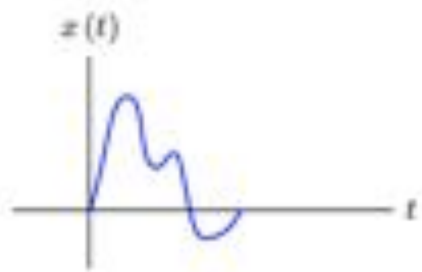
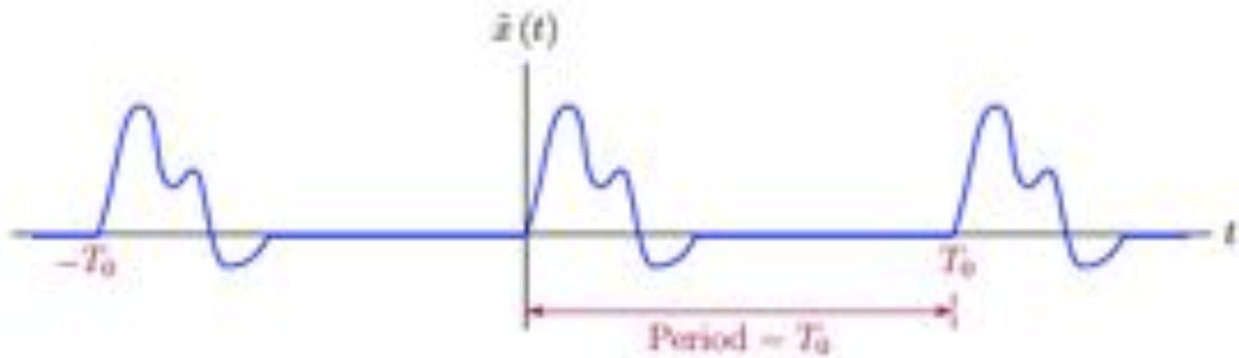
4.3 Analysis of Non-periodic Continuous-Time Signals

A non-periodic signal $x(t)$:



We view this ***non-periodic signal*** as a "***periodic signal***" with **period as infinite large**.



4.3 Analysis of Non-periodic Continuous-Time Signals

For periodic signal:

$$c_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-jk\omega_0 t}\,dt$$

For non-periodic signal: we make ***T_0 -> ∞***

$$c_k T_0 = \int_{-T_0/2}^{T_0/2} x(t)e^{-jk\omega_0 t}\,dt$$

$$X(\omega) = \lim_{T_0 -> \infty}[c_k T_0] = \lim_{T_0 -> \infty}\left[\int_{-T_0/2}^{T_0/2} x(t)e^{-jk\omega_0 t}\,dt\right]$$

$$= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

where $w = kw_0$

4.3 Analysis of Non-periodic Continuous-Time Signals

Fourier transform for continuous-time signals

Synthesis equation: (Inverse transform)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Analysis equation: (Forward transform)

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Shorthand notation:

$$X(\omega) = \Im\{x(t)\} \qquad x(t) = \Im^{-1}\{X(w)\}$$

$$x(t) \xleftrightarrow{\Im} X(w)$$

4.3 Analysis of Non-periodic Continuous-Time Signals

Fourier transform for continuous-time signals (using f instead of ω)

Synthesis equation: (Inverse transform)

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

Analysis equation: (Forward transform)

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

***x*(*t*)** : waveform in **time** domain

***X*(*f*)** : same waveform in **frequency** domain

4.3 Analysis of Non-periodic Continuous-Time Signals

How to understand Fourier transform?

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft}\, dt$$

For *X(f)*, at any frequency ***f***, all the **x(t)** (***t*** from -∞ to ∞) has contribution.

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft}\, dt$$

Similar For *x(t)*, at any frequency ***t***, all the **X(f)** (***f*** from -∞ to ∞) has contribution.

4.3 Analysis of Non-periodic Continuous-Time Signals

Time Domain

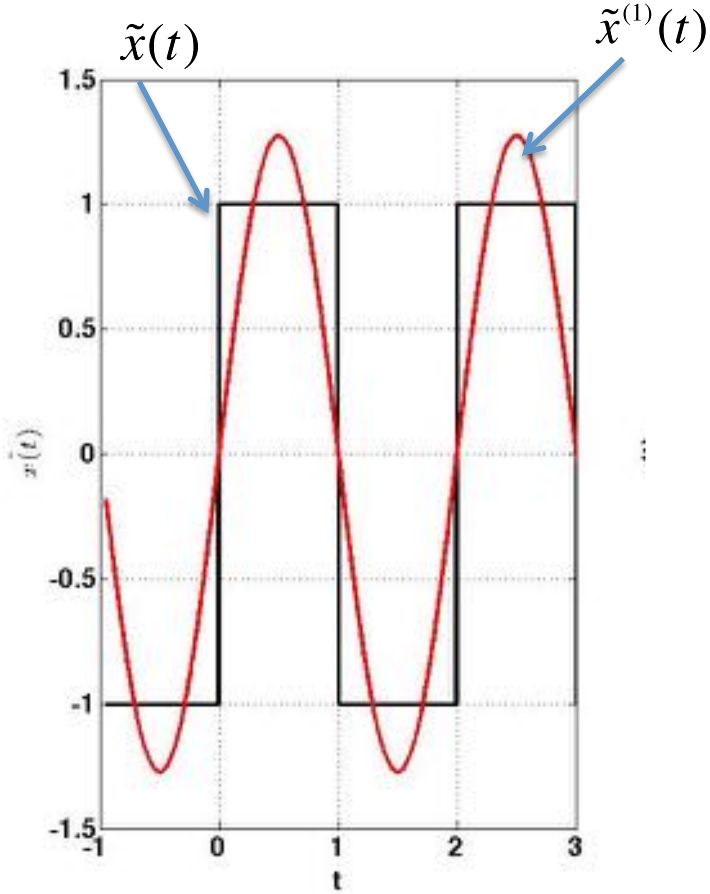
How to understand **Fourier Series** and **Fourier Transform ?**

