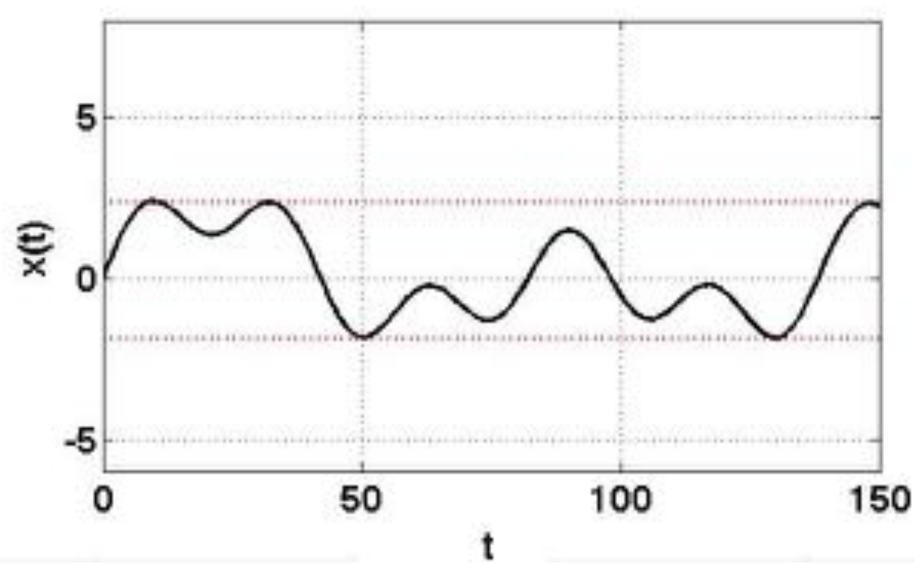
1.3 Continuous-Time Signals

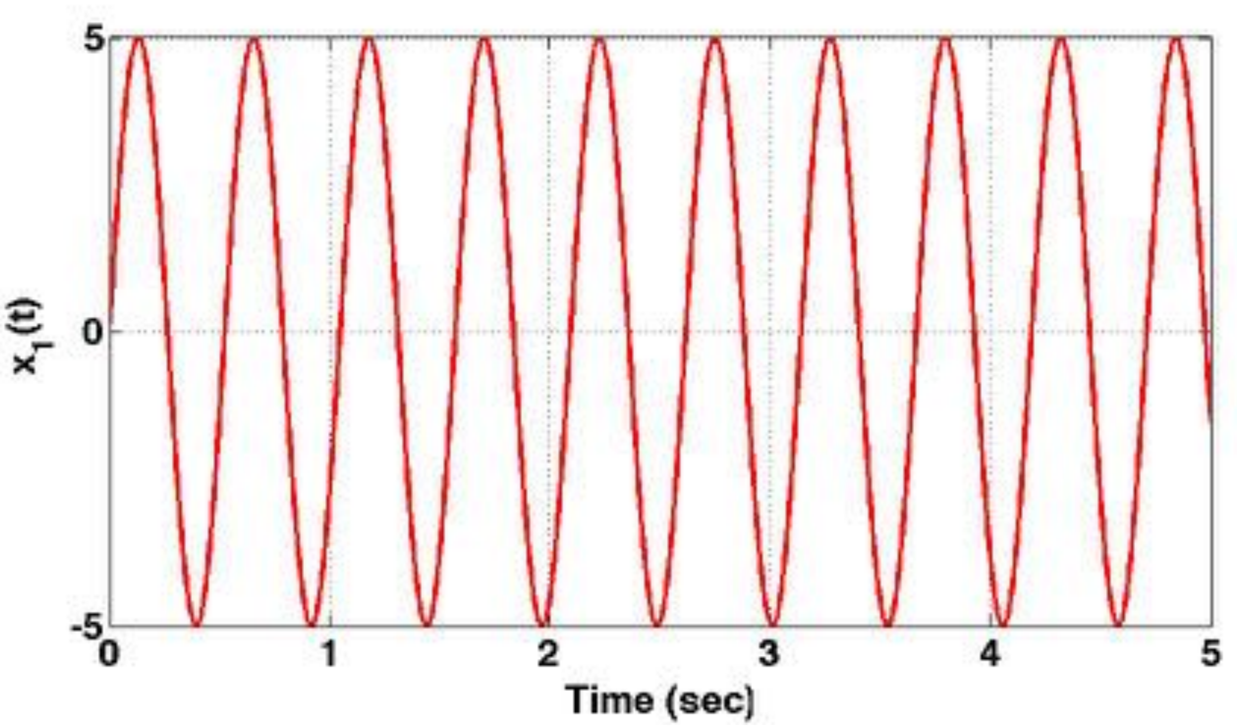
Consider ***x*(*t*)**, a mathematical function of time chosen to approximate the strength of the physical quantity at the time instant ***t***. in this relationship, ***t*** is the ***independent*** variable, and ***x*** is the ***dependent*** variable. The signal ***x(t)*** is referred to as a ***continuous-time*** signal or an ***analog*** signal.



1.3 Continuous-Time Signals

Some signals can be described ***analytically***. For example

$$x(t) = 5\sin(12t)$$



1.3 Continuous-Time Signals

Matlab code of x(t) = 5sin(12t):

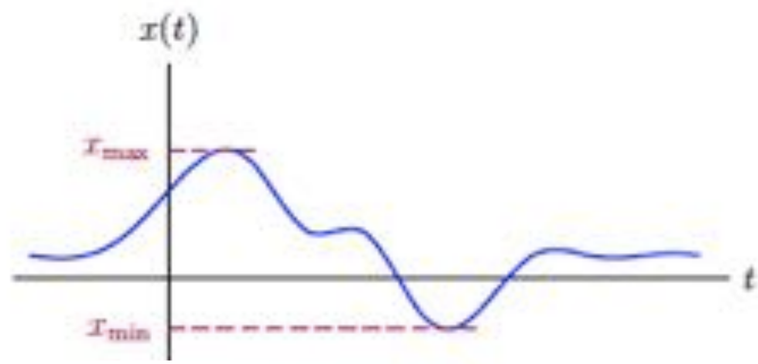
```
clear
close all
% Script : matex_1_1a
%
% Construct a vector of time instants.
t = linspace(0,5,1000);
% Compute the signal at time instants in vector "t".
x1 = 5*sin(12*t);

hh = plot(t,x1);
set(hh,'LineWidth',3,'Color','r');
hh = xlabel('Time (sec)');
set(hh,'FontSize',26,'FontWeight','bold');
hh = ylabel('x_1(t)');
set(hh,'FontSize',26,'FontWeight','bold');
set(gca,'FontSize',26,'FontWeight','bold');
grid
```

