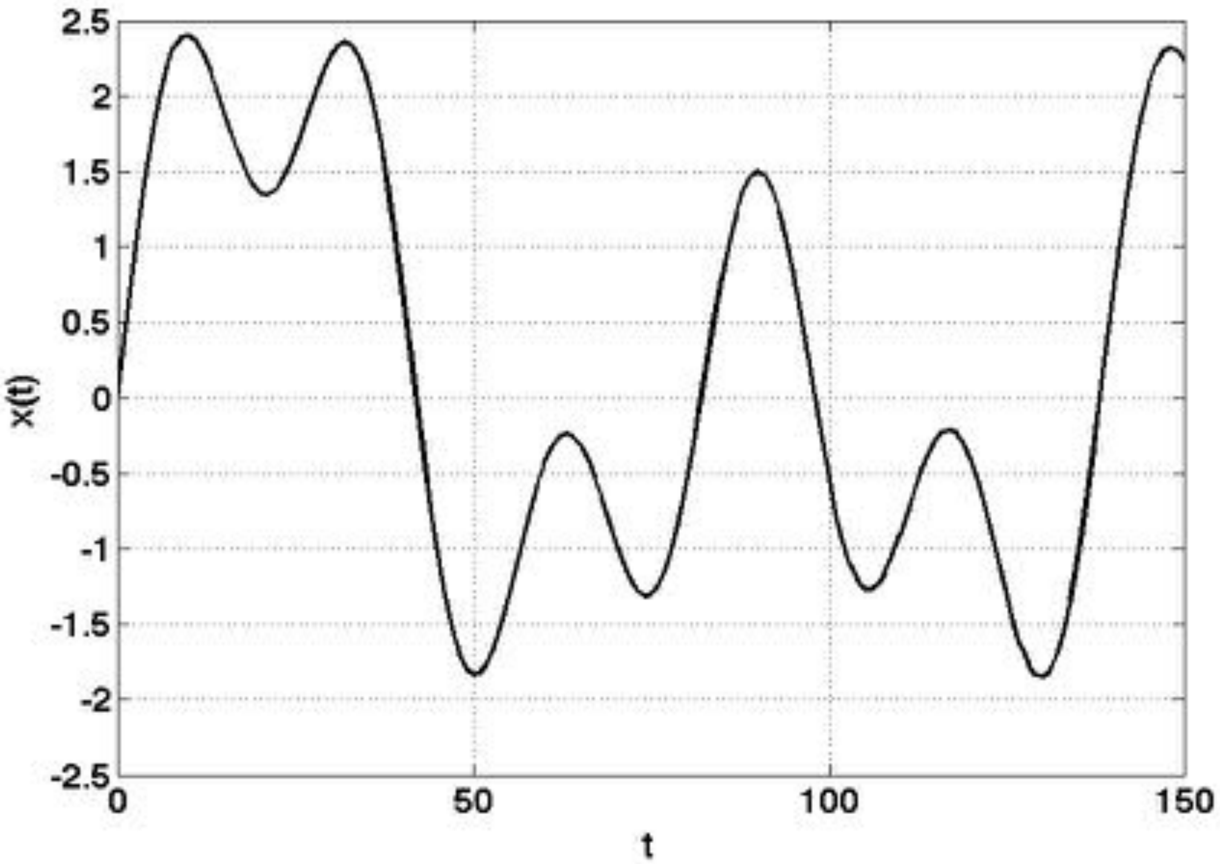
Chapter 4. Fourier Analysis for Continuous-Time Signals and Systems

Chapter Objectives

1. Learn techniques for representing ***continuous-time periodic*** signals using ***orthogonal*** sets of ***periodic basis*** functions.
2. Study properties of ***exponential, trigonometric*** and ***compact Fourier series***, and conditions for their existence.
3. Learn the ***Fourier transform*** for ***non-periodic*** signal as an extension of Fourier series for periodic signals
4. Study the ***properties*** of the Fourier transform. Understand the concepts of ***energy*** and ***power spectral density***.

4.1 Introduction

How to deal with a complicated signal ?