$$\tilde{x}^{(1)}(t) = b_1 \sin(\omega_0 t)$$

Time "t" is from

-∞ to ∞

we call it time domain



$$b_1 = \frac{4A}{\pi}$$

Only one frequency is used in this example: $\omega_0 = \frac{2\pi}{T_0}$

or $f_0 = 1/T_0$

4.3 Analysis of Non-periodic Continuous-Time Signals

Time Domain

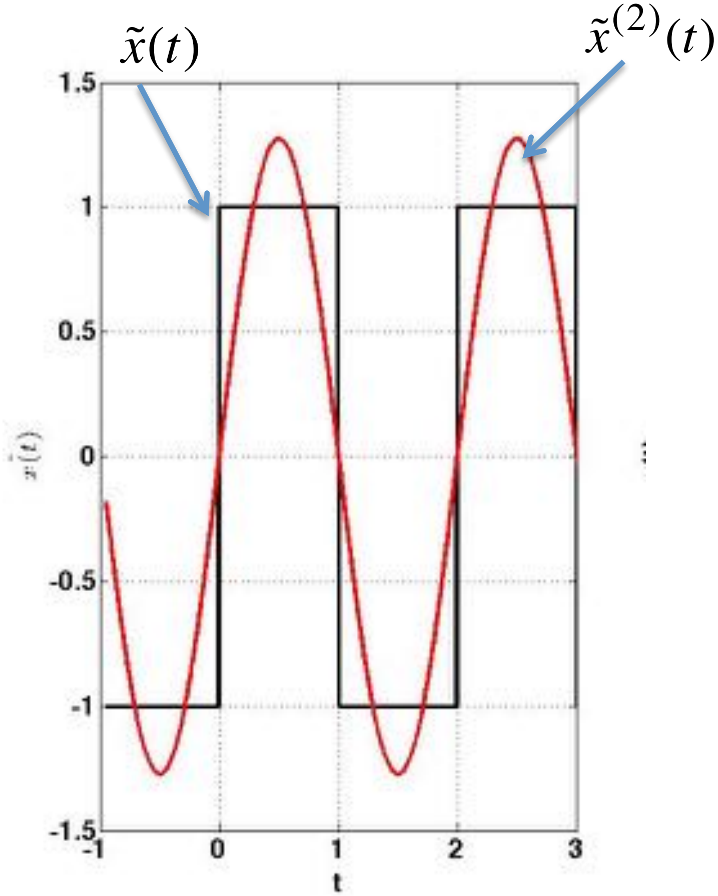
How to understand **Fourier Series** and **Fourier Transform ?**

$$\tilde{x}^{(2)}(t) = b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t)$$

Time "t" is from

-∞ to ∞

we call it time domain



$$b_1 = \frac{4A}{\pi}$$

$$b_2 = 0$$

two frequencies are used in this example: $\omega_0 = \dfrac{2\pi}{T_0}$

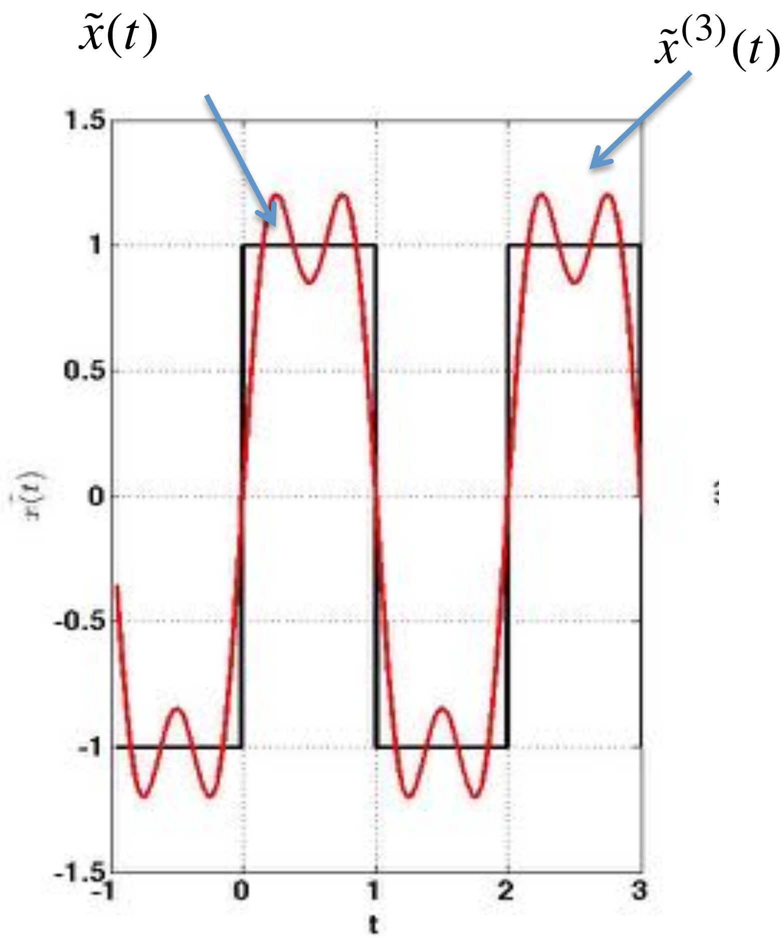
and $2\omega_0$

4.3 Analysis of Non-periodic Continuous-Time Signals

Time Domain

How to understand **Fourier Series** and **Fourier Transform ?**

$$\tilde{x}^{(3)}(t) = b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + b_3 \sin(3\omega_0 t)$$