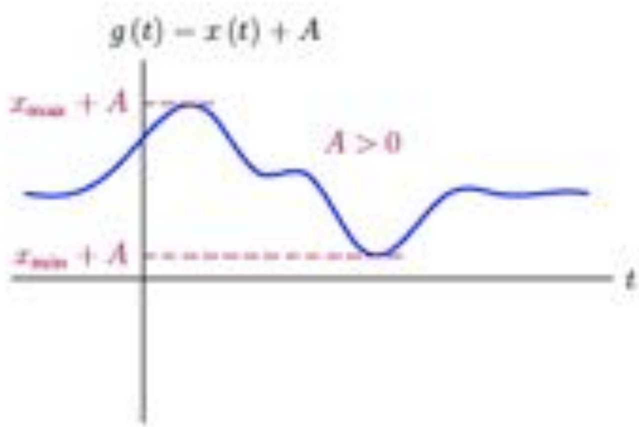
1.3.1 Signal Operations

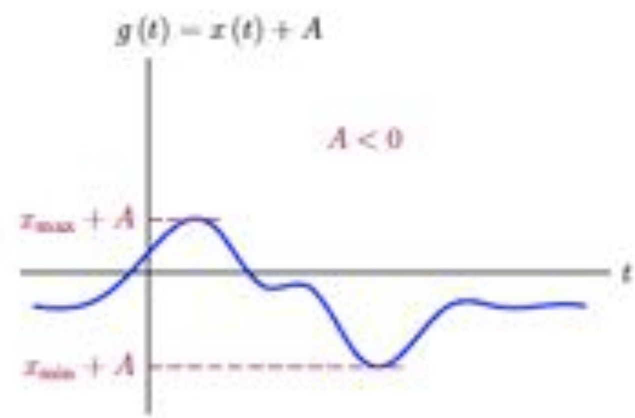
Arithmetic Operations

Addition of a constant offset ***A*** to the signal ***x(t)***

$$g(t) = x(t) + A$$





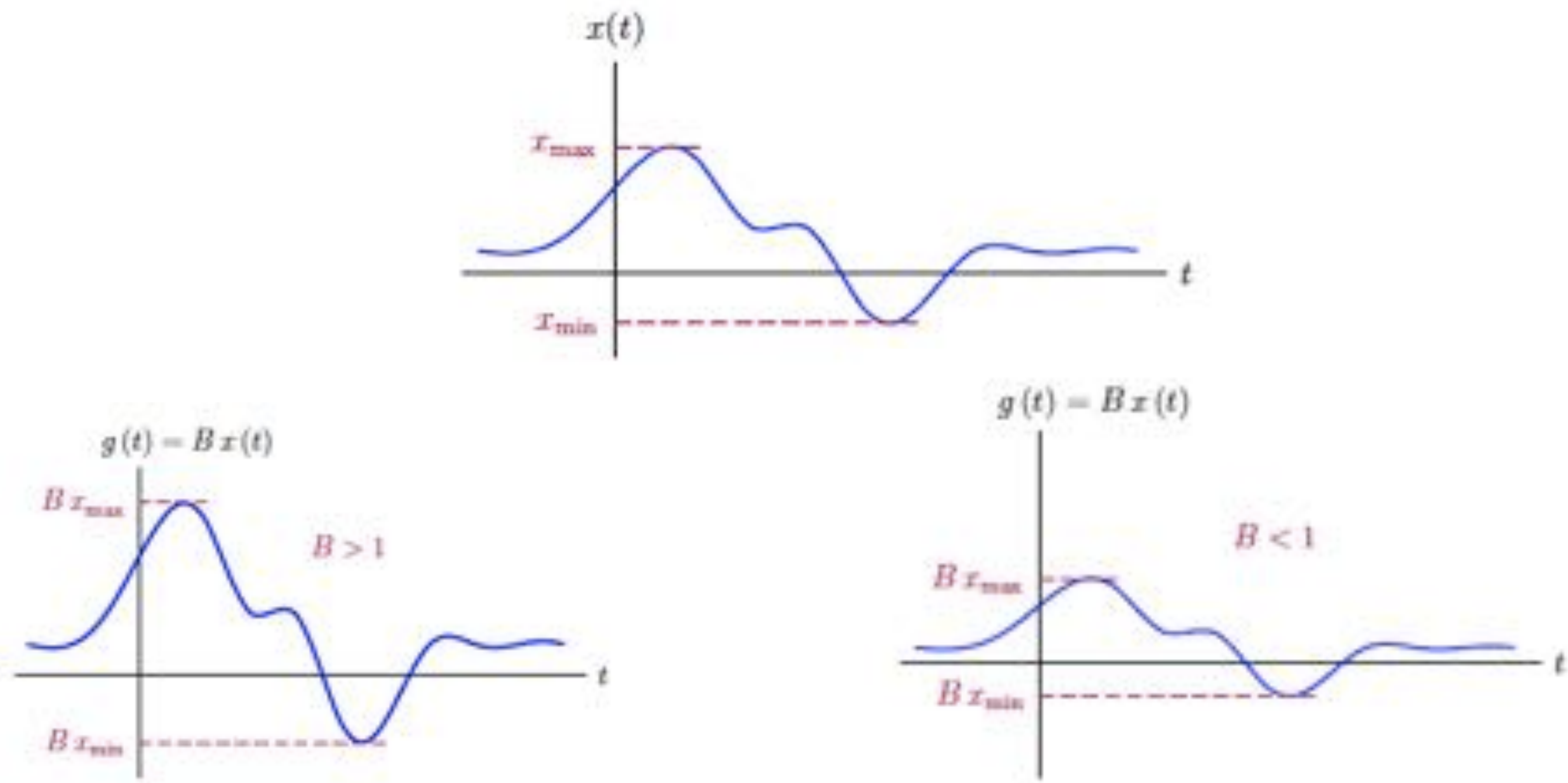


1.3.1 Signal Operations

Arithmetic Operations

Multiplication of a constant gain ***B*** to the signal ***x(t)***

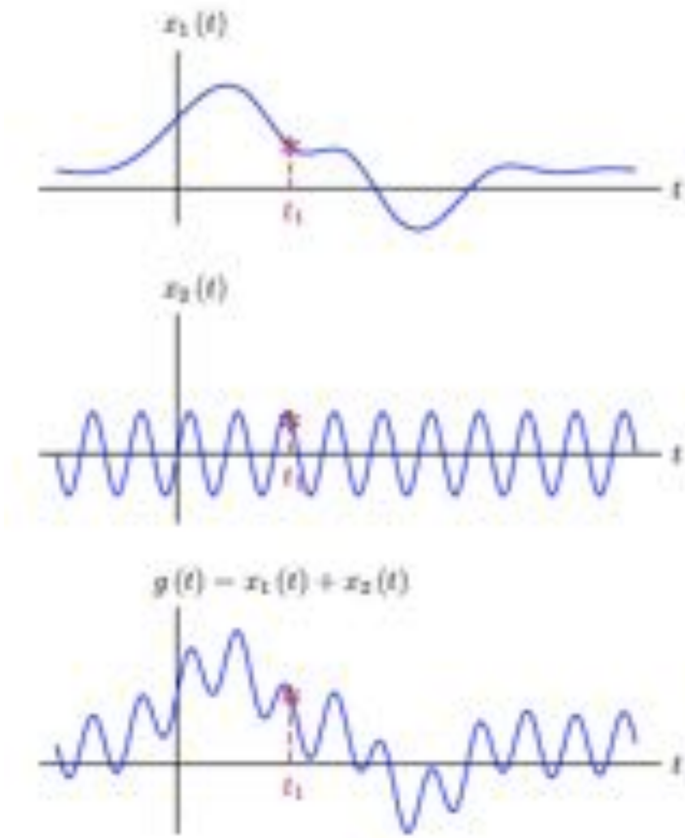
$$g(t) = Bx(t)$$



1.3.1 Signal Operations

Arithmetic Operations

Summation of two signals $x_1(t)$ and $x_2(t)$



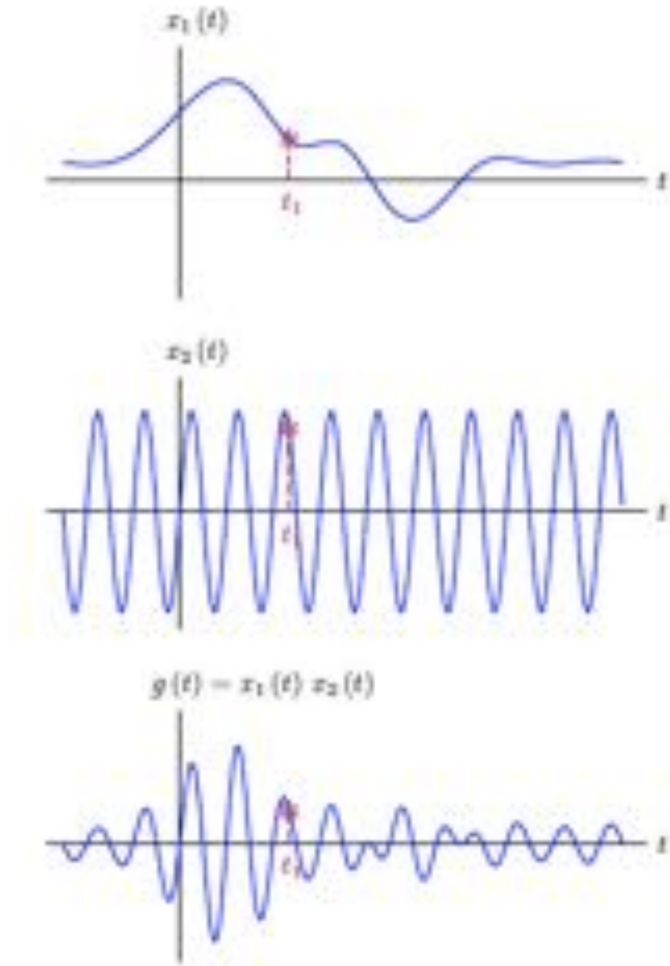
$$g(t) = x_1(t) + x_2(t)$$

1.3.1 Signal Operations

Arithmetic Operations

Multiplication of two signals $x_1(t)$ and $x_2(t)$

$$g(t) = x_1(t) * x_2(t)$$

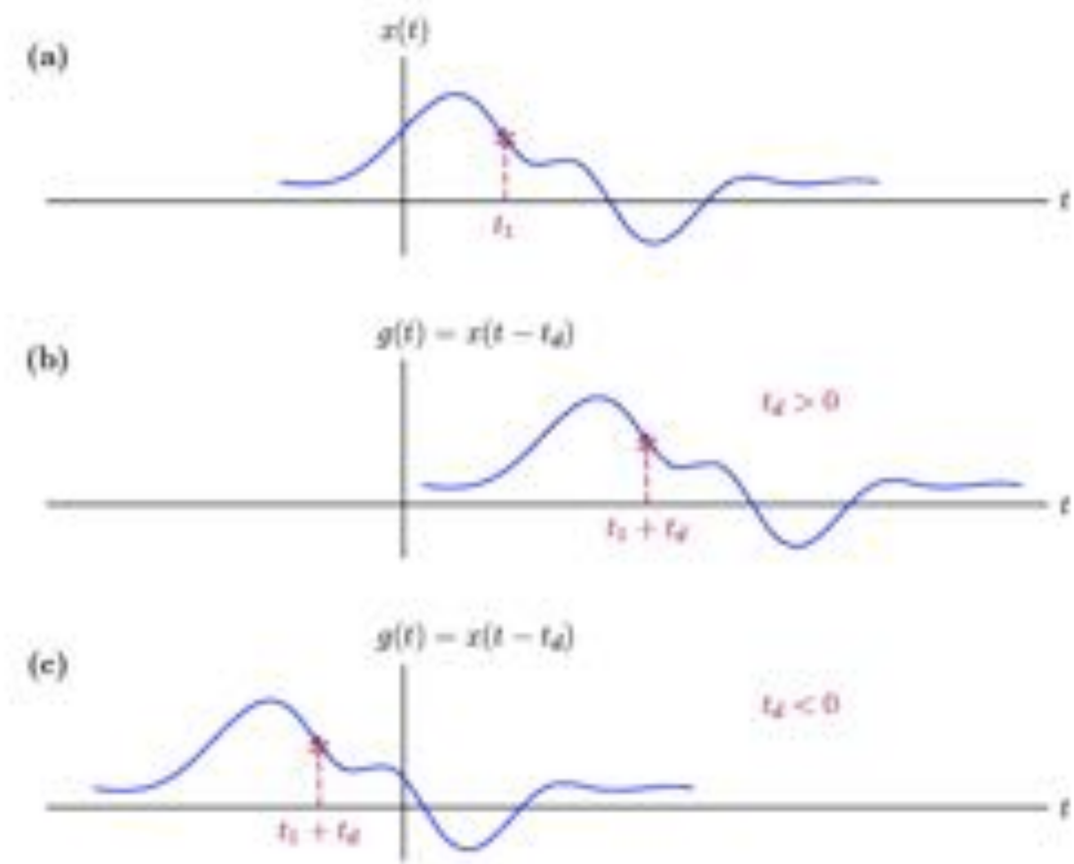


1.3.1 Signal Operations

Time shifting

A ***time shifted*** version of the signal ***x(t)*** can be obtained through

$g(t) = x(t-t_d)$

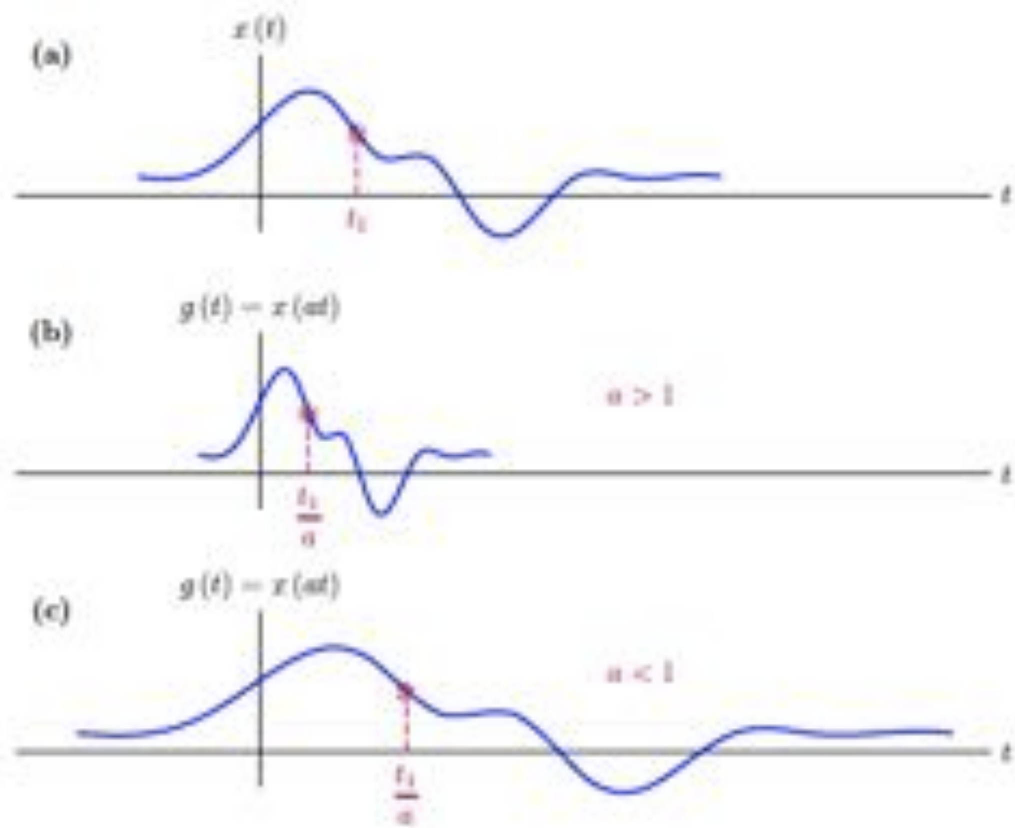


1.3.1 Signal Operations

Time scaling

A ***time scaling*** version of the signal $x(t)$ can be obtained through

g(t) = x(at)

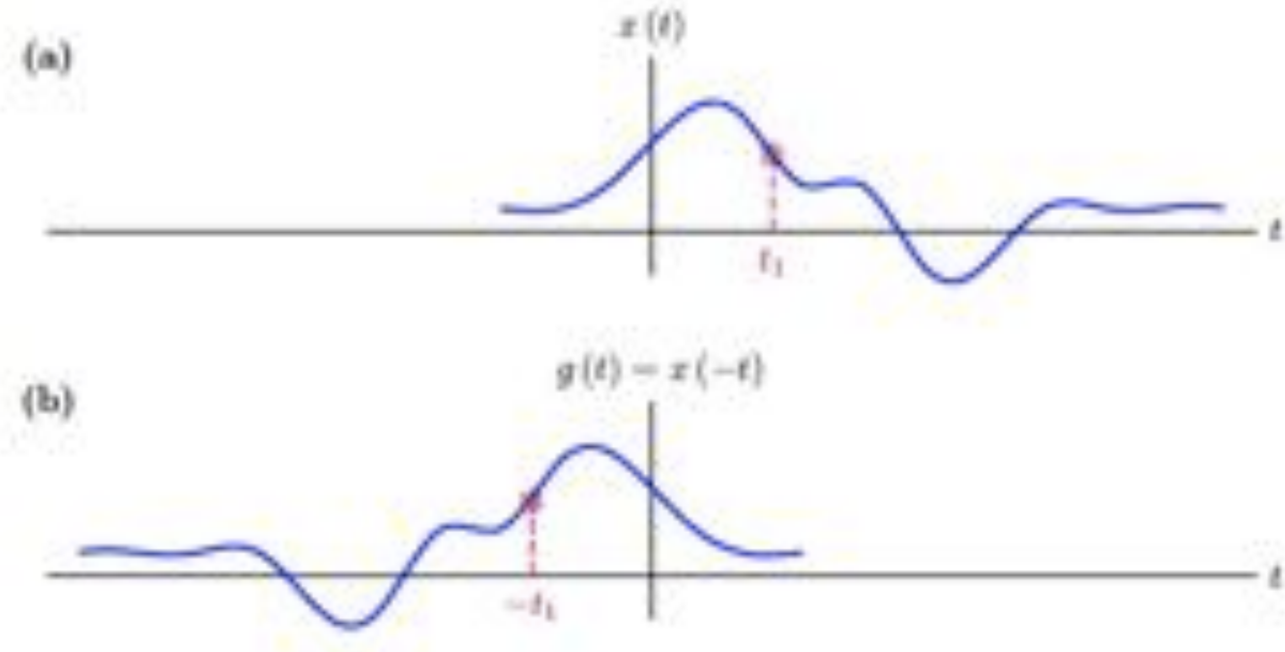


1.3.1 Signal Operations

Time reversal

A ***time shifted*** version of the signal $x(t)$ can be obtained through

$$g(t) = x(-t)$$



1.3.2 Basic Building Blocks for Continuous-time Signals

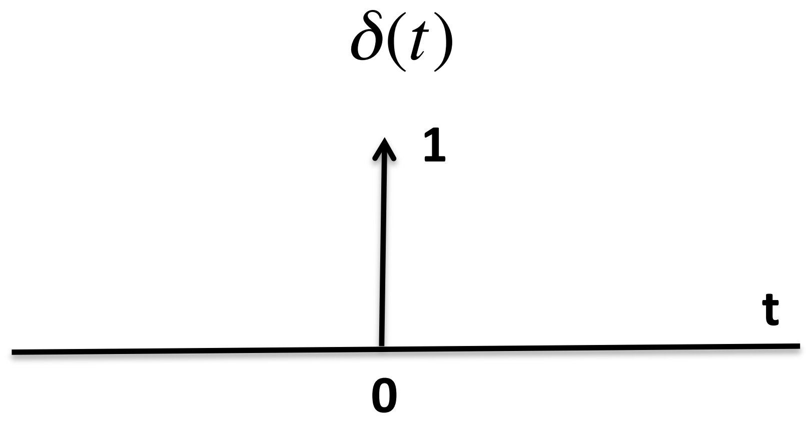
Basic building blocks

- ✧ **Unit-impulse function**
- ✧ **Unit-step function**
- ✧ **Unit-pulse function**
- ✧ **Unit-ramp function**
- ✧ **Unit-triangle function**
- ✧ **Sinusoidal signals**

1.3.2 Basic Building Blocks for Continuous-time Signals

unit-impulse function

An arrow is used to indicate the **location** of that undefined amplitude.



