= \omega_0 T_s \\ F_0 &= f_0 T_s \\ \Omega_0 &= 2\pi F_0 \end{aligned}$$





1.4.3 Impulse Decomposition for Discrete-time Signals

Consider an arbitrary discrete-time signal ***x*[*n*]**. Let us define a new signal $\boldsymbol{x_k[n]}$ by using the ***k***-th sample of the signal ***x*[*n*]** in conjunction with a time shifted unit-impulse function as

$$x_k[n] = x[k]\delta[n-k] = \begin{cases} x[k], & n = k \\ 0, & n \neq k \end{cases}$$

1.4.3 Impulse Decomposition for Discrete-time Signals

For the signal ***x***[***n***]

$$x[n] = \{3.7,\ \underset{\uparrow}{1.3},\ -1.5,\ 3.4,\ 5.9\}$$

The components $\boldsymbol{x_k}[\boldsymbol{n}]$ are:

$$\begin{array}{rcllllll}
x_{-1}[n] & = & \{3.7, & \underset{\uparrow}{0}, & 0, & 0, & 0\} \\
x_0[n] & = & \{0, & \underset{\uparrow}{1.3}, & 0, & 0, & 0\} \\
x_1[n] & = & \{0, & \underset{\uparrow}{0}, & -1.5, & 0, & 0\} \\
x_2[n] & = & \{0, & \underset{\uparrow}{0}, & 0, & 3.4, & 0\} \\
x_3[n] & = & \{0, & \underset{\uparrow}{0}, & 0, & 0, & 5.9\}
\end{array}$$

The signal ***x***[***n***] can be reconstructed by adding these components together:

$$\boxed{x[n] = \sum_{k=-\infty}^{\infty} x_k[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]}$$

1.4.4 Signal Classification

Real vs. complex signals

$$x[n] = x_r[n] + jx_i[n]$$ **Cartesian form**

or

$$x[n] = |x[n]|e^{j\angle x[n]}$$ **Polar form**

$$|x[n]| = \left[x_r^2[n] + x_i^2[n]\right]^{1/2} \quad \text{and} \quad \angle x[n] = \tan^{-1}\left[\frac{x_i[n]}{x_r[n]}\right]$$

$$x_r[n] = |x[n]|\cos\left[\angle x[n]\right] \quad \text{and} \quad x_i[n] = |x[n]|\sin\left[\angle x[n]\right]$$

1.4.4 Signal Classification

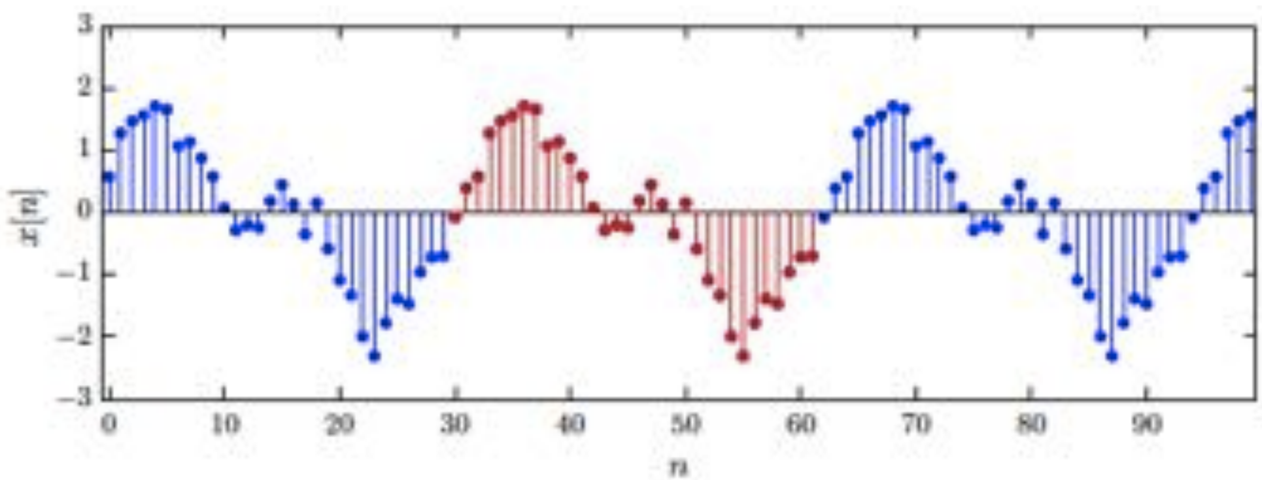
Periodic vs. non-periodic signals

A signal is said to be ***periodic*** if it satisfies

$$x[n+N] = x[n]$$

For all integer ***n***, and for a specific value of $N \neq 0$.

The ***N*** is referred as the ***period*** of the signal



If a signal is periodic with period *N*, then it is also periodic with periods of *2N*, *3N*, …., *kN*, …., where *k* is any integer

1.4.4 Signal Classification

Periodicity of discrete-time sinusoidal signals

A sinusoidal signal is said to be ***periodic*** if it satisfies:

$$A\cos(2\pi F_0 n + \theta) = A\cos(2\pi F_0[n+N] + \theta)$$

$$= A\cos(2\pi F_0 n + 2\pi F N + \theta)$$

For this equation to hold, the arguments of the cosine functions must differ by an integer multiple of *2π*. This results in

$$2\pi F_0 N = 2\pi k$$

And consequently

$$N = \frac{k}{F_0}$$

The obtained ***N*** must be an **integer value**

1.4.4 Signal Classification

Example 1.6: Periodicity of a discrete-time sinusoidal signal

1.4.4 Signal Classification

Example 1.7: Periodicity of the two-tone discrete-time signal

1.4.5 Energy and Power Definitions

Energy of a signal

The ***energy*** of a ***real-valued*** signal ***x[n]***:

$$E_x = \sum_{n=-\infty}^{\infty} x^2[n]$$

If the results of the summation can be computed

The ***energy*** of a ***complex*** signal ***x[n]***:

$$E_x = \sum_{n=-\infty}^{\infty} \left| x[n] \right|^2$$

If the results of the summation can be computed

1.4.5 Energy and Power Definitions

Time averaging operator

We use the operator **< >** to indicate time average.