Time "t" is from
-∞ to ∞
we call it time domain

three frequencies are used in this example: $\omega_0 = \dfrac{2\pi}{T_0}$

and $2\omega_0$ $3\omega_0$

$$b_1 = \frac{4A}{\pi}$$

$$b_2 = 0$$

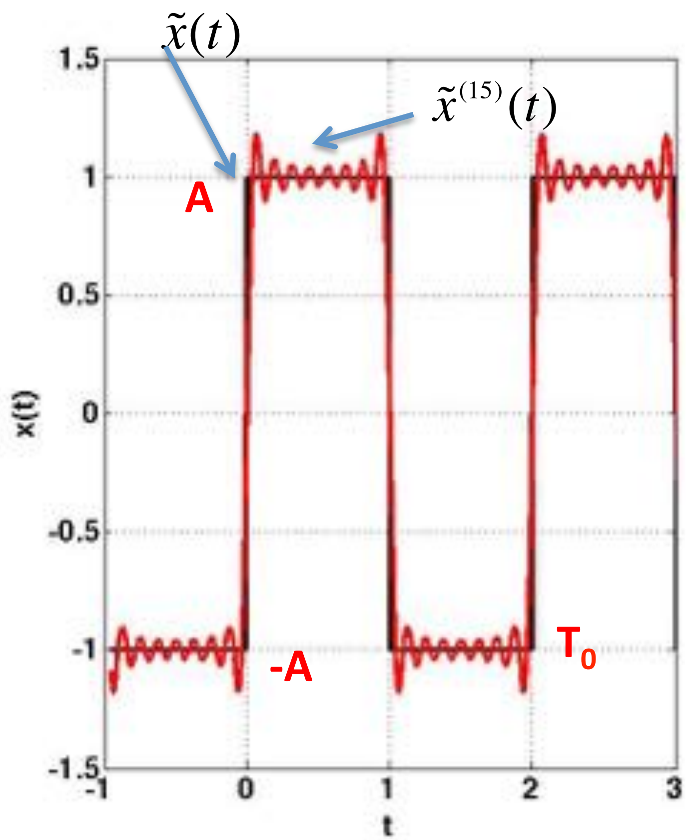
$$b_3 = \frac{4A}{3\pi}$$

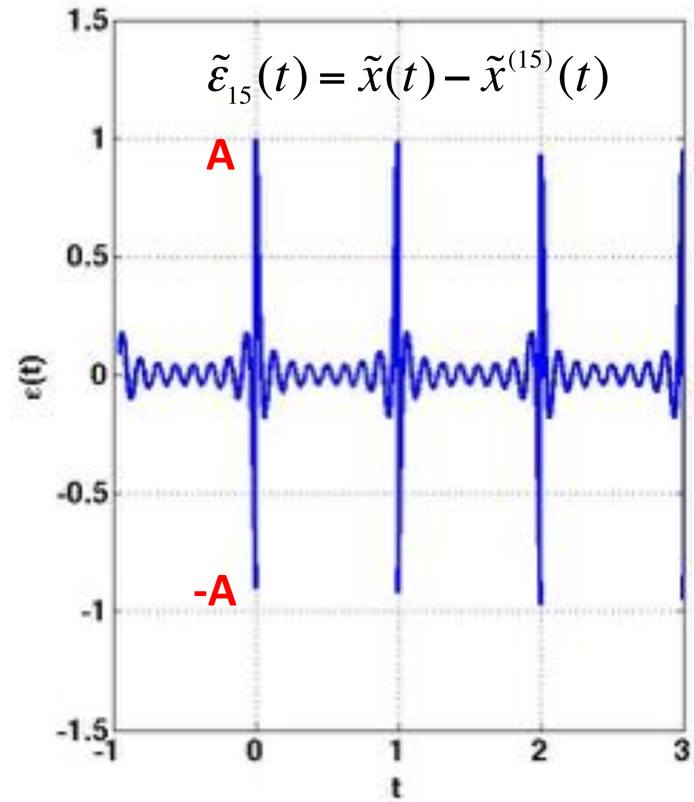
4.2.1 Approximating a periodic signal with trigonometric functions

Let's try a **15-frequency** approximation to $\tilde{x}(t)$ and see if the ***approximate error*** can be reduced.

$$\tilde{x}^{(15)}(t) = b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + \quad . \quad . \quad . \quad + b_{15} \sin(15\omega_0 t)$$

$$\tilde{\varepsilon}_{15}(t) = \tilde{x}(t) - \tilde{x}^{(15)}(t)$$





4.3 Analysis of Non-periodic Continuous-Time Signals

Time Domain

How to understand **Fourier Series** and **Fourier Transform ?**

$$\tilde{x}^{(15)}(t) = b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + \quad . \quad . \quad . \quad + b_{15} \sin(15\omega_0 t)$$

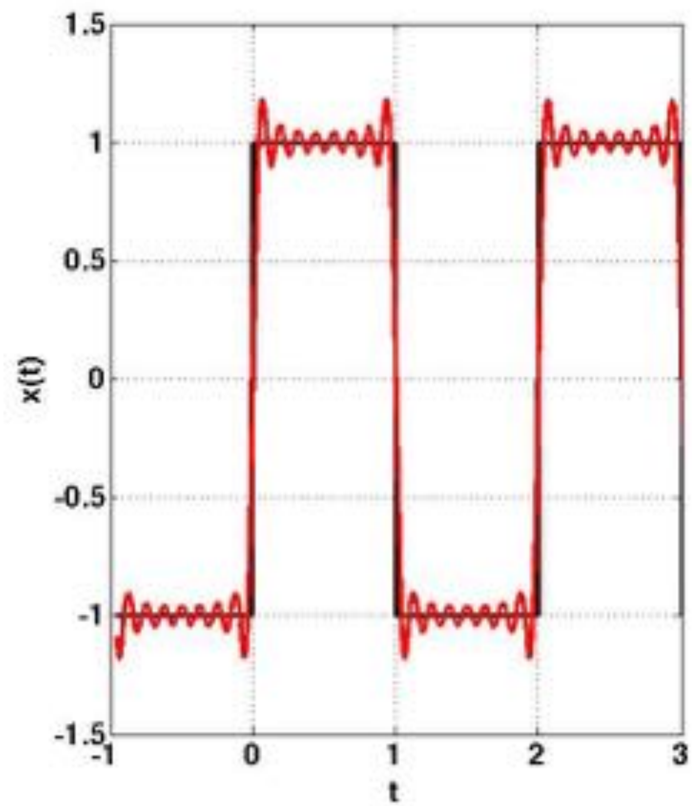
Time "t" is from

-∞ to ∞

we call it time domain

15 frequencies are used in this example: $\omega_0 = \dfrac{2\pi}{T_0}$

and $2\omega_0$ $3\omega_0$, ….



