4.4 Energy and Power in Frequency Domain

Parseval's Theorem:

For a periodic power signal x(t)

$$\frac{1}{T_0}\int_{t_0}^{t_0+T_0}|x(t)|^2\,dt = \sum_{k=-\infty}^{\infty}|c_k|^2$$

Where C_k is the Exponential Fourier Series (EFS) Coefficients

For a non-periodic power signal

$$\int_{-\infty}^{\infty}|x(t)|^2\,dt = \int_{-\infty}^{\infty}|X(f)|^2\,df$$

4.4 Energy and Power in Frequency Domain

Power Spectral Density:

For a periodic power signal x(t)

$$\frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

Normalized **average power**

Power at the frequency component at $f = kf_0$

Power Spectral Density:

$$S_x(f) = \sum_{k=-\infty}^{\infty} |c_k|^2\, \delta(f - kf_0)$$

$$S_x(w) = 2\pi \sum_{k=-\infty}^{\infty} |c_k|^2\, \delta(f - kw_0)$$

4.4 Energy and Power in Frequency Domain

Energy Spectral Density:

For a non-periodic power signal

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

Normalized **signal energy**

Energy density

4.4 Energy and Power in Frequency Domain

Power Spectral Density:

For a periodic power signal x(t)

$$S_x(f) = \sum_{k=-\infty}^{\infty} |c_k|^2 \delta(f - kf_0) \qquad S_x(w) = 2\pi \sum_{k=-\infty}^{\infty} |c_k|^2 \delta(f - kw_0)$$

For a non-periodic power signal

$$S_x(f) = \lim_{T->\infty} \left[\frac{1}{T} |X_T(f)|^2 \right]$$

Where $X_T(f)$ is the Fourier transform of the truncated signal $x(t)$

$$x_T(t) = \begin{cases} x(t), & -T/2 < x < T/2 \\ 0, & otherwise \end{cases}$$

4.4 Energy and Power in Frequency Domain

Autocorrelation Function:

For a energy signal *x*(*t*) the autocorrelation function is

$$R_{xx}(\tau) = < x(t)x(t+\tau) > = \int_{-\infty}^{\infty} x(t)x(t+\tau)\,dt$$

For a periodic power signal *x*(*t*) with period of T_0

$$R_{xx}(\tau) = < x(t)x(t+\tau) > = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)x(t+\tau)\,dt$$

For a non-periodic power signal

$$R_{xx}(\tau) = < x(t)x(t+\tau) > = \lim_{T_0 -> \infty}\left[\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)x(t+\tau)\,dt\right]$$

4.4 Energy and Power in Frequency Domain

Fourier Pair

Power spectral density <---------> Autocorrelation function

T_0

Periodic signal

Lim T->∞

Non-periodic signal

Energy spectral density <---------> Autocorrelation function

4.4 Energy and Power in Frequency Domain

Properties of Autocorrelation Function:

Lag zero has the maximum value

$$R_{xx}(0) \geq \left| R_{xx}(\tau) \right|$$

Even symmetry

$$R_{xx}(-\tau) = R_{xx}(\tau)$$

Periodic

$$x(t) = x(t + kT) \Longrightarrow R_{xx}(\tau) = R_{xx}(\tau + kT)$$

For all integers *K*

4.5 System Function Concept

System function (frequency response)

Fourier Transform

Impulse response (***h(t)***) ⟷ System function (***H(w)***)

$$H(w) = \Im\{h(t)\} = \int_{-\infty}^{\infty} h(t)e^{-jwt}\,dt$$

In general , *H*(*w*) is a complex function of *w*, and can be written in polar form as:

$$H(w) = |H(w)|e^{j\Theta(w)}$$

4.5 System Function Concept

CTLTI systems with non-periodic input signals

Signal-system interaction

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\, d\lambda$$

Assume that

- the system is stable ensuring that $H(\omega)$ converges, and
- the input signal has a Fourier transform.

$$Y(\omega) = H(\omega)\, X(\omega)$$

$$|Y(\omega)| = |H(\omega)|\, |X(\omega)|$$

$$\angle Y(\omega) = \angle X(\omega) + \Theta(\omega)$$

4.5 System Function Concept

Power Transfer Function

We also can obtain the relationship between the ***power spectral density*** (***PSD***) at the input, and the output for a linear time-invariant network as:

$$P_y(f) = \lim_{T \to \infty} \frac{1}{T} \left| Y_T(f) \right|^2$$

using $$Y(f) = X(f)H(f)$$

we get $$P_y(f) = \left| H(f) \right|^2 \lim_{T \to \infty} \frac{1}{T} \left| X_T(f) \right|^2$$

or $$P_y(f) = \left| H(f) \right|^2 P_x(f)$$

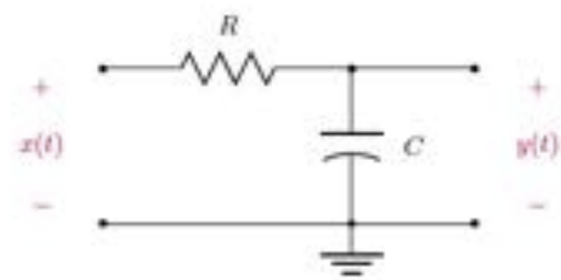
Power transfer function $$G_h(f) = \frac{P_y(f)}{P_x(f)} = \left| H(f) \right|^2$$

4.5 System Function Concept

Example 4.47

Pulse response of RC circuit revisited

Consider the RC circuit shown. Let $f_c = 1/RC = 80$ Hz. Determine the Fourier transform of the response of the system to the unit pulse input signal $x(t) = \Pi(t)$.



Solution:

The system function of the RC circuit was found in Example 4.43 to be

$$H(f) = \frac{1}{1 + j(f/f_c)}$$

The transform of the input signal is

$$X(f) = \text{sinc}(f)$$

Using $f_c = 80$ Hz, the transform of the output signal is

$$Y(f) = H(f)\,X(f) = \frac{1}{1 + j\left(\frac{f}{80}\right)}\,\text{sinc}(f)$$

4.5 System Function Concept

Example 4.47 (continued)

Magnitude of the output transform:

$$|Y(f)| = \frac{1}{\sqrt{1+\left(\frac{f}{80}\right)^2}} \, |\mathrm{sinc}(f)|$$

Phase of the output transform

$$\measuredangle Y(f) = -\tan^{-1}\left(\frac{f}{80}\right) + \measuredangle[\mathrm{sinc}(f)]$$