1.3.2 Basic Building Blocks for Continuous-time Signals

unit-impulse function

$$\delta(t) = \begin{cases} 0 \quad if \quad t \neq 0 \\ undefined \quad if \quad t = 0 \end{cases} \qquad (1.16)$$

and

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \qquad (1.17)$$

Note: Eqn. 1.16 by itself represents an incomplete definition of the function $\delta(t)$ since the amplitude of it is defined only when $t \neq 0$, and is undefined at the time instant *t = 0.* The Eqn. 1.17 fills this void.

1.3.2 Basic Building Blocks for Continuous-time Signals

unit-impulse function

Scaling and time shifting

$$a\delta(t - t_1) = \begin{cases} 0 \quad if \quad t \neq t_1 \\ undefined \quad if \quad t = t_1 \end{cases}$$

and

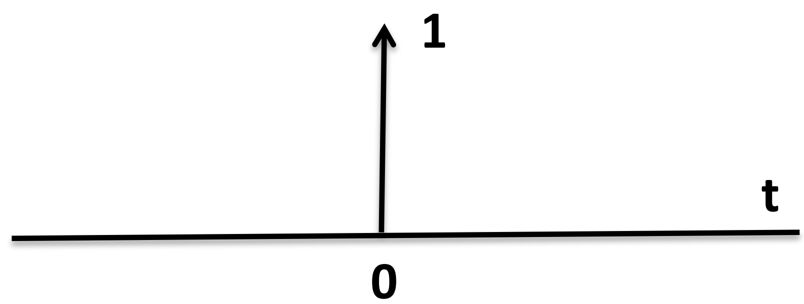
$$\int_{-\infty}^{\infty} a\delta(t - t_1)\, dt = a$$

1.3.2 Basic Building Blocks for Continuous-time Signals

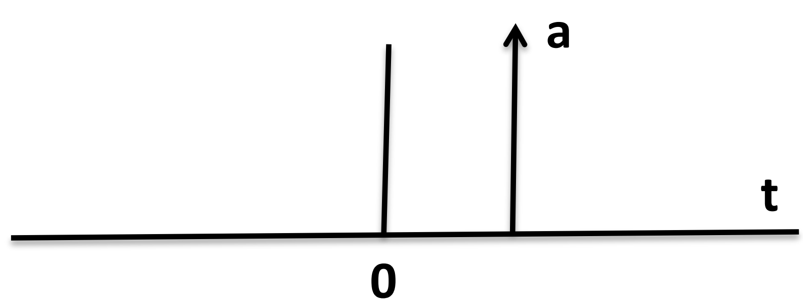
unit-impulse function

An arrow is used to indicate the **location** of that undefined amplitude.

$$\delta(t)$$



$$a\delta(t-t_1)$$

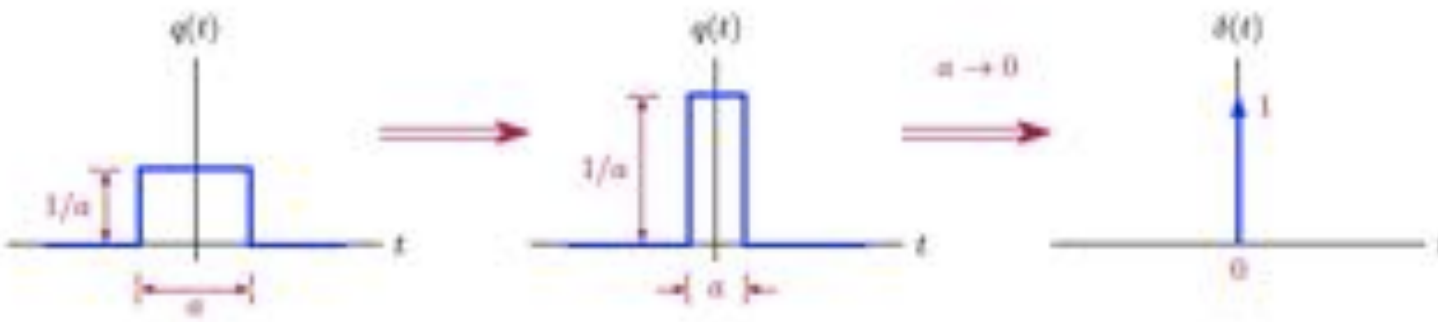


1.3.2 Basic Building Blocks for Continuous-time Signals

unit-impulse function

Obtaining ***unit-impulse*** function from a ***rectangular pulse***

$$\text{Let} \quad q(t) = \begin{cases} \dfrac{1}{a}, & |t| < \dfrac{a}{2} \\ 0, & |t| > \dfrac{a}{2} \end{cases}$$



(a) (b) (c)

$$\delta(t) = \lim_{a \to 0} [q(t)]$$

1.3.2 Basic Building Blocks for Continuous-time Signals

The impulse function has **two** fundamental properties that are useful

✧ ***Sampling property*** of the impulse function

$$f(t)\delta(t-t_1) = f(t_1)\delta(t-t_1)$$

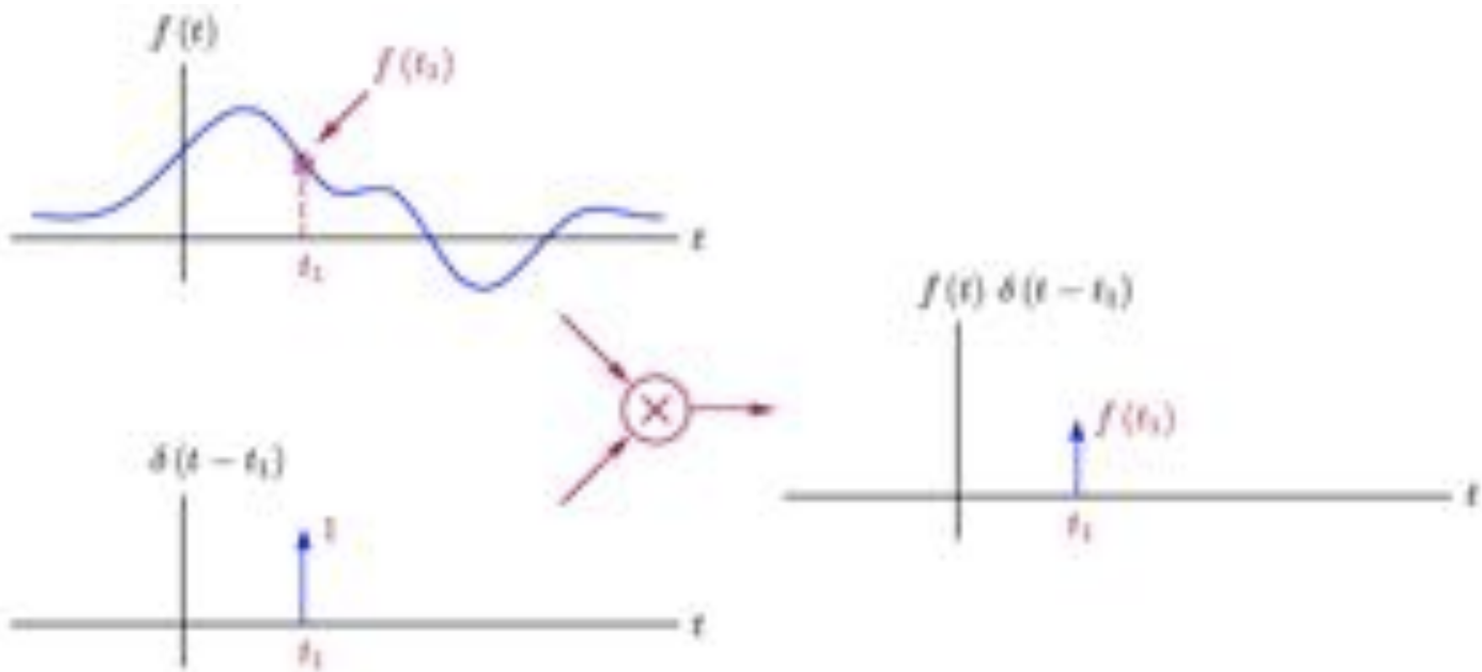
✧ ***Sifting property*** of the impulse function

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_1)dt = f(t_1)$$

1.3.2 Basic Building Blocks for Continuous-time Signals

Sampling property of the unit-impulse function

$$f(t)\delta(t-t_1)=f(t_1)\delta(t-t_1)$$



The function $f(t)$ must be continuous at $t = t_1$

1.3.2 Basic Building Blocks for Continuous-time Signals

Sifting property of the unit-impulse function

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_1)dt = f(t_1)$$

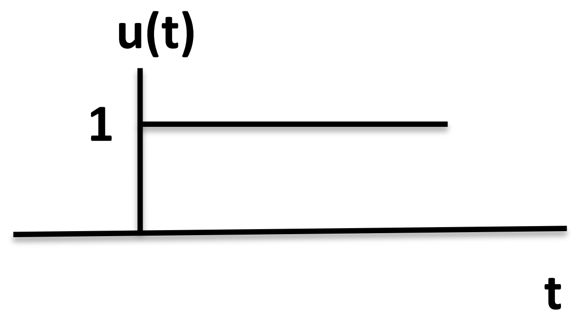
$$\int_{t_1-\Delta t}^{t_1+\Delta t} f(t)\delta(t-t_1)dt = f(t_1)$$

The function ***f*(*t*)** must be **continuous** at $t = t_1$. Also, $\Delta t > 0$

1.3.2 Basic Building Blocks for Continuous-time Signals

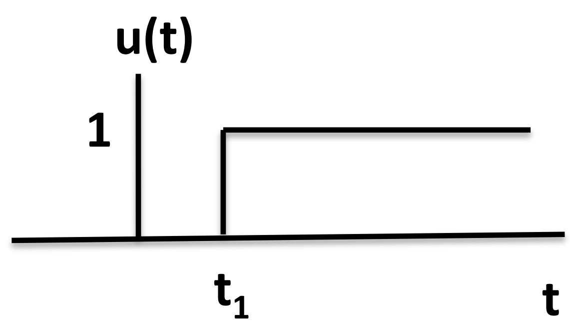
unit-step function

$$u(t) = \begin{cases} 1 & if \quad t > 0 \\ 0 & if \quad t < 0 \end{cases} \qquad (1.30)$$