$$b_1 = \frac{4A}{\pi}$$

$$b_2 = 0$$

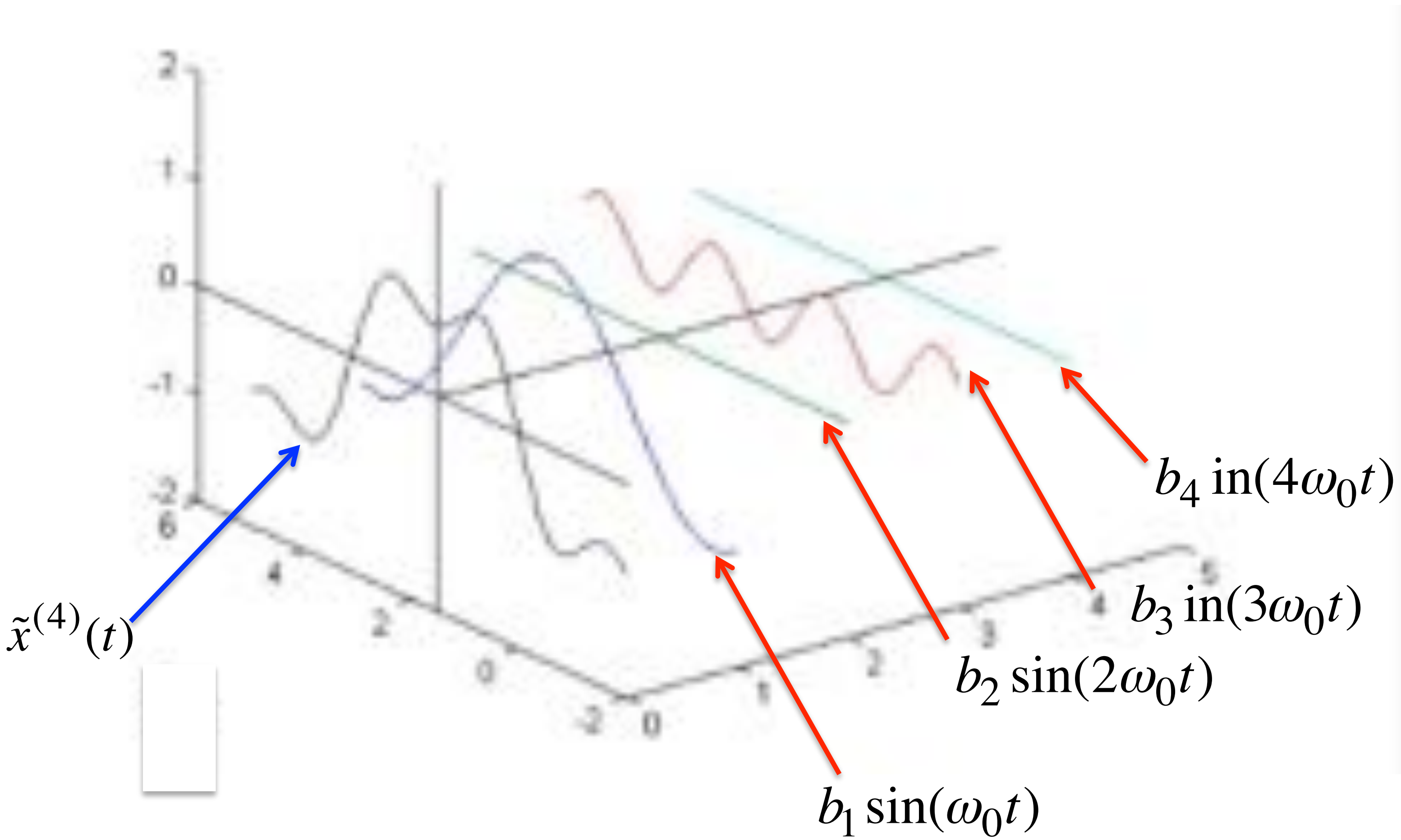
$$b_3 = \frac{4A}{3\pi}$$

$$b_4 = 0$$

$$b_5 = \frac{4A}{5\pi}$$

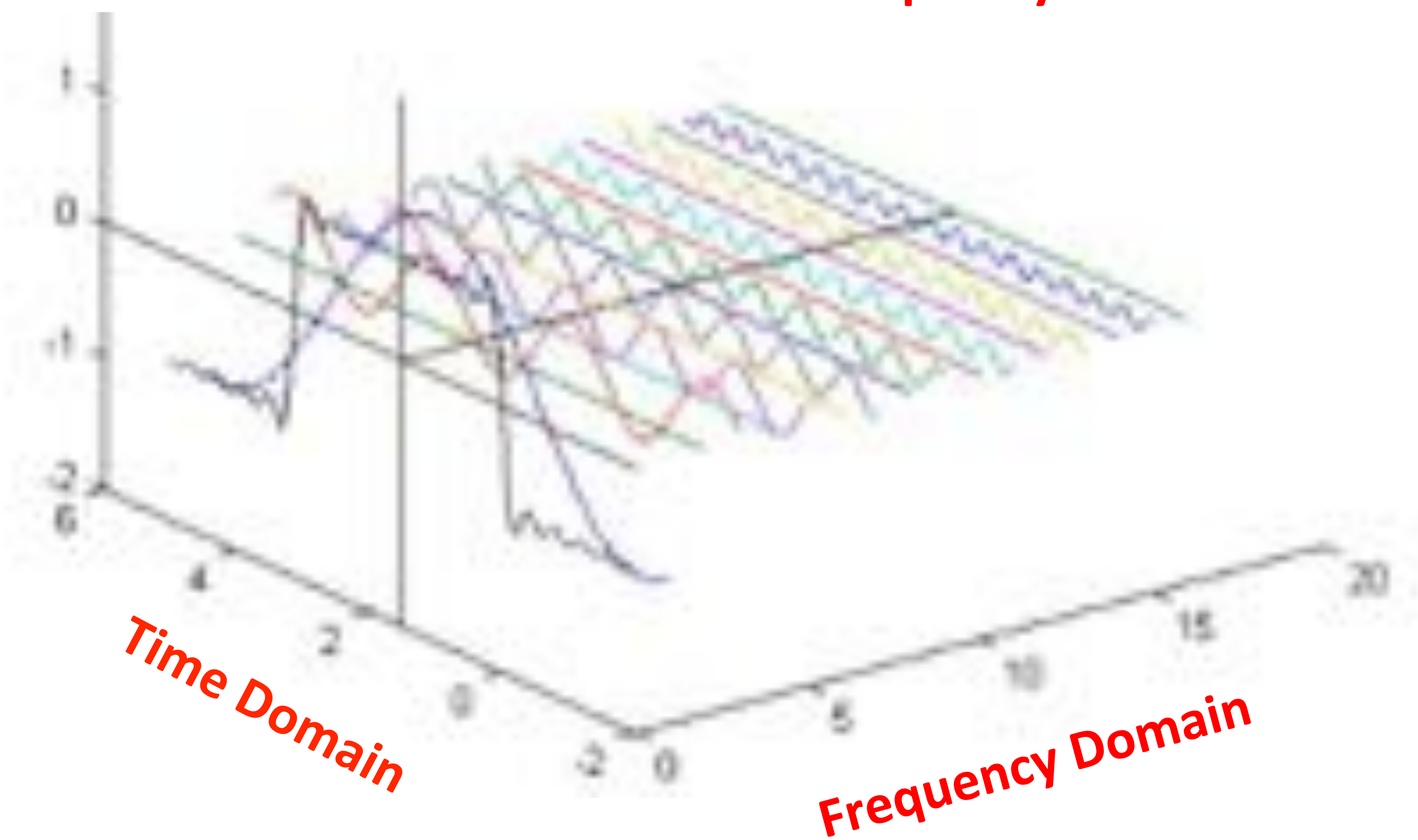
4.3 Analysis of Non-periodic Continuous-Time Signals