- If the signal ***x[n]*** is ***periodic*** with period ***N***, its **time average** can be computed as.

$$\langle x[n] \rangle = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$$

- If the signal ***x[n]*** is ***non-periodic***, its **time average** can be computed as.

$$\langle x[n] \rangle = \lim_{M \to \infty} \left[\frac{1}{2M+1} \sum_{n=-M}^{M} x[n] \right]$$

1.3.5 Energy and Power Definitions

Power of a signal

Normalized avg. power (real signal)

$$P_x = \left\langle x^2[n] \right\rangle$$

Periodic signal:

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} x^2[n]$$

Non-periodic signal:

$$P_x = \lim_{M \to \infty} \left[\frac{1}{2M+1} \sum_{n=-M}^{M} x^2[n] \right]$$

Normalized avg. power (complex signal)

$$P_x = \left\langle |x[n]|^2 \right\rangle$$

Periodic signal:

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} \left|x[n]\right|^2$$

Non-periodic signal:

$$P_x = \lim_{M \to \infty} \left[\frac{1}{2M+1} \sum_{n=-M}^{M} \left|x[n]\right|^2 \right]$$

1.4.5 Energy and Power Definitions

Energy Signals vs. Power signals

✧ ***Energy signals*** are those that have **finite energy** and **zero power**

$$E_x < \infty \quad \text{and} \quad P_x = 0$$

✧ ***Power signals*** are those that have **finite power** and **infinite energy**

$$E_x \rightarrow \infty \quad \text{and} \quad P_x < \infty$$

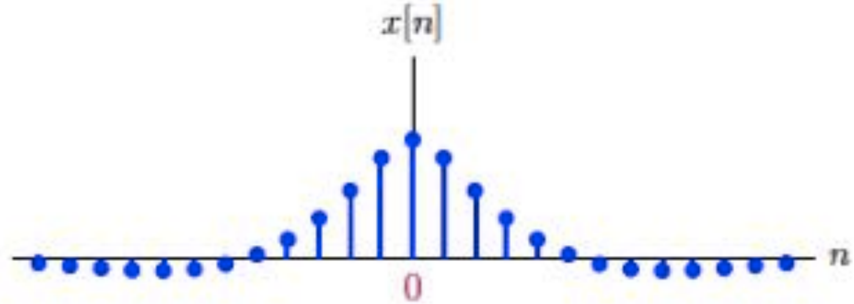
1.4.6 Symmetry Properties

even and odd symmetry

✧ A **real-value** signal is said to have **even symmetry** if it has the property

$$x[-n] = x[n]$$

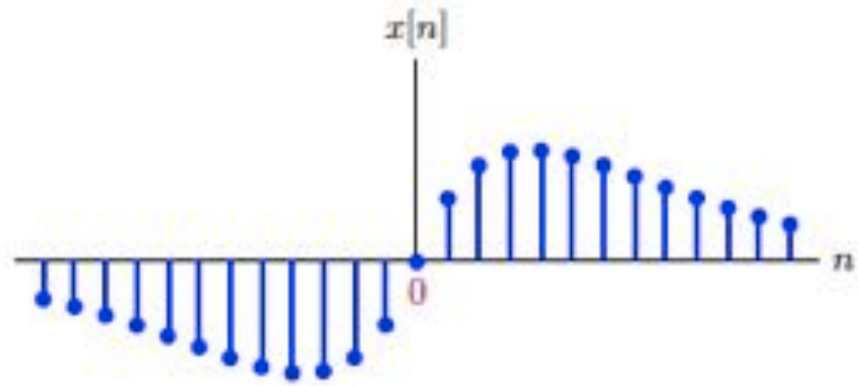
For all integer values of ***n***



✧ A **real-value** signal is said to have **odd symmetry** if it has the property

$$x[-n] = -x[n]$$

For all integer values of ***n***



1.4.6 Symmetry Properties

Decomposition into even and odd components

$$x[n] = x_e[n] + x_o[n]$$

✧ **Even component**

$$x_e[n] = \frac{x[n] + x[-n]}{2} \qquad x_e[-n] = x_e[n]$$

✧ **odd component**

$$x_o[n] = \frac{x[n] - x[-n]}{2} \qquad x_o[-n] = -x_o[n]$$

1.4.6 Symmetry Properties

Symmetry properties for *complex* signals

- A **complex-value** signal is said to have **conjugate symmetry** if it satisfies

$$x[-n] = x^*[n] \qquad \text{for all integer } n.$$

- A **complex-value** signal is said to have **conjugate antisymmetry** if it satisfies

$$x[-n] = -x^*[n] \qquad \text{for all integer } n.$$

$$x[n] = x_E[n] + x_O[n]$$

Conjugate symmetric component

$$x_E[n] = \frac{x[n] + x^*[-n]}{2}$$

Conjugate antisymmetric component

$$x_O[n] = \frac{x[n] - x^*[-n]}{2}$$