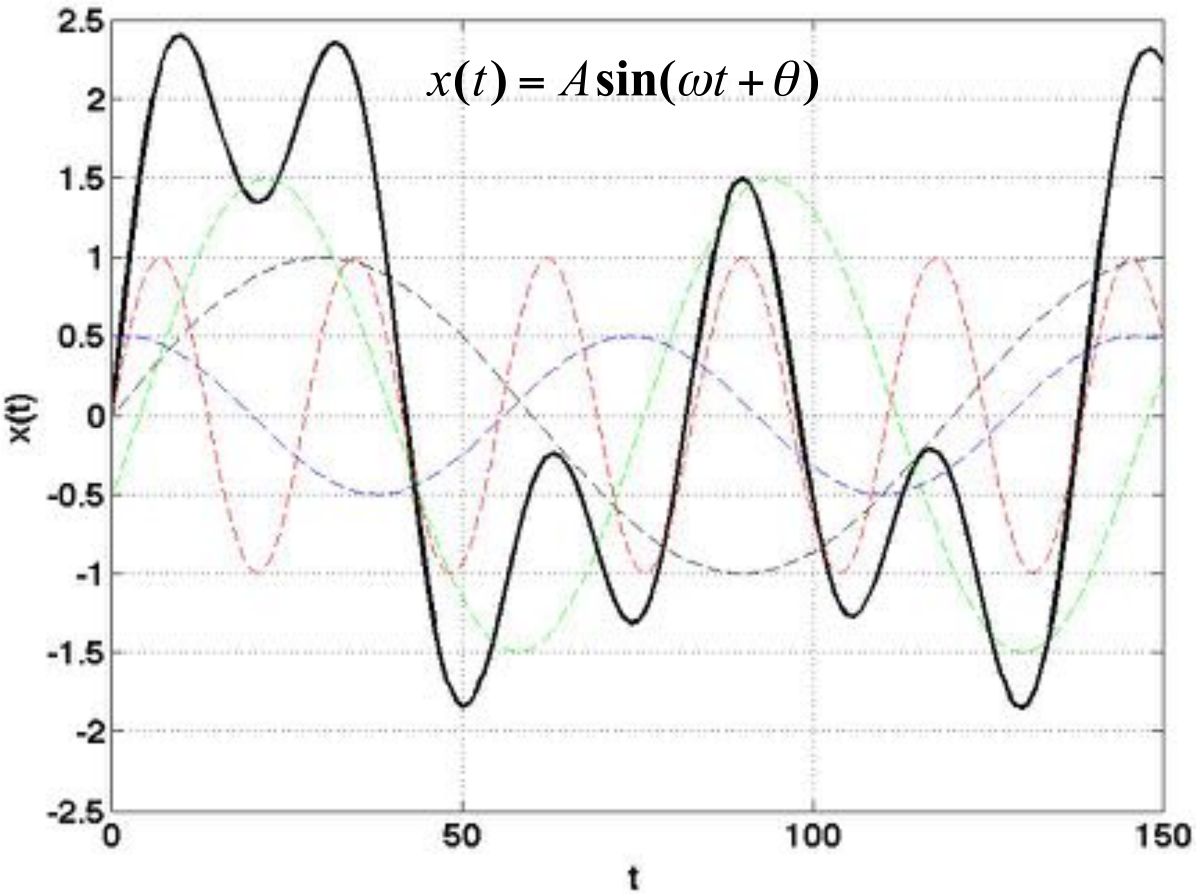


4.1 Introduction

How to deal with a complicated signal ?

Using a series of ***sin(t)***, or ***cos(t)*** functions.

