Chapter 5. Fourier Analysis for Discrete-Time Signals and Systems

Chapter Objectives

1. Learn techniques for representing ***discrete-time periodic*** signals using ***orthogonal*** sets of ***periodic basis*** functions.
2. Study properties of ***exponential, trigonometric*** and ***compact Fourier series***, and conditions for their existence.
3. Learn the ***Fourier transform*** for ***non-periodic*** signal as an extension of Fourier series for periodic signals
4. Study the ***properties*** of the Fourier transform. Understand the concepts of ***energy*** and ***power spectral density***.

5.2 Exponential Fourier Series (EFS)

Continue-Time Fourier Series

Synthesis equation:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Analysis equation:

$$c_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} \tilde{x}(t) e^{-jk\omega_0 t}\, dt$$

Discrete-Time Fourier Series

Synthesis equation:

$$\tilde{x}[n] = \sum_{k=0}^{N-1} c_k e^{j(2\pi/N)kn}$$

Analysis equation:

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}$$

5.2.7 Properties of Fourier Series

Linearity

Continue-Time Fourier Series

$$x(t) \overset{\Im}{\longleftrightarrow} c_k$$

$$y(t) \overset{\Im}{\longleftrightarrow} d_k$$

$$a_1 x(t) + a_2 y(t) \overset{\Im}{\longleftrightarrow} a_1 c_k + a_2 d_k$$

Discrete-Time Fourier Series

$$x[n] \overset{\Im}{\longleftrightarrow} c_k$$

$$y[n] \overset{\Im}{\longleftrightarrow} d_k$$

$$a_1 x[n] + a_2 y[n] \overset{\Im}{\longleftrightarrow} a_1 c_k + a_2 d_k$$

Where a_1 and a_2 are any two constants

5.2.7 Properties of Fourier Series

Time shift

Continue-Time Fourier Series

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jkw_0 t}$$

$$\tilde{x}(t-\tau) = \sum_{k=-\infty}^{\infty} [c_k e^{-jkw_0 \tau}] e^{jkw_0 t}$$

Discrete-Time Fourier Series

$$\tilde{x}[n] = \sum_{k=0}^{N-1} c_k e^{j(2\pi/N)kn}$$

$$\tilde{x}[n-m] = \sum_{k=0}^{N-1} c_k e^{j(2\pi/N)kn} e^{-j(2\pi/N)km}$$

5.3 Analysis of Non-periodic Continuous-Time Signals

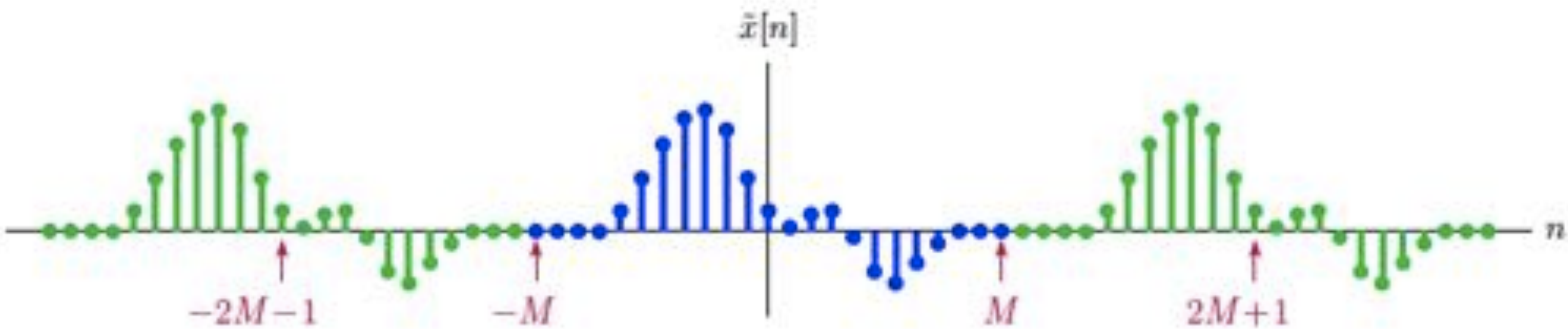
Discrete-Time Fourier Transform

Synthesis equation (inverse):

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}\, d\Omega$$

Analysis equation (forward):

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$



$$\frac{2\pi k}{2M+1} -> \Omega$$

5.3 Analysis of Non-periodic Continuous-Time Signals

Continue-Time Fourier Transform

Synthesis equation (inverse):

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{jwt}\,dw$$

Analysis equation (forward):

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-jwt}\,dt$$

Discrete-Time Fourier Transform

Synthesis equation (inverse):

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}\,d\Omega$$

Analysis equation (forward):

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

5.3.2 Existence of Fourier Transform

Is it always possible to determine the Fourier series coefficients?