Time shift of the unit-step function

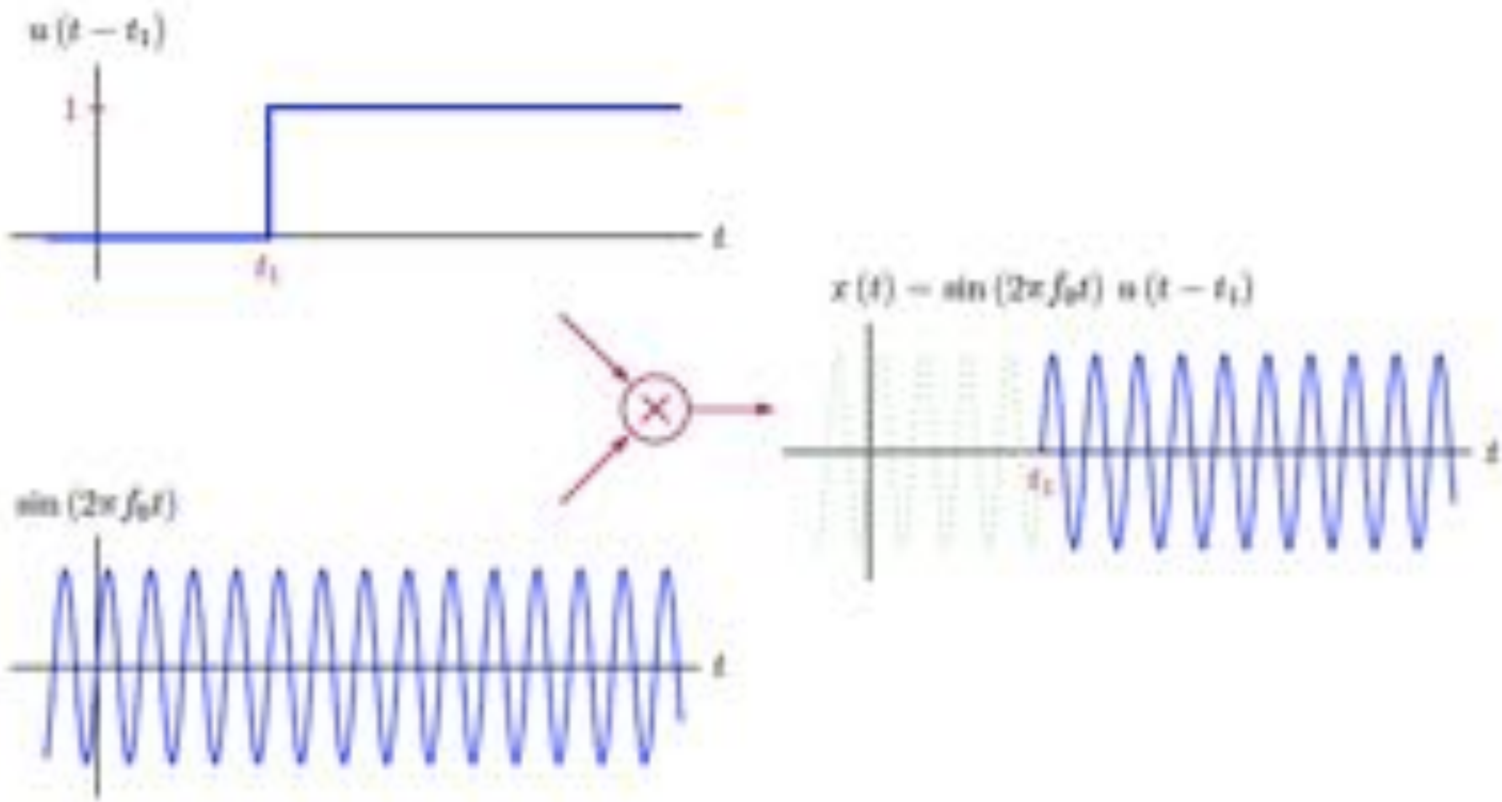
$$u(t - t_1) = \begin{cases} 1 & if \quad t > t_1 \\ 0 & if \quad t < t_1 \end{cases} \qquad (1.31)$$



1.3.2 Basic Building Blocks for Continuous-time Signals

Using the unit-step function to turn a signal on at a specified time instant

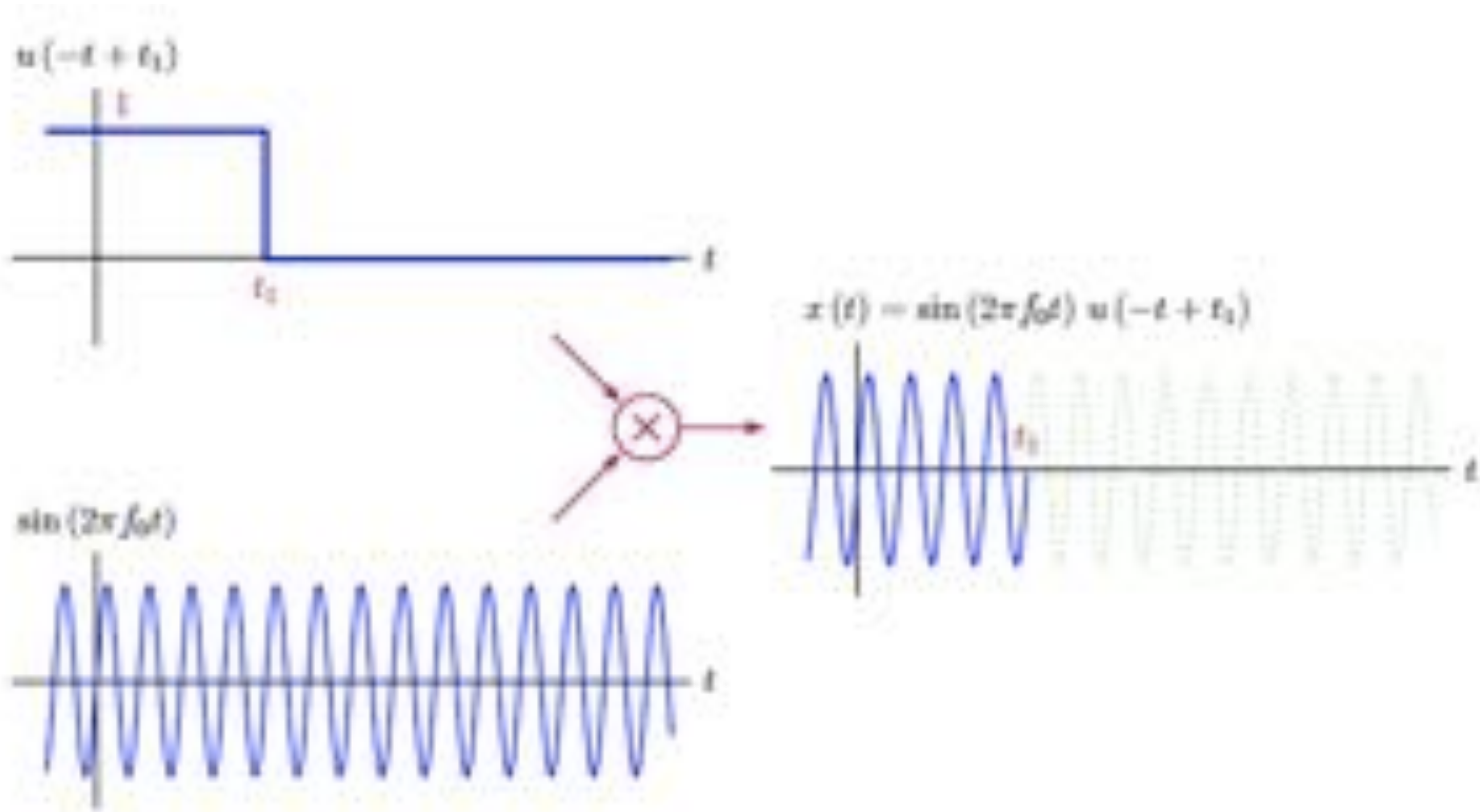
$$x(t) = \sin(2\pi f_0 t)u(t - t_1) = \begin{cases} \sin(2\pi f_0 t) & if \quad t > t_1 \\ 0 & if \quad t < t_1 \end{cases}$$



1.3.2 Basic Building Blocks for Continuous-time Signals

Using the unit-step function to turn a signal off at a specified time instant

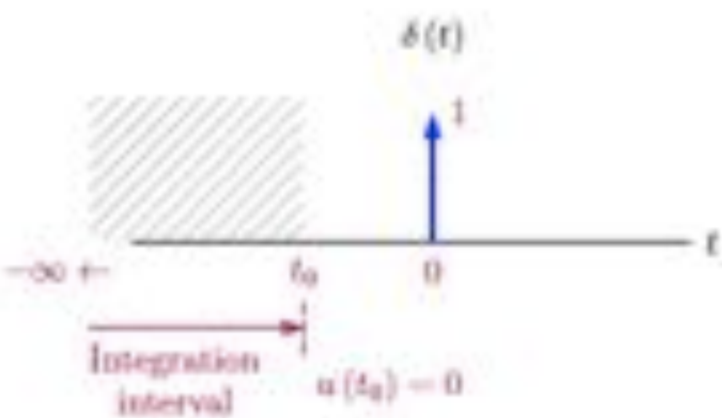
$$x(t) = \sin(2\pi f_0 t)u(-t+t_1) = \begin{cases} \sin(2\pi f_0 t) & if \quad t < t_1 \\ 0 & if \quad t > t_1 \end{cases}$$

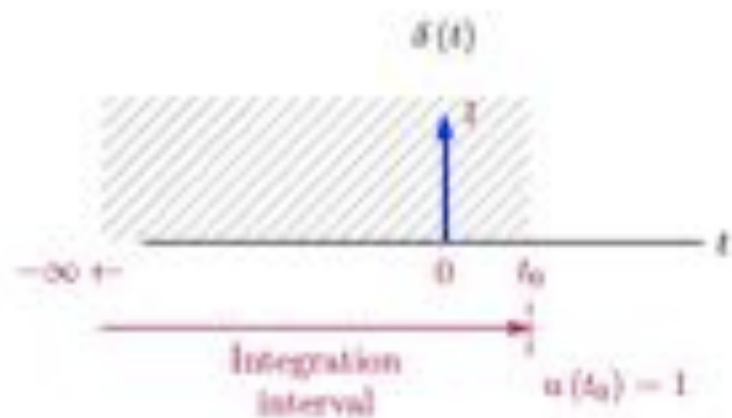


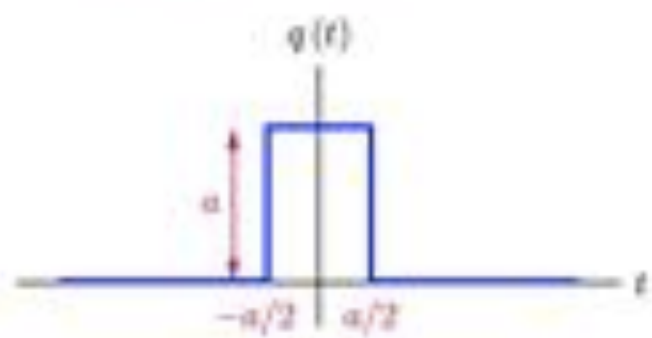
1.3.2 Basic Building Blocks for Continuous-time Signals

The **relationship** between ***unit-step*** and ***unit-impulse*** functions

$$u(t)=\int_{-\infty}^{\infty}\delta(\lambda)d\lambda \quad \Longleftrightarrow \quad \delta(t)=\frac{du}{dt}$$