Time Domain



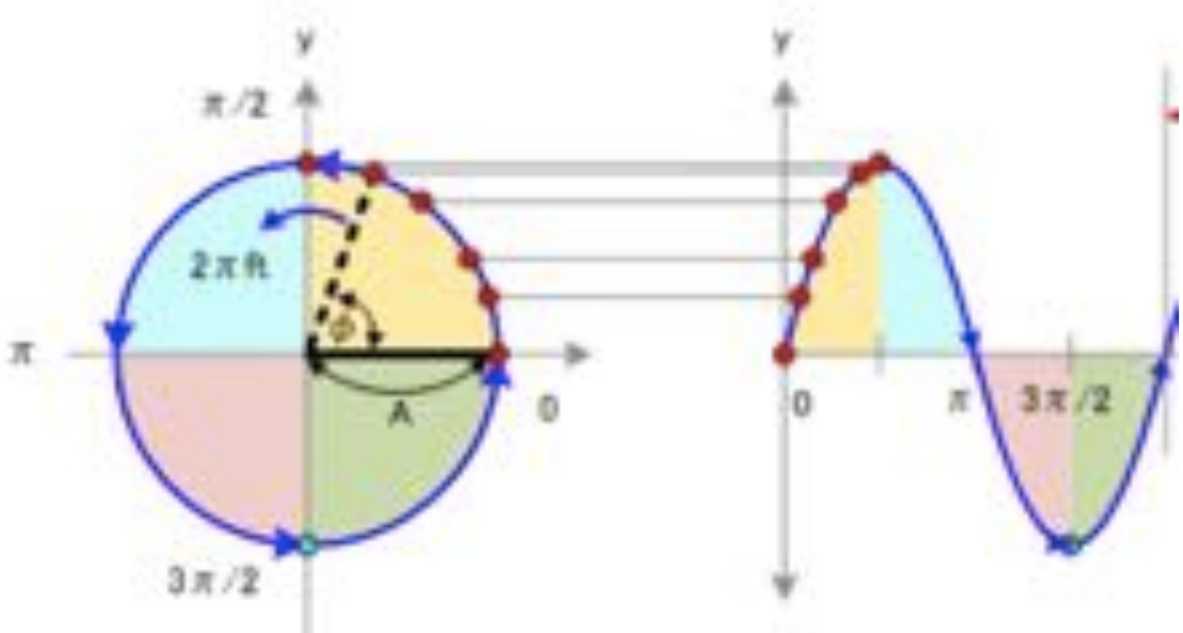
4.3 Analysis of Non-periodic Continuous-Time Signals