Fourier Series and Fourier Transform

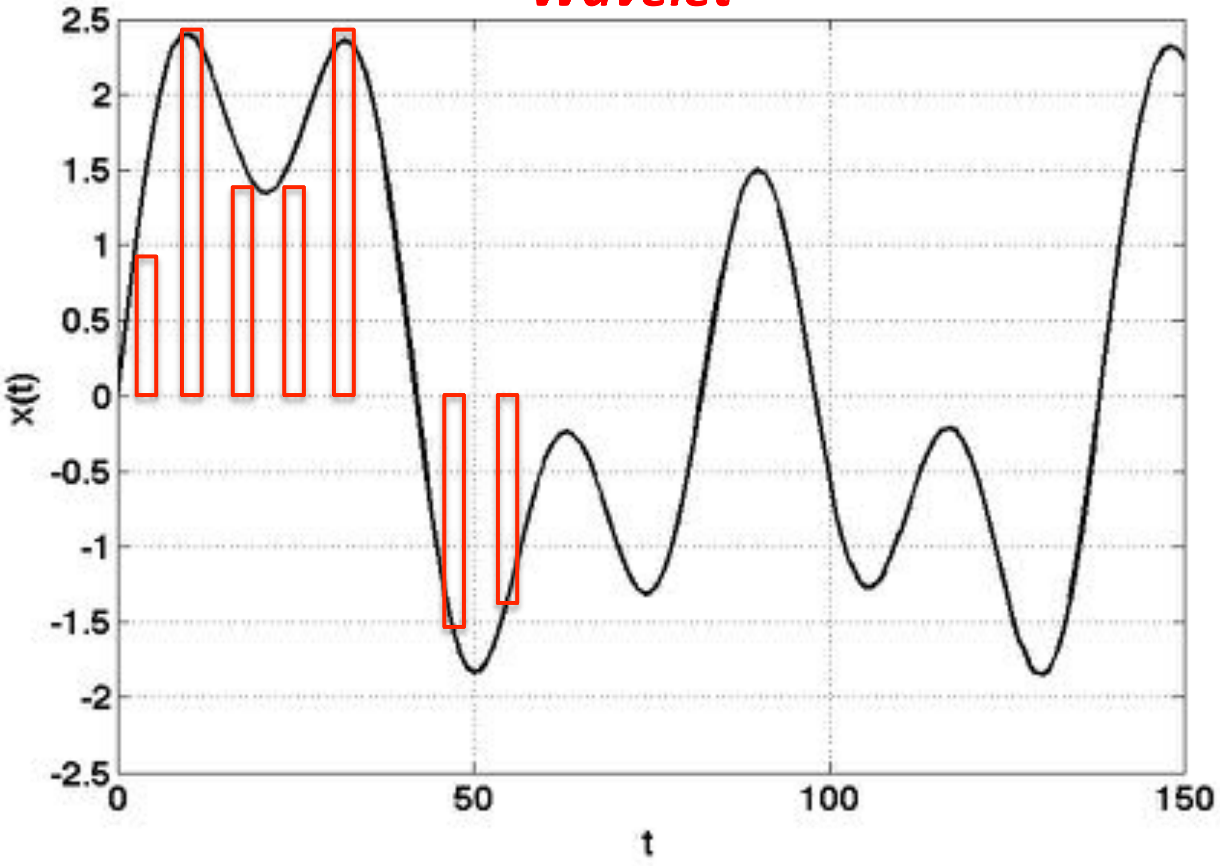


4.1 Introduction

How to deal with a complicated signal ?

Using a series of ***rectangular pulses***

Wavelet



4.1 Introduction

What is Fourier, Fourier Series, and Fourier Transform ??

Fourier is a man, a genius

Name: Jean Baptiste Joseph Fourier

Year: 1768-1830

Nationality: French

Fields: Mathematician, physicist, historian

4.1 Introduction

Fourier Series and Fourier Transformer

A weighted summation of *Sines* and *Cosines* of different frequencies can be used to represent periodic (***Fourier Series***), or non-periodic (***Fourier Transform***) functions.

Is this true?

People didn't believe that, including Lagrange, Laplace, Poisson, and other big wigs.

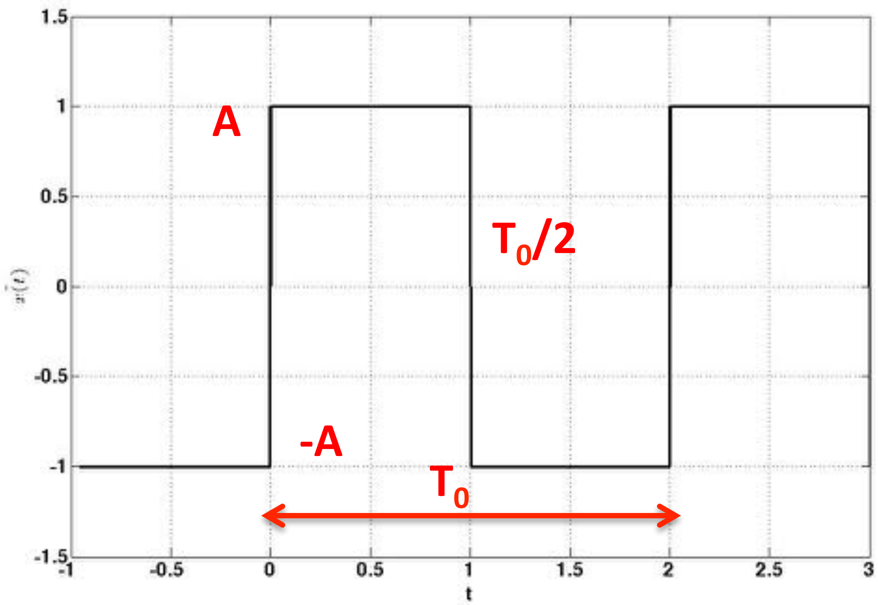
But, yes, this is true, this is great !!

4.2.1 Approximating a periodic signal with trigonometric functions

For a periodic signal $\tilde{x}(t)$ which is periodic with period T_0 has the property

$$\tilde{x}(t+T) = \tilde{x}(t)$$

for all t



Periodic square-wave signal

4.2.1 Approximating a periodic signal with trigonometric functions

Suppose that we wish to approximate this signal using just one trigonometric function

1.) **Q**: Should we use a ***sine*** or a ***cosine*** function?

A: choose ***sine*** function. Because both $\tilde{x}(t)$ and a ***sine*** function ($b_1 \sin(\omega t)$) are odd symmetry. The fundamental period of ***sine*** function is $\mathbf{T_0}$ as $\tilde{x}(t)$.

$$\tilde{x}^{(1)}(t) = b_1 \sin(\omega_0 t)$$

2.) **Q**: How should we adjust the parameters of the trigonometric function?