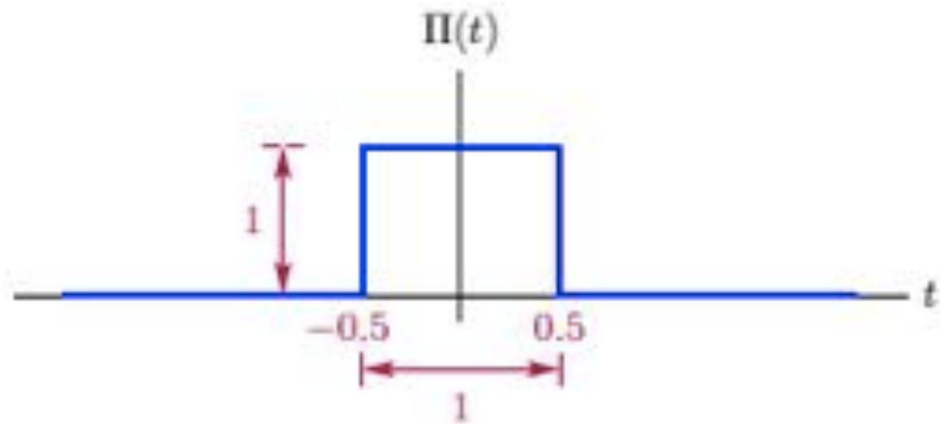




1.3.2 Basic Building Blocks for Continuous-time Signals

unit-pulse function

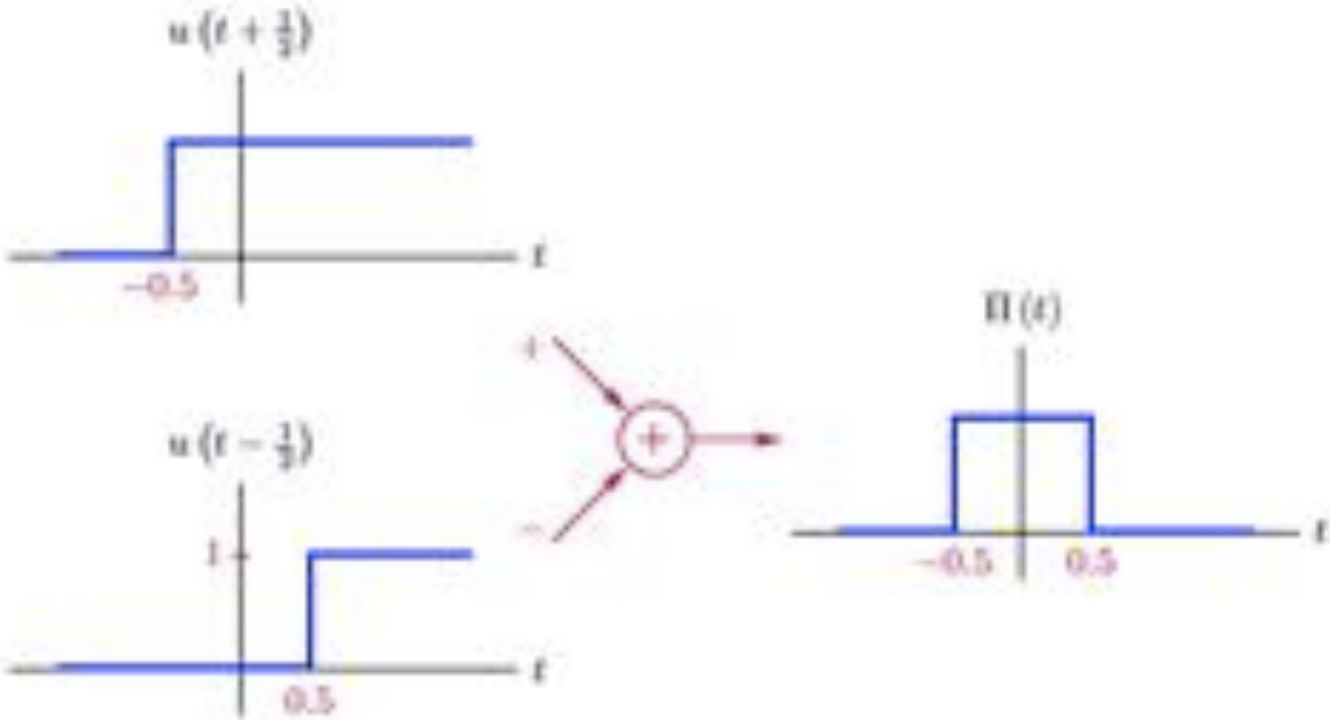
$$\prod(t) = \begin{cases} 1, & |t| < 1/2 \\ 0, & |t| > 1/2 \end{cases}$$



1.3.2 Basic Building Blocks for Continuous-time Signals

Constructing a unit-pulse function from unit-step functions

$$\prod(t) = u(t + \frac{1}{2}) - u(t - \frac{1}{2})$$



1.3.2 Basic Building Blocks for Continuous-time Signals

Constructing a unit-pulse function from unit-impulse functions

$$\prod(t) = u(t+\frac{1}{2}) - u(t-\frac{1}{2}) = \int_{-\infty}^{t+1/2} \delta(\lambda)d\lambda - \int_{-\infty}^{t-1/2} \delta(\lambda)d\lambda = \int_{t-1/2}^{t+1/2} \delta(\lambda)d\lambda$$

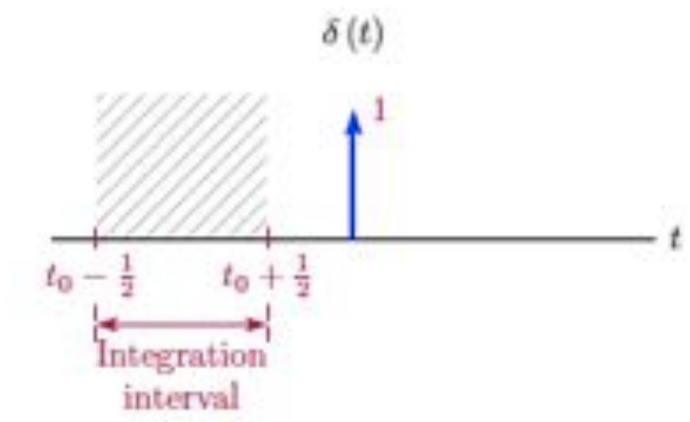
$$\int_{t-1/2}^{t+1/2} \delta(\lambda)d\lambda = \begin{cases} 1, & t-\frac{1}{2} < 0, and, t+\frac{1}{2} > 0 \\ 0, & otherwise \end{cases}$$

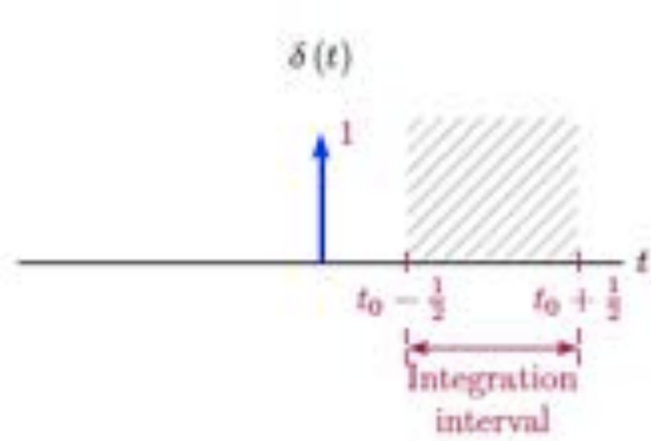
$$= \begin{cases} 1, & -\frac{1}{2} < t < \frac{1}{2} \\ 0, & otherwise \end{cases}$$

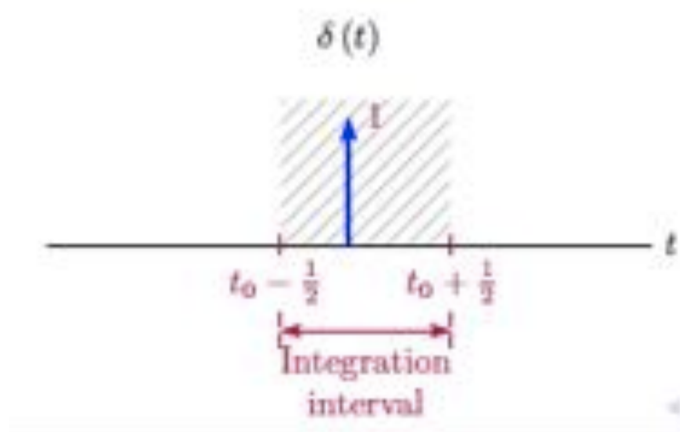
1.3.2 Basic Building Blocks for Continuous-time Signals

Constructing a unit-pulse function from unit-impulse functions

$$\prod(t) = \int_{t-1/2}^{t+1/2} \delta(\lambda) d\lambda$$







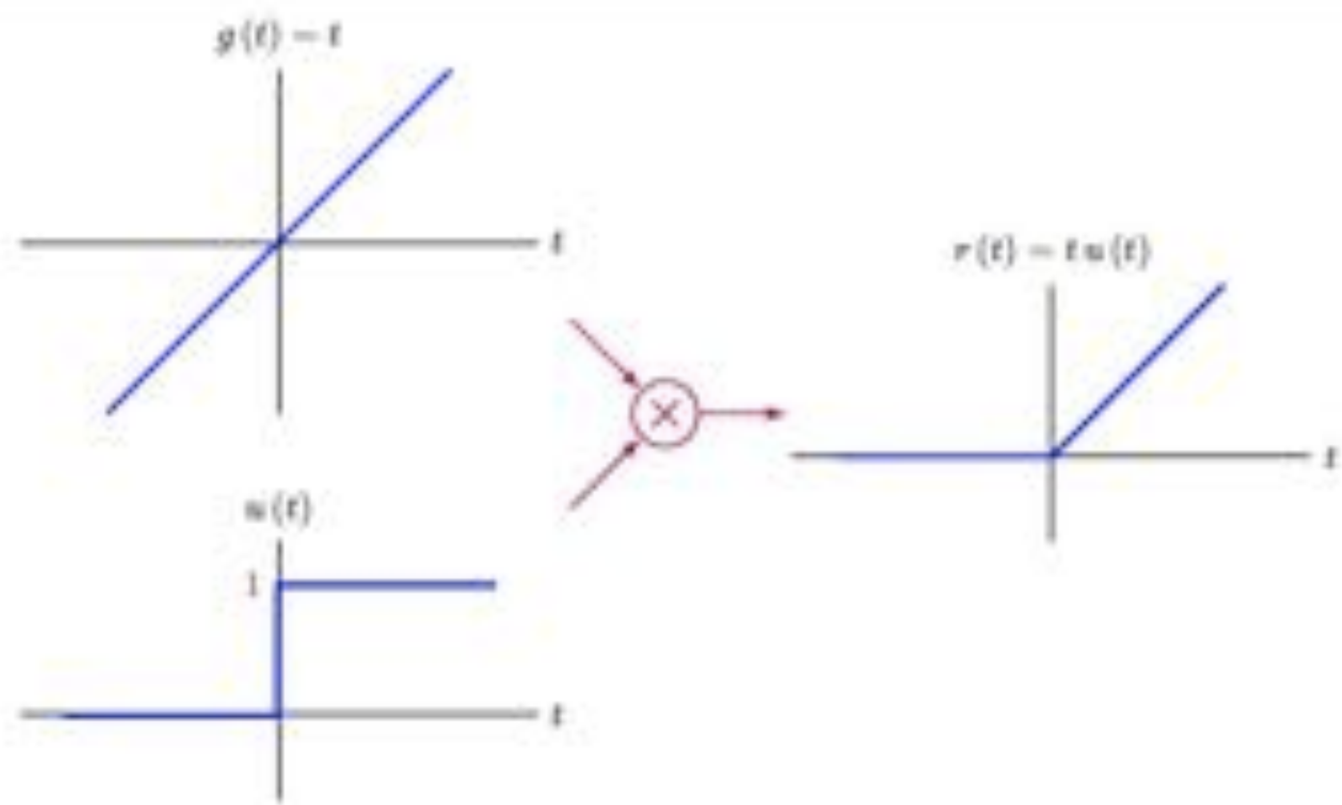
1.3.2 Basic Building Blocks for Continuous-time Signals