Time Domain and Frequency Domain



4.3 Analysis of Non-periodic Continuous-Time Signals

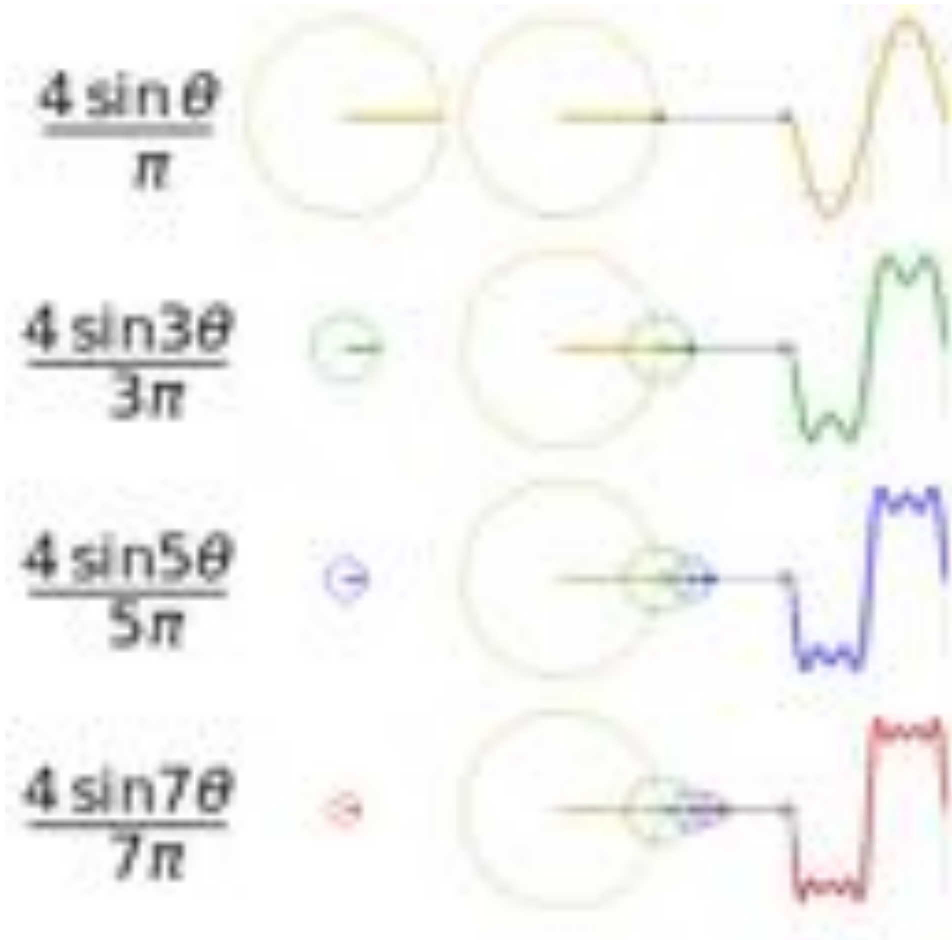
Time Domain and Frequency Domain



sin(x) function can be viewed as a circle projected onto a line

4.3 Analysis of Non-periodic Continuous-Time Signals

Time Domain and Frequency Domain



4.3 Analysis of Non-periodic Continuous-Time Signals

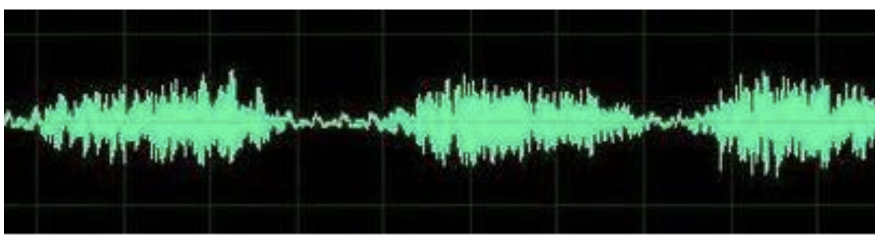
time domain

$x(t)$

frequency domain

$X(f)$

Example of Music



4.3.2 Existence of Fourier Transform

Is it always possible to determine the Fourier series coefficients?

Dirichlet Condition

3F

✧ Finite absolute value: $\int_{-\infty}^{\infty} |x(t)| dt < \infty$

✧ Finite number of discontinuities in $\tilde{x}(t)$

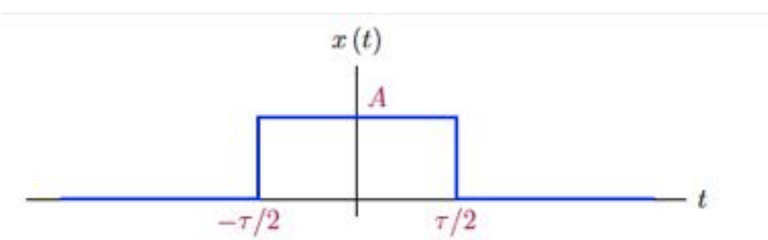
✧ Finite number of minima and maxima in one period

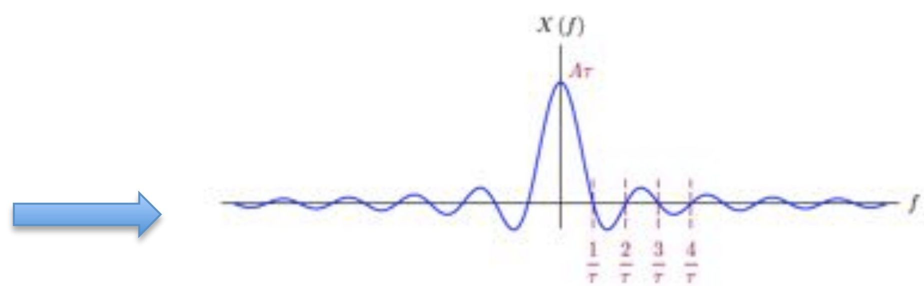
4.3.2 Existence of Fourier Transform

Example 4.12 Fourier Transform of a Rectangular Pulse

Using the forward Fourier transform integral, find the Fourier transform of the isolated rectangular pulse signal

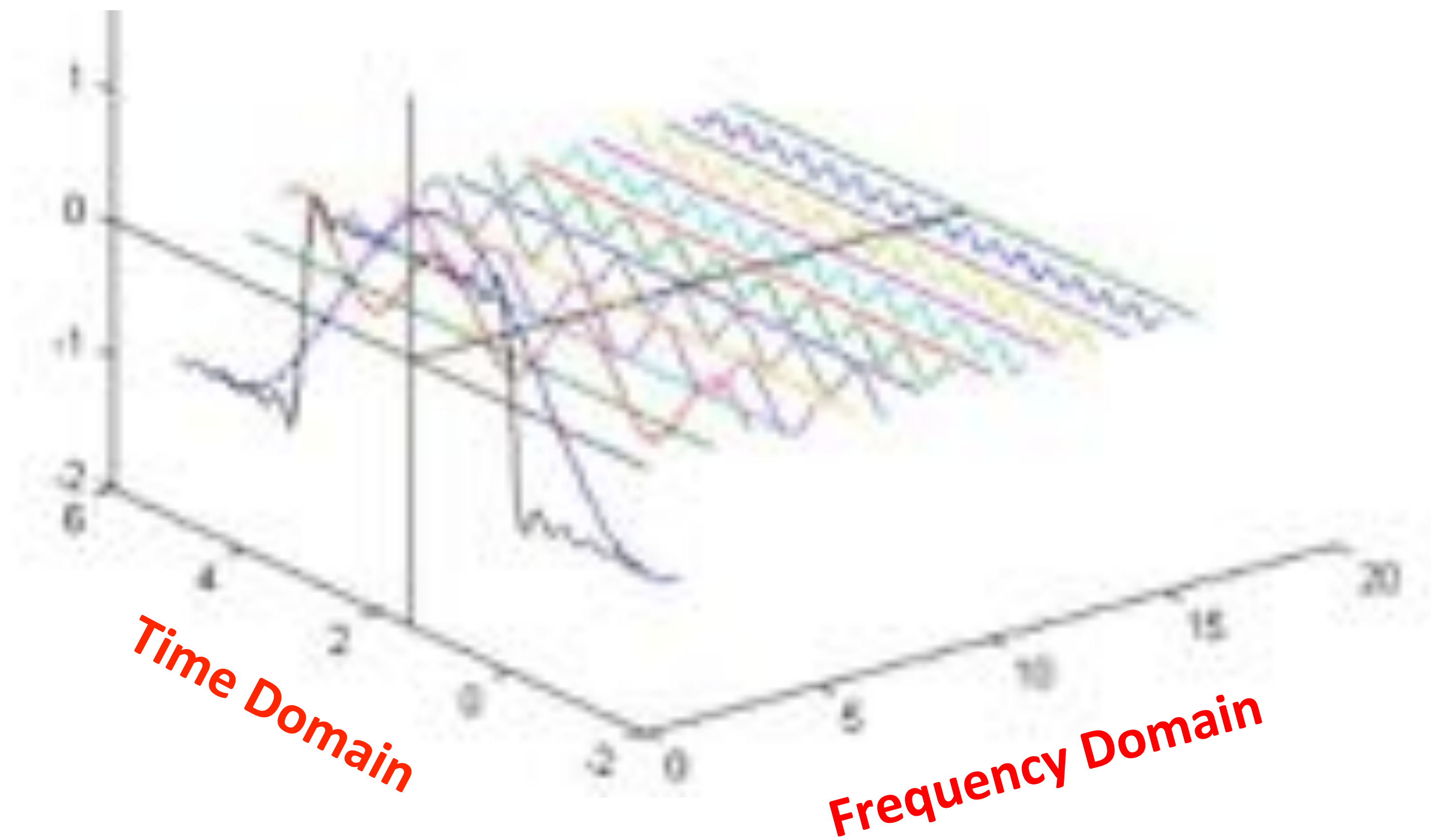
$$x(t) = A\prod\left(\frac{t}{\tau}\right)$$