- ✧ Absolute summable: $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$
- ✧ Square - summable: $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$

5.3.5 Properties of Fourier Transform

Linearity:

$$x_1[n] \overset{\Im}{\longleftrightarrow} X_1(\Omega) \qquad \text{and} \qquad x_2[n] \overset{\Im}{\longleftrightarrow} X_2(\Omega)$$

$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \overset{\Im}{\longleftrightarrow} \alpha_1 X_1(\Omega) + \alpha_2 X_2(\Omega)$$

Where a_1 and a_2 are any two constants

Periodicity:

$$X(\Omega + 2\pi r) = X(\Omega)$$

for all integers ***r***

5.3.5 Properties of Fourier Transform

Time Shifting:

$$x[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} X(\Omega) \quad \Longrightarrow \quad x[n-m] \stackrel{\mathfrak{F}}{\longleftrightarrow} X(\Omega)e^{-j\Omega m}$$

Frequency Shifting:

$$x[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} X(\Omega) \quad \Longrightarrow \quad x[n]e^{-j\Omega_0 n} \stackrel{\mathfrak{F}}{\longleftrightarrow} X(\Omega - \Omega_0)$$

5.3.5 Properties of Fourier Transform

Convolution Property:

$$x_1[n] \overset{\mathfrak{F}}{\longleftrightarrow} X_1(\Omega) \qquad x_2[n] \overset{\mathfrak{F}}{\longleftrightarrow} X_2(\Omega)$$

$$\Downarrow$$

$$x_1[n] * x_2[n] \overset{\mathfrak{F}}{\longleftrightarrow} X_1(\Omega) X_2(\Omega) \qquad X_1(\Omega) * X_2(\Omega) \overset{\mathfrak{F}}{\longleftrightarrow} x_1[n] x_2[n]$$

5.4 Energy and Power in Frequency Domain

Parseval's Theorem:

For a periodic power signal x(t)

Continue-Time

$$\frac{1}{T_0}\int_{t_0}^{t_0+T_0}|x(t)|^2\,dt = \sum_{k=-\infty}^{\infty}|c_k|^2$$

Discrete-Time

$$\frac{1}{N}\sum_{k=0}^{N-1}|x[n]|^2 = \sum_{k=0}^{N-1}|c_k|^2$$

For a non-periodic power signal

Continue-Time

$$\int_{-\infty}^{\infty}|x(t)|^2\,dt = \int_{-\infty}^{\infty}|X(f)|^2\,df$$

Discrete-Time

$$\sum_{k=0}^{N-1}|x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|X(\Omega)|^2\,d\Omega$$

5.4 Energy and Power in Frequency Domain

Power Spectral Density:

$$S_x(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} |c_k|^2 \, \delta(\Omega - k\Omega_0)$$

5.4 Energy and Power in Frequency Domain

Autocorrelation Function:

For a energy signal *x*(*t*) the autocorrelation function is

$$r_{xx}[m] = \sum_{n=-\infty}^{\infty} x[n]x[n+m]$$

5.5 System Function Concept

System function (frequency response)

Fourier Transform

Impulse response (***h[n]***) ⟷ System function (***H(Ω)***)

$$H(\Omega) = \Im\{h[n]\} = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n}$$

In general , *H*(*w*) is a complex function of *w*, and can be written in polar form as:

$$H(\Omega) = |H(\Omega)|e^{j\Theta(\Omega)}$$

5.8 Discrete Fourier Transform

DTFS | **DTFT** | **DFT**

Synthesis equation (inverse):

DTFS:

$$x[n] = \sum_{k=0}^{N-1} c_k e^{j(2\pi/N)kn}$$

n = 0, 1, …., N-1

DTFT:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n}\, d\Omega$$

DFT:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j(2\pi/N)kn}$$

n = 0, 1, …., N-1

Analysis equation (forward):

DTFS:

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}$$

k = 0, 1, …., N-1

DTFT:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

DFT:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}$$

k = 0, 1, …., N-1

5.8 Discrete Fourier Transform

Relationship of the DFT to the DTFT

DTFT

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

DFT

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)kn}$$

The DFT of a length-N signal is equal to its DTFT evaluated at a set of *N* angular frequencies equally spaced in the interval [*0, 2π*). Let an indexed set of angular frequencies be defined as

$$\Omega_k = \frac{2\pi k}{N}, \quad k = 0,1,\ldots\ldots,N-1$$

$$X[k] = X(\Omega) = \sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)kn}$$

5.8 Discrete Fourier Transform

<u>Why do we need DFT?</u>

- The signal x[n] and its DFT X[k] each have N samples, making the discrete Fourier transform practical for computer implementation.

- Fast and efficient algorithm, know as fast Fourier transforms (FFTs), are available for the computation of the DFT.

- DFT can be used for approximating other forms of Fourier series and transforms for both continuous-time and discrete-time system.

- Dedicated processors are available for fast and efficient. computation of the DFT with minimal or no programming needed.