unit-ramp function

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

or, equivalently

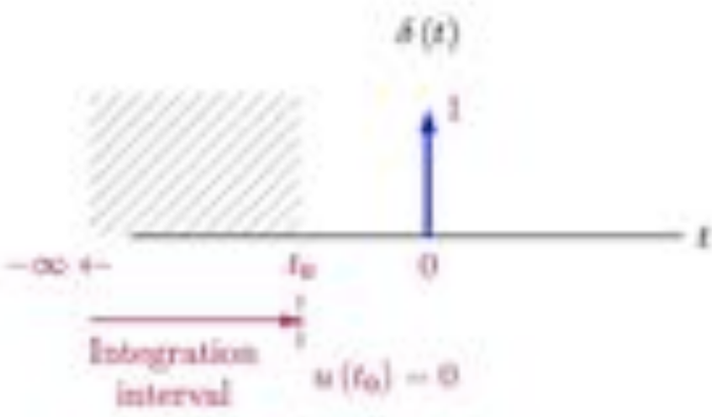
$$r(t) = tu(t)$$

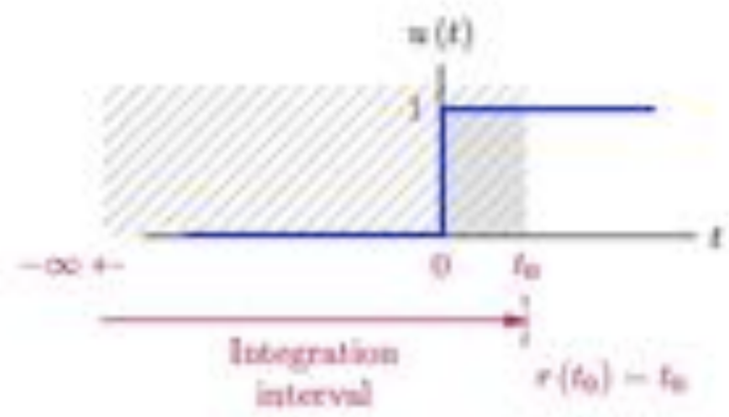


1.3.2 Basic Building Blocks for Continuous-time Signals

Constructing a unit-ramp function from a unit-step

$$r(t) = \int_{-\infty}^{\infty} u(\lambda)\, d\lambda$$

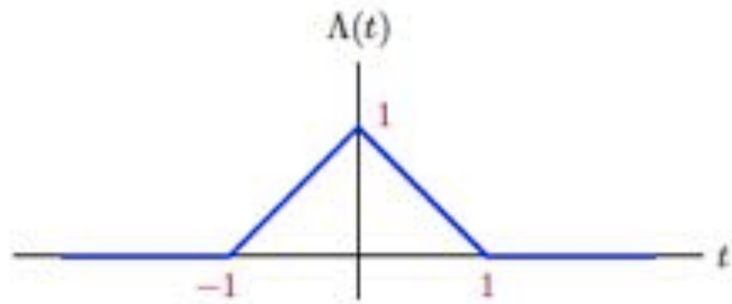




1.3.2 Basic Building Blocks for Continuous-time Signals

unit-triangle function

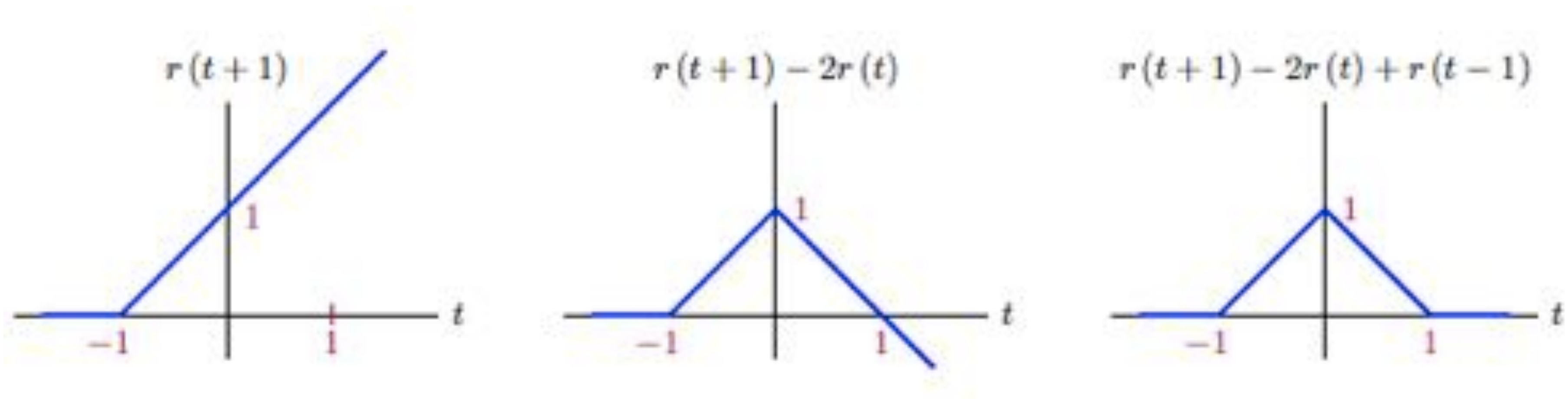
$$\Lambda(t) = \begin{cases} t+1, & -1 \le t < 0 \\ -t+1, & 0 \le t < 1 \\ 0, & otherwise \end{cases}$$



1.3.2 Basic Building Blocks for Continuous-time Signals

Constructing a unit-triangle using unit-ramp functions

$$\Lambda(t) = r(t+1) - 2r(t) + r(t+1)$$

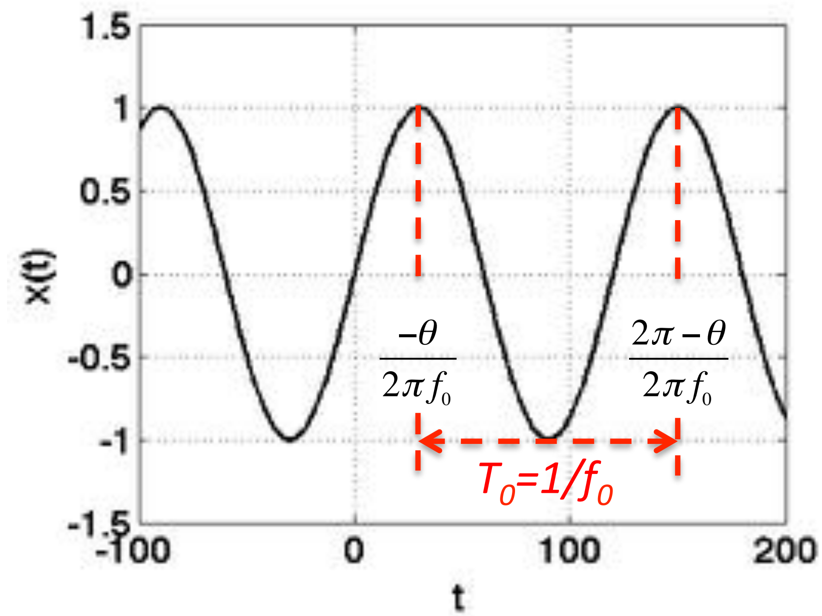


1.3.2 Basic Building Blocks for Continuous-time Signals

Sinusoidal function

$$x(t) = A\cos(\omega_0 t + \theta)$$

Where ***A*** is the ***amplitude*** of the signal, and ω_0 is the ***radian frequency*** which has the unit of *radians per second*, abbreviated as *rad/s*. The parameter θ is the ***initial phase angle*** in *radians*. The radian frequency can be expressed as $\omega_0 = 2\pi f_0$ where f_0 is the frequency in Hz.

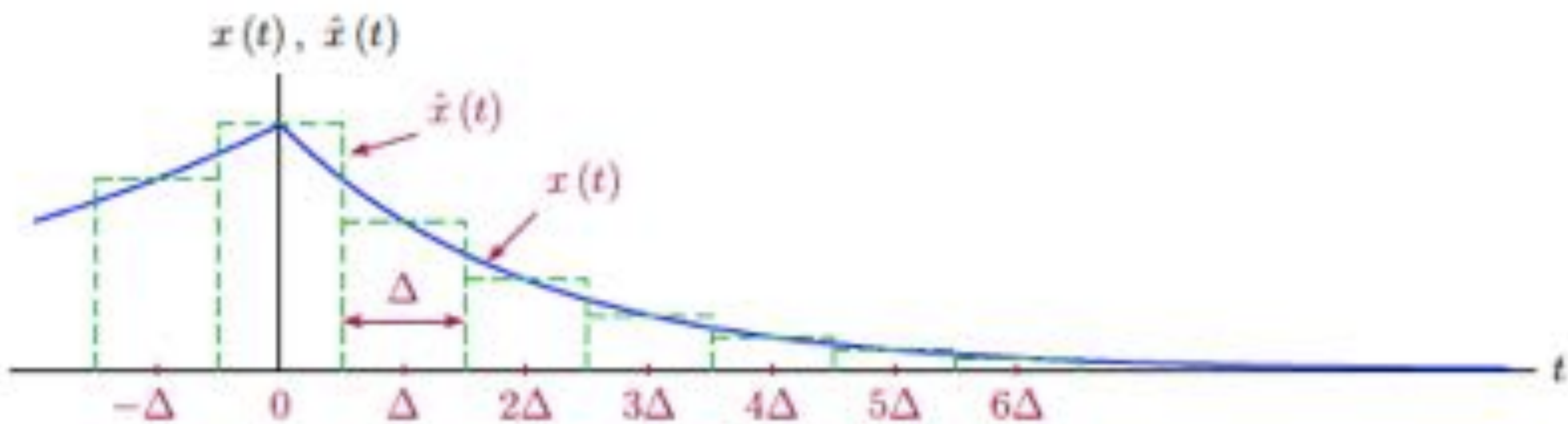


The ***A*** controls the **peak value** of the signal, and the θ affects the **peaks locations**

1.3.3 Impulse decomposition for continuous-time signals

rough approximation to the signal $x(t)$

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(n\Delta)\prod\left(\frac{t-n\Delta}{\Delta}\right)$$



Take the limit as $\Delta \to 0$ $\quad x(t) = \lim_{\Delta \to 0}\left[\hat{x}(t)\right]$

$$= \int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)d\lambda$$

1.3.4 Signal Classification

Real vs. complex signals

- A real signal is one in which the amplitude is ***real-value*** at all time instants.

$$x(t) = u \qquad \text{where } u \text{ is the voltage}$$

- A complex signal is one in which the amplitude may also have an ***imaginary*** part.

$$x(t) = x_r(t) + x_i(t) \qquad \textbf{Cartesian form}$$

or

$$x(t) = |x(t)|e^{j\angle x(t)} \qquad \textbf{Polar form}$$

$$|x(t)| = \left[x_r^2(t) + x_i^2(t)\right]^{1/2} \quad \text{and} \quad \angle x(t) = \tan^{-1}\left[\frac{x_i(t)}{x_r(t)}\right]$$

$$x_r(t) = |x(t)|\cos\left[\angle x(t)\right] \quad \text{and} \quad x_i(t) = |x(t)|\sin\left[\angle x(t)\right]$$

1.3.4 Signal Classification

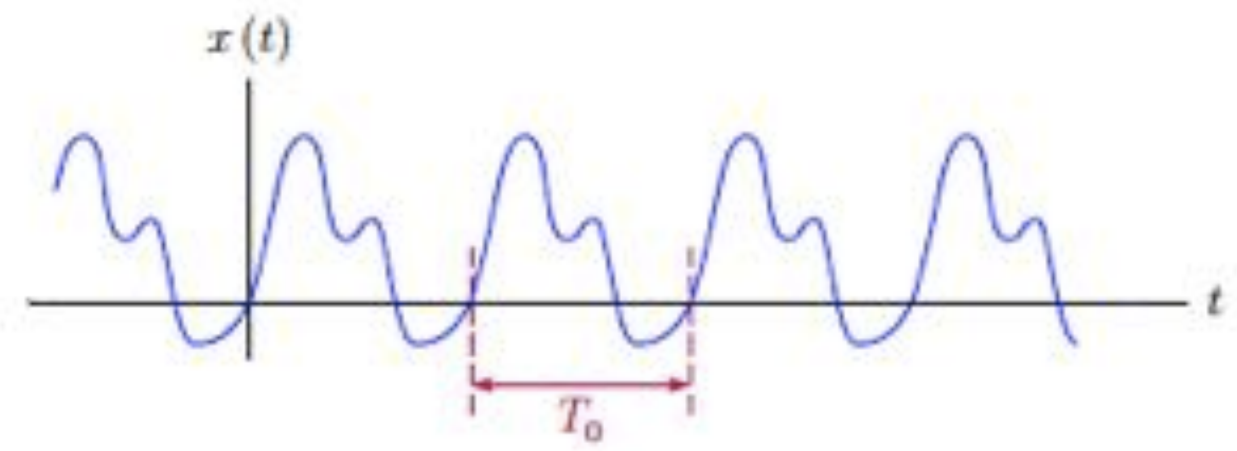
Periodic vs. non-periodic signals

A signal is said to be ***periodic*** if it satisfies

$$x(t+T_0) = x(t)$$

For all time instants ***t***, and for a specific value of $T_0 \neq 0$.

The $\boldsymbol{T_0}$ is referred as the ***period*** of the signal



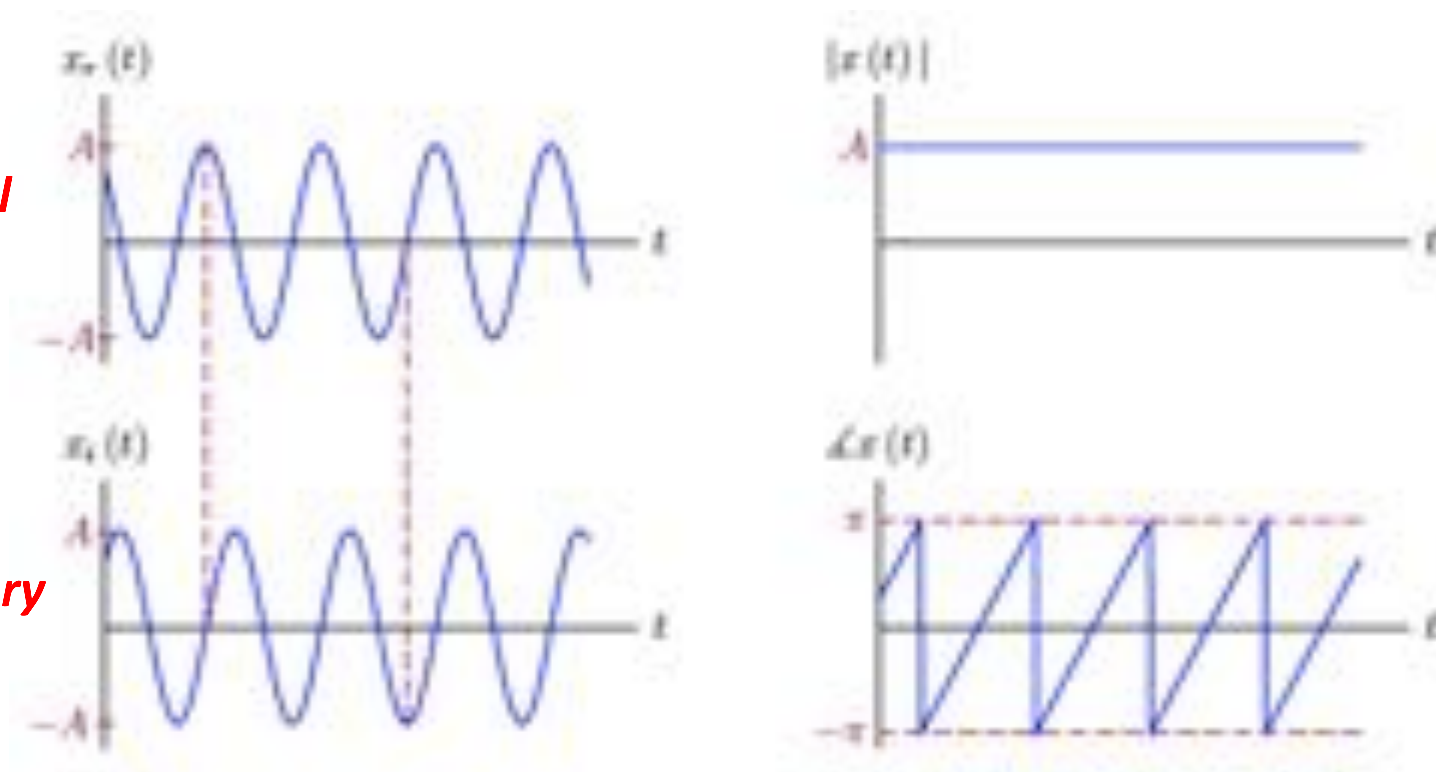
If a signal is periodic with period T_0, then it is also periodic with periods of $2T_0$, $3T_0$, …., kT_0, …., where k is any integer