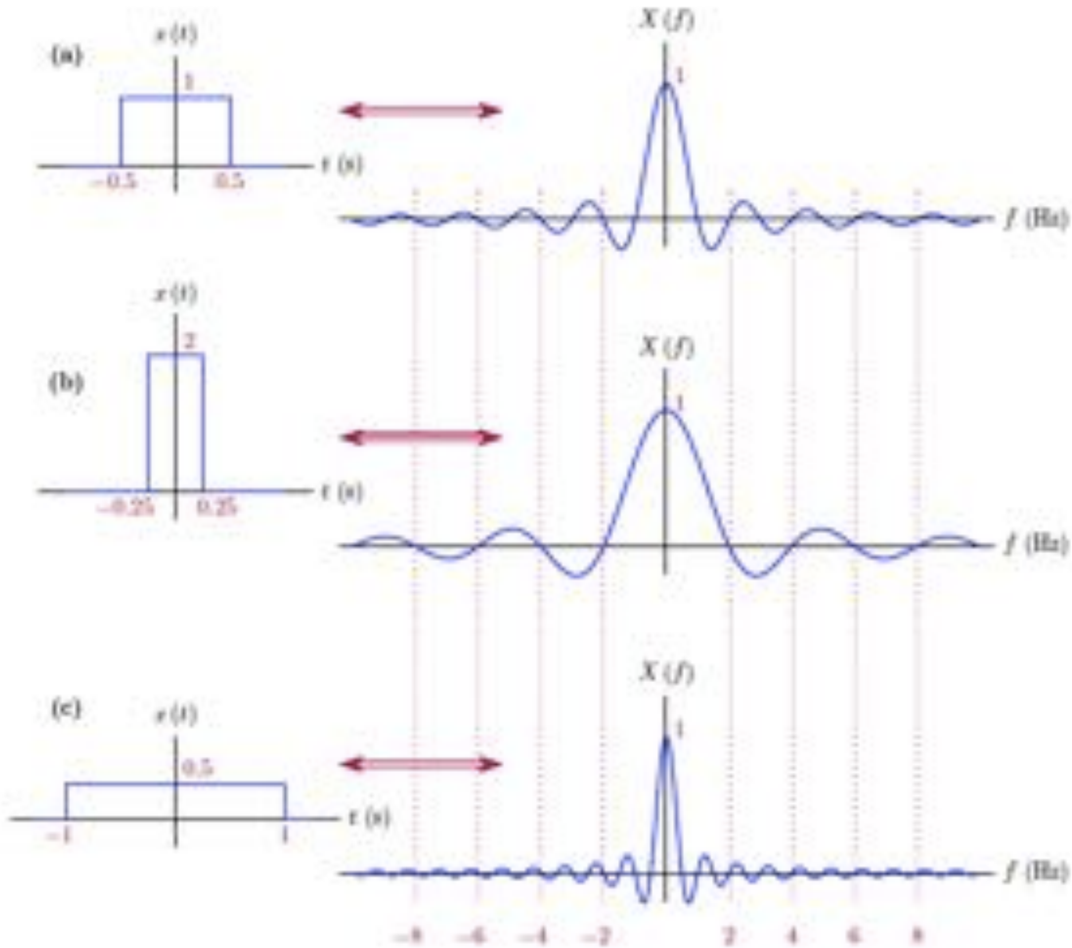
4.3.2 Existence of Fourier Transform

Example 4.12 Fourier Transform of a Rectangular Pulse



4.3.2 Existence of Fourier Transform

Example 4.12 Fourier Transform of a Rectangular Pulse



4.3.5 Properties of Fourier Transform

Linearity:

$$x_1(t) \xleftrightarrow{\Im} X_1(w) \quad \text{and} \quad x_2(t) \xleftrightarrow{\Im} X_2(w)$$

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) \xleftrightarrow{\Im} \alpha_1 X_1(w) + \alpha_2 X_2(w)$$

Where a_1 and a_2 are any two constants

Duality:

$$x(t) \xleftrightarrow{\Im} X(w) \quad \Rightarrow \quad X(t) \xleftrightarrow{\Im} 2\pi x(-w)$$

$$x(t) \xleftrightarrow{\Im} X(f) \quad \Rightarrow \quad X(t) \xleftrightarrow{\Im} x(-f)$$

4.3.5 Properties of Fourier Transform

Symmetry of Fourier Transform:

x(*t*): Real, $\operatorname{Im}\{x(t)\} = 0 \Longrightarrow X^*(w) = X(-w)$

x(*t*): Image, $\operatorname{Re}\{x(t)\} = 0 \Longrightarrow X^*(w) = -X(-w)$

Time Shifting:

$$x(t) \xleftrightarrow{\Im} X(w) \Longrightarrow x(t-\tau) \xleftrightarrow{\Im} X(w)e^{-jw\tau}$$

Frequency Shifting:

$$x(t) \xleftrightarrow{\Im} X(w) \Longrightarrow x(t)e^{-jw_0 t} \xleftrightarrow{\Im} X(w - w_0)$$

4.3.5 Properties of Fourier Transform

Modulation Property:

$$x(t) \overset{\Im}{\longleftrightarrow} X(w)$$

$$x(t)\cos(w_0 t) \overset{\Im}{\longleftrightarrow} \frac{1}{2}\left[X\left(w-w_0\right)+X\left(w+w_0\right)\right]$$

Or $$x(t)\cos(w_0 t) \overset{\Im}{\longleftrightarrow} \frac{1}{2}\left[X\left(f-f_0\right)+X\left(f+f_0\right)\right]$$

$$x(t)\sin(w_0 t) \overset{\Im}{\longleftrightarrow} \frac{1}{2}\left[X\left(w-w_0\right)e^{-j\pi/2}+X\left(w+w_0\right)e^{j\pi/2}\right]$$