1.3.4 Signal Classification

Example 1.4: Working with a complex periodic signal

real

imaginary



magnitude

phase

1.3.4 Signal Classification

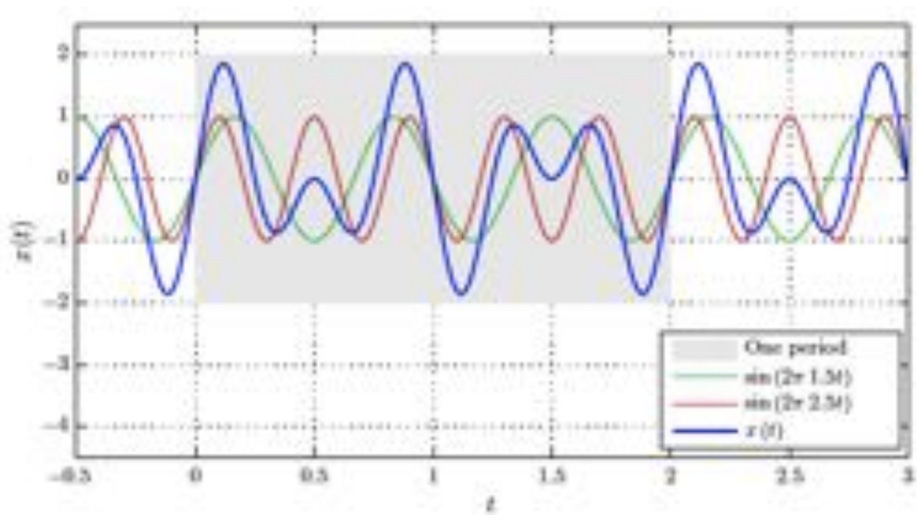
***Example 1.6**. Discuss the periodicity of the signals*

$$x(t) = \sin(2\pi 1.5t) + \sin(2\pi 2.5t)$$

For this signal, the ***fundamental*** frequency is f_0 = 0.5Hz. The two signal frequencies can be expressed as

$$f_1 = 1.5\text{Hz} = 3f_0 \quad \textit{and} \quad f_2 = 2.5\text{Hz} = 5f_0$$

The resulting fundamental period is $T_0 = 1/f_0$ = 2 seconds. Within one period of $x(t)$ there are m_1 = 3 full cycles of the first sinusoid and m_2 = 5 cycles of the second sinusoid. This is illustrated in following figure



1.3.4 Signal Classification

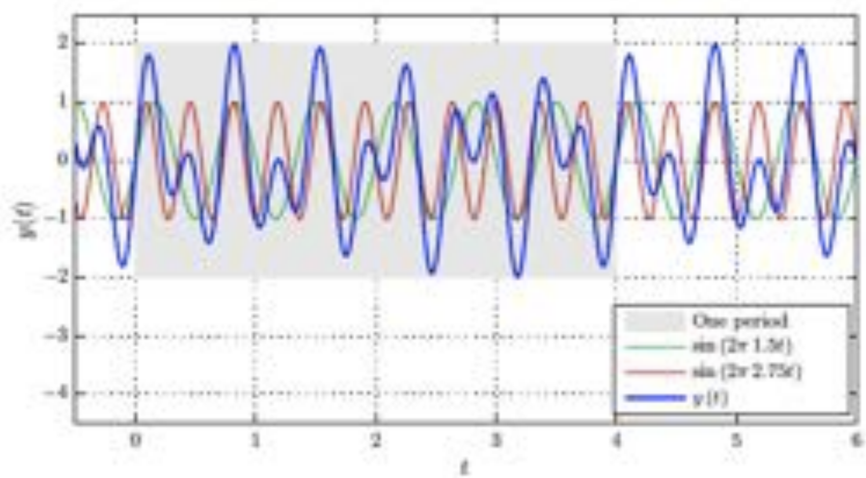
Example 1.6. *Discuss the periodicity of the signals*

$$y(t) = \sin(2\pi 1.5t) + \sin(2\pi 2.75t)$$

For this signal, the ***fundamental*** frequency is f_0 = 0.25Hz. The two signal frequencies can be expressed as

$$f_1 = 1.5\text{Hz} = 6f_0 \quad \textit{and} \quad f_2 = 2.75\text{Hz} = 11f_0$$

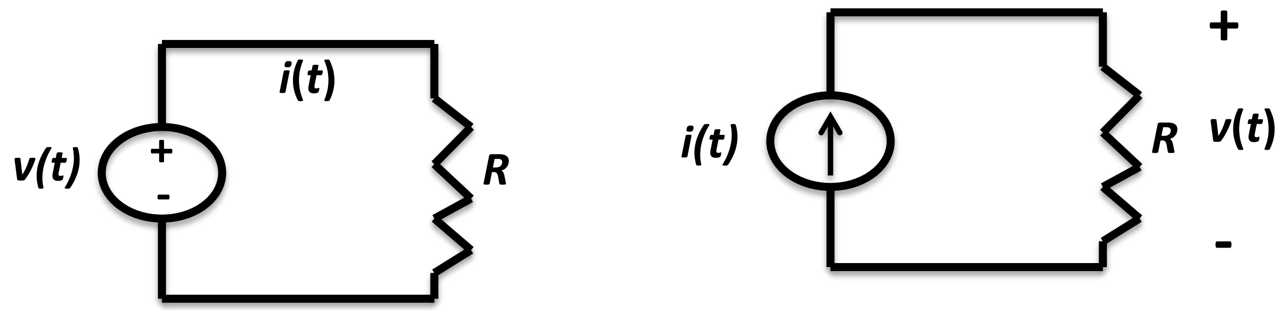
The resulting fundamental period is $T_0 = 1/f_0$ = 4 seconds. Within one period of $x(t)$ there are m_1 = 6 full cycles of the first sinusoid and m_2 = 11 cycles of the second sinusoid. This is illustrated in following figure



1.3.5 Energy and Power Definitions

Energy of a signal

With physical signals and systems, the concept of ***energy*** is associated with a signal that is applied to a load.



If we wanted to use the voltage ***v(t)*** as our basis in energy calculations:

$$E = \int_{-\infty}^{\infty} v(t)i(t)\,dt = \int_{-\infty}^{\infty} \frac{v^2(t)}{R}\,dt$$

Alternatively, we wanted to use the current ***i(t)*** as our basis in energy calculations:

$$E = \int_{-\infty}^{\infty} v(t)i(t)\,dt = \int_{-\infty}^{\infty} Ri^2(t)\,dt$$

1.3.5 Energy and Power Definitions

Normalized energy of a signal

The ***normalized energy*** of a ***real-valued*** signal ***x(t)***:

$$E = \int_{-\infty}^{\infty} x^2(t)\, dt$$

If the integral can be computed

The ***normalized energy*** of a ***complex*** signal ***x(t)***:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

If the integral can be computed

1.3.5 Energy and Power Definitions

Time averaging operator

We use the operator **< >** to indicate time average.

- If the signal ***x(t)*** is ***periodic*** with period T_0, its **time average** can be computed as.

$$\left\langle x(t) \right\rangle = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)dt$$

- If the signal ***x(t)*** is ***non-periodic***, its **time average** can be computed as.

$$\left\langle x(t) \right\rangle = \lim_{T \to \infty} \left[\frac{1}{T} \int_{-T/2}^{T/2} x(t)dt \right]$$

1.3.5 Energy and Power Definitions

Time averaging operator

✧ The ***normalized average power*** for a ***periodic signal*** with period T_0

$$P_x = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x^2(t)dt$$

✧ The ***normalized average power*** for a ***non-periodic signal***

$$P_x = \lim_{T\to\infty}\left[\frac{1}{T}\int_{-T/2}^{T/2} x^2(t)dt\right]$$

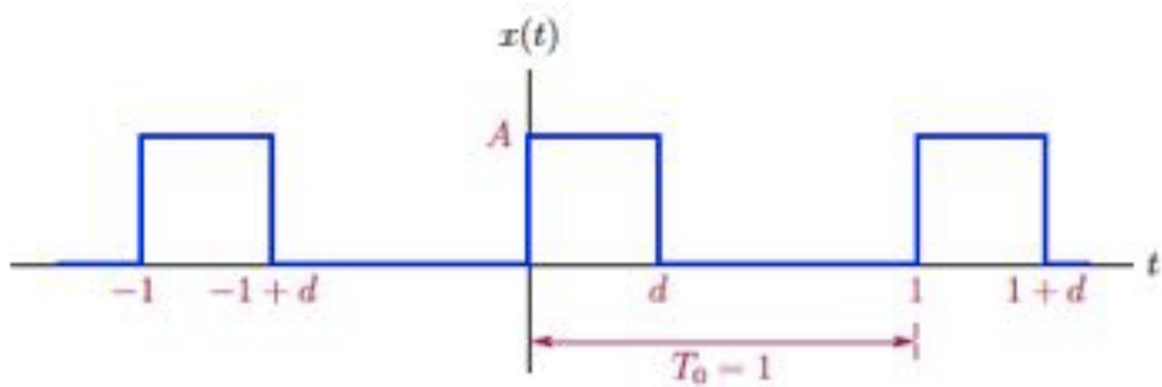
1.3.5 Energy and Power Definitions

Example 1.8. *Time average of a pulse train*

Compute the time average of a periodic pulse train with an amplitude of *A* and a period of $T_0 = 1$ s, defined by the equations

$$x(t) = \begin{cases} A, & 0 < t < d \\ 0, & d < t < 1 \end{cases}$$

and $x(t+kT_0) = x(t+k) = x(t)$ for all t, and all integers k, the signal $x(t)$ is shown as below



1.3.5 Energy and Power Definitions

***Example 1.8**. Time average of a pulse train*

Solution:

The time average of $x(t)$ can be calculated as

$$\langle x(t) \rangle = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\,dt$$

where $T_0 = 1$

$$\langle x(t) \rangle = \int_0^1 x(t)\,dt = \int_0^d (A)\,dt + \int_d^1 (0)\,dt = Ad$$

1.3.5 Energy and Power Definitions

Power of a signal

The ***instantaneous power*** dissipated in the load resistor would be.

$$p_{inst}(t) = v(t)i(t)$$

If the load is chose to have a value of $R = 1\Omega$, the ***normalized instantaneous power*** can be

$$p_{norm}(t) = x^2(t)$$

Normalized instantaneous power (real signal)

$$p_{\mathrm{norm}}(t) = x^2(t)$$

Normalized instantaneous power (complex signal)

$$p_{\mathrm{norm}}(t) = |x(t)|^2$$

Normalized average power (real signal)

$$P_x = \left\langle x^2(t) \right\rangle$$

Normalized average power (complex signal)

$$P_x = \left\langle |x(t)|^2 \right\rangle$$

1.3.5 Energy and Power Definitions

Energy Signals vs. Power signals

✧ ***Energy signals*** are those that have **finite energy** and **zero power**

$$E_x < \infty \quad \text{and} \quad P_x = 0$$

✧ ***Power signals*** are those that have **finite power** and **infinite energy**

$$E_x \rightarrow \infty \quad \text{and} \quad P_x < \infty$$

1.3.5 Energy and Power Definitions

RMS value of a signal

The root-mean-square (RMS) value of a signal ***x(t)*** is defined as

$$X_{RMS} = \left[\left\langle x^2(t) \right\rangle \right]^{1/2}$$

1.3.5 Energy and Power Definitions

Example 1.11. *RMS value of a sinusoidal signal*

Determine the RMS value of the signal

$$x(t) = A\sin(2\pi f_0 t + \theta)$$

Solution:

Recall example 1.9, the normalized average power of *x*(*t*) is

$$P_x = \frac{A^2}{2}$$

The RMS value of this signal is

$$X_{RMS} = \sqrt{P_x} = \frac{A}{\sqrt{2}}$$

1.3.6 Symmetry Properties

- ✧ Some signals ***have*** certain **symmetry properties** that could be utilized in a variety of ways in the analysis.