More general format

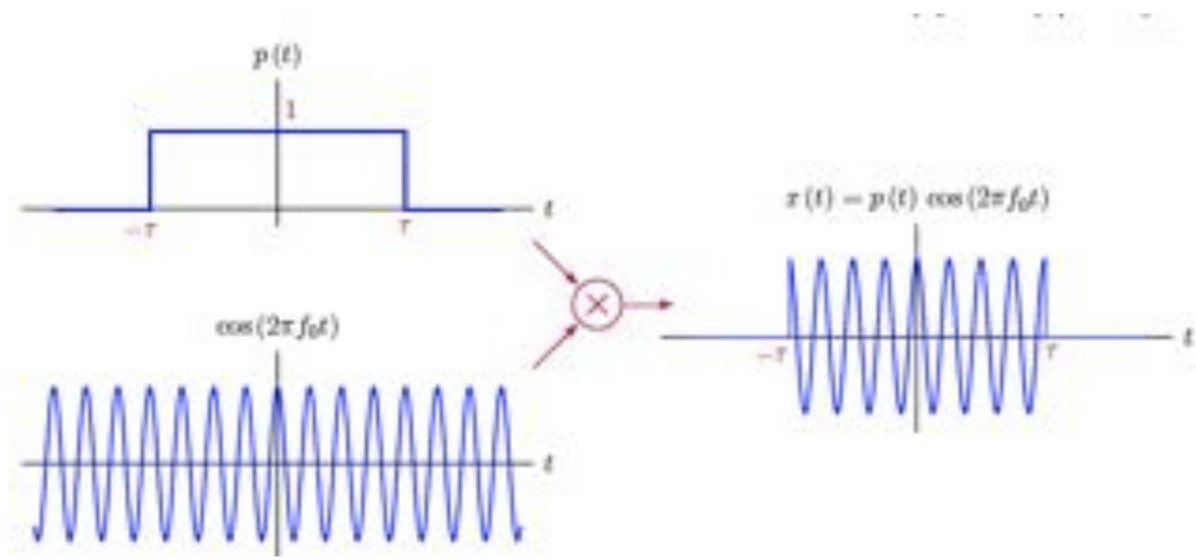
$$x(t)\cos(w_0 t+\theta) \overset{\Im}{\longleftrightarrow} \frac{1}{2}\left[e^{j\theta}X\left(f-f_0\right)+e^{-j\theta}X\left(f+f_0\right)\right]$$

4.3.5 Properties of Fourier Transform

Modulation Property:

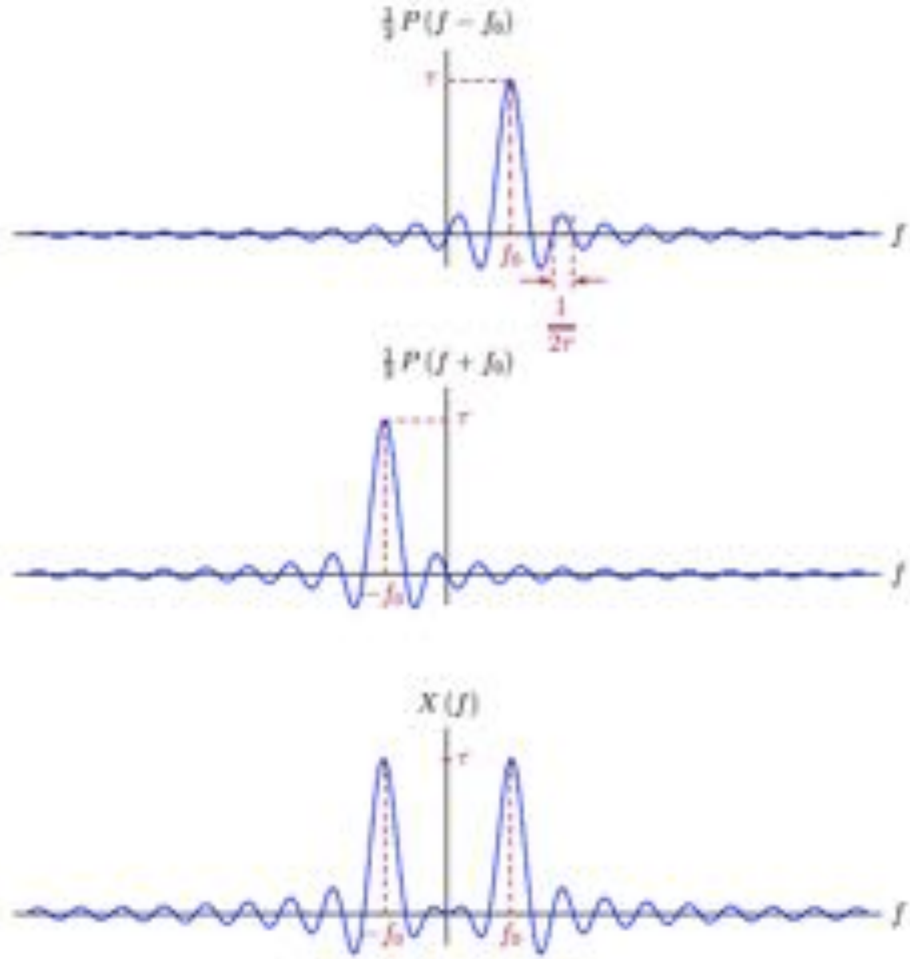
Find the Fourier Transform of the modulated pulse given by

$$x(t) = \begin{cases} \cos(2\pi f_0 t), & |t| < \tau \\ 0, & |t| < \tau \end{cases}$$



4.3.5 Properties of Fourier Transform

Modulation Property:



4.3.5 Properties of Fourier Transform

Convolution Property:

$$x_1(t) \xleftrightarrow{\mathfrak{F}} X_1(w) \qquad x_2(t) \xleftrightarrow{\mathfrak{F}} X_2(w)$$

$$\Downarrow$$

$$x_1(t) * x_2(t) \xleftrightarrow{\mathfrak{F}} X_1(w)X_2(w) \qquad X_1(w) * X_2(w) \xleftrightarrow{\mathfrak{F}} x_1(t)x_2(t)$$