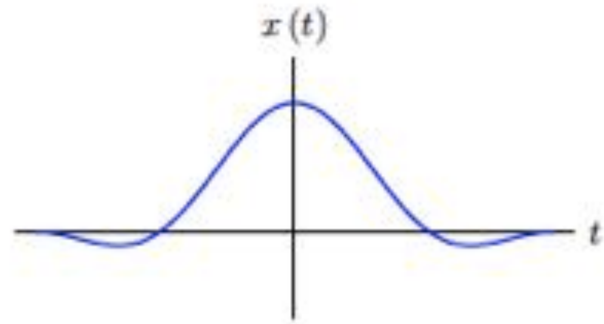
- ✧ More importantly, a signal that may ***not have*** any **symmetry properties** can ***still be written*** as a **linear combination** of signals with **certain symmetry properties**

1.3.6 Symmetry Properties

even and odd symmetry

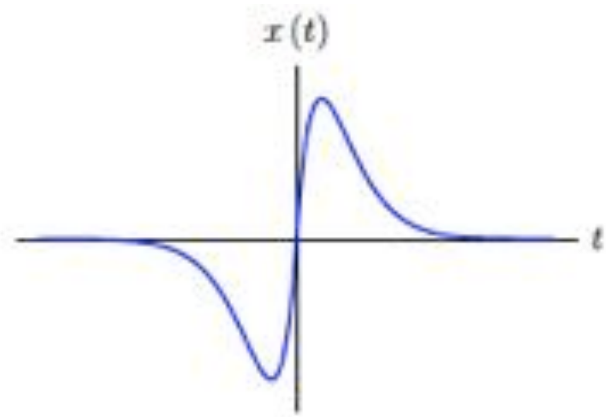
- A **real-value** signal is said to have **even symmetry** if it has the property

$$x(-t) = x(t)$$



- A **real-value** signal is said to have **odd symmetry** if it has the property

$$x(-t) = -x(t)$$



1.3.6 Symmetry Properties

Decomposition into even and odd components

$$x(t) = x_e(t) + x_o(t)$$

✧ **Even component**

$$x_e(t) = \frac{x(t) + x(-t)}{2} \qquad x_e(-t) = x_e(t)$$

✧ **odd component**

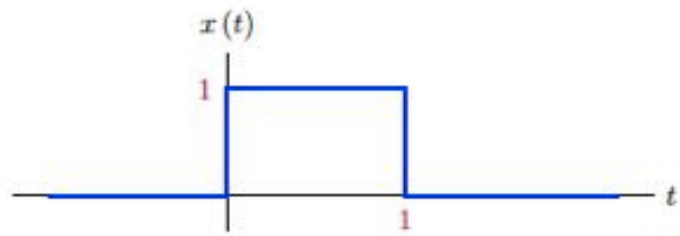
$$x_o(t) = \frac{x(t) - x(-t)}{2} \qquad x_o(-t) = -x_o(t)$$

1.3.6 Symmetry Properties

Example 1.13*. Even and odd component of a rectangular pulse*

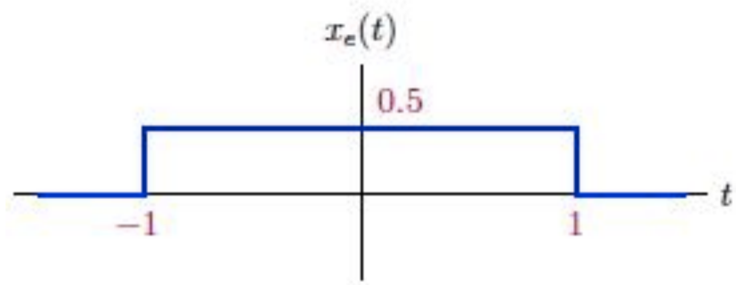
Determine the odd and even components of the rectangular pulse signal

$$x(t) = \prod(t - \frac{1}{2}) = \begin{cases} 1, & 0 < t < 1 \\ 0, & otherwise \end{cases}$$

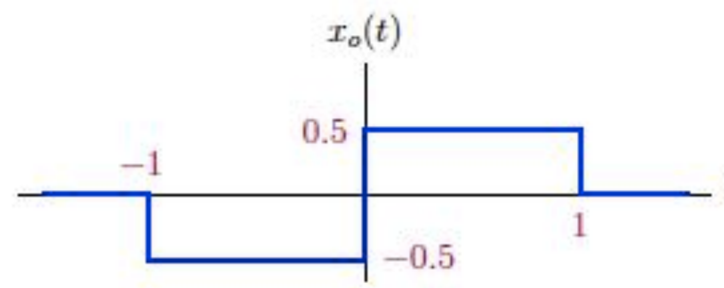


Solution:

$$x_e(t) = \frac{\prod(t - \frac{1}{2}) + \prod(-t - \frac{1}{2})}{2} = \frac{1}{2}\prod(\frac{t}{2})$$



$$x_o(t) = \frac{\prod(t - \frac{1}{2}) - \prod(-t - \frac{1}{2})}{2}$$

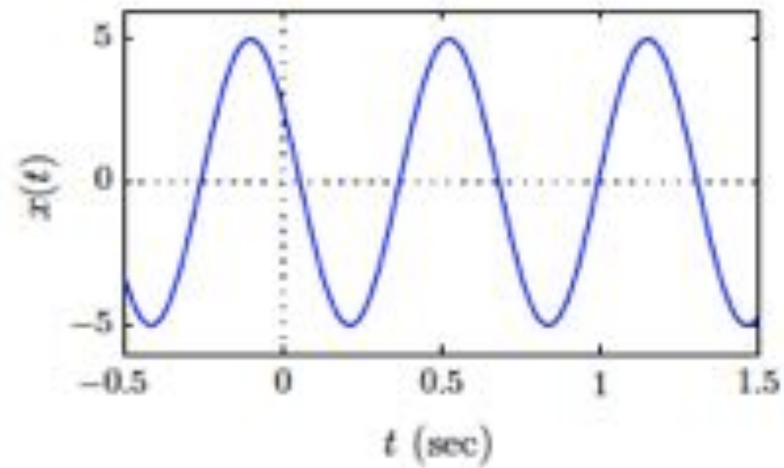


1.3.6 Symmetry Properties

***Example 1.13**. Even and odd component of a sinusoidal signal*

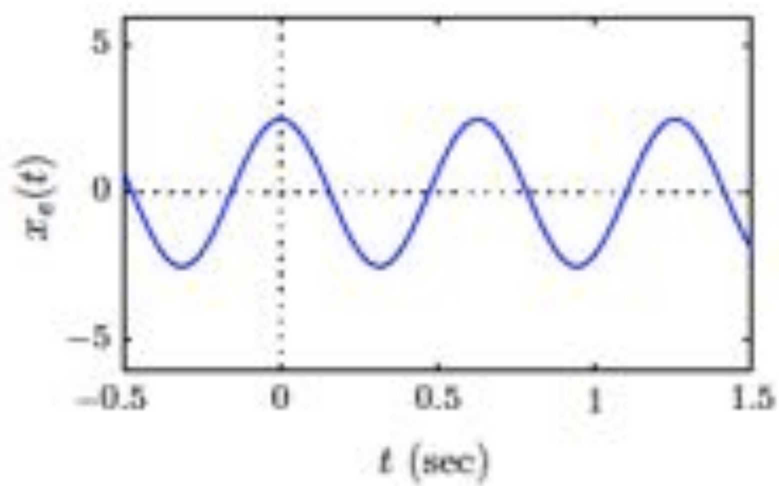
Determine the odd and even components of the rectangular pulse signal

$$x(t) = 5\cos(10t + \pi / 3)$$



Solution:

$$x_e(t) = 2.5\cos(10t)$$

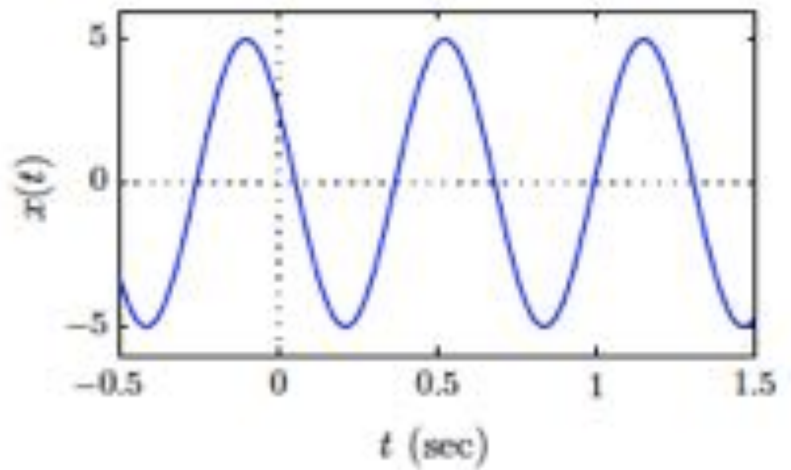


1.3.6 Symmetry Properties

***Example 1.13**. Even and odd component of a sinusoidal signal*

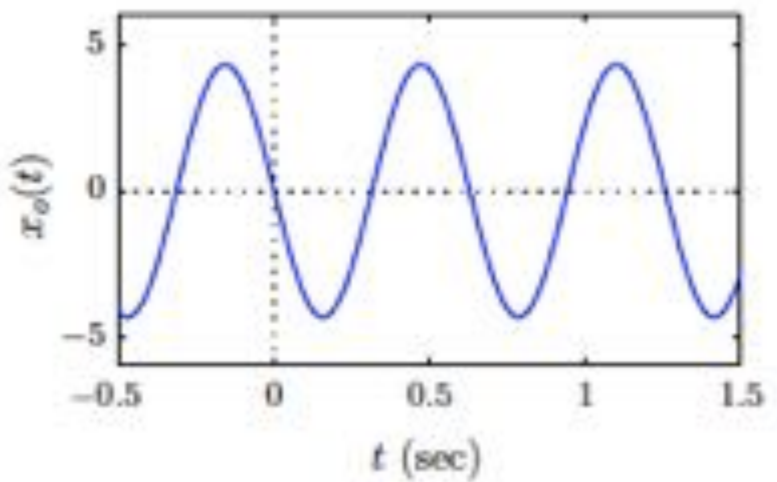
Determine the odd and even components of the rectangular pulse signal

$$x(t) = 5\cos(10t + \pi/3)$$



Solution:

$$x_o(t) = -4.3301\sin(10t)$$



1.3.6 Symmetry Properties

Symmetry properties for *complex* signals

- A **complex-value** signal is said to have **conjugate symmetry** if it satisfies

$$x(-t) = x^*(t) \qquad \text{for all } t.$$

- A **complex-value** signal is said to have **conjugate antisymmetry** if it satisfies

$$x(-t) = -x^*(t) \qquad \text{for all } t.$$

$$x(t) = x_E(t) + x_O(t)$$

Conjugate symmetric component

$$x_E(t) = \frac{x(t) + x^*(-t)}{2}$$

Conjugate antisymmetric component

$$x_O(t) = \frac{x(t) - x^*(-t)}{2}$$

1.3.6 Symmetry Properties

***Example 1.15**. Symmetry of a complex exponential signal*

Consider the complex exponential signal

$$x(t) = Ae^{j\omega t} \qquad \text{A: real}$$

What symmetry property does this signal have, if any?

Solution:

Time reverse the signal: $x(-t) = Ae^{-j\omega t}$

Conjugate the signal: $x^*(t) = (Ae^{j\omega t})^* = Ae^{-j\omega t}$

Since $x(-t) = x^*(t)$, the signal is conjugate symmetric

1.3.7 Graphical representation of sinusoidal signals using phasor

$$x(t) = A\cos(2\pi f_0 t + \theta)$$

Let the phasor *X* be defined as

$$X \triangleq Ae^{j\theta}$$

so that

$$x(t) = \mathrm{Re}\left\{Ae^{j(2\pi f_0 t + \theta)}\right\} = \mathrm{Re}\left\{Xe^{j2\pi f_0 t}\right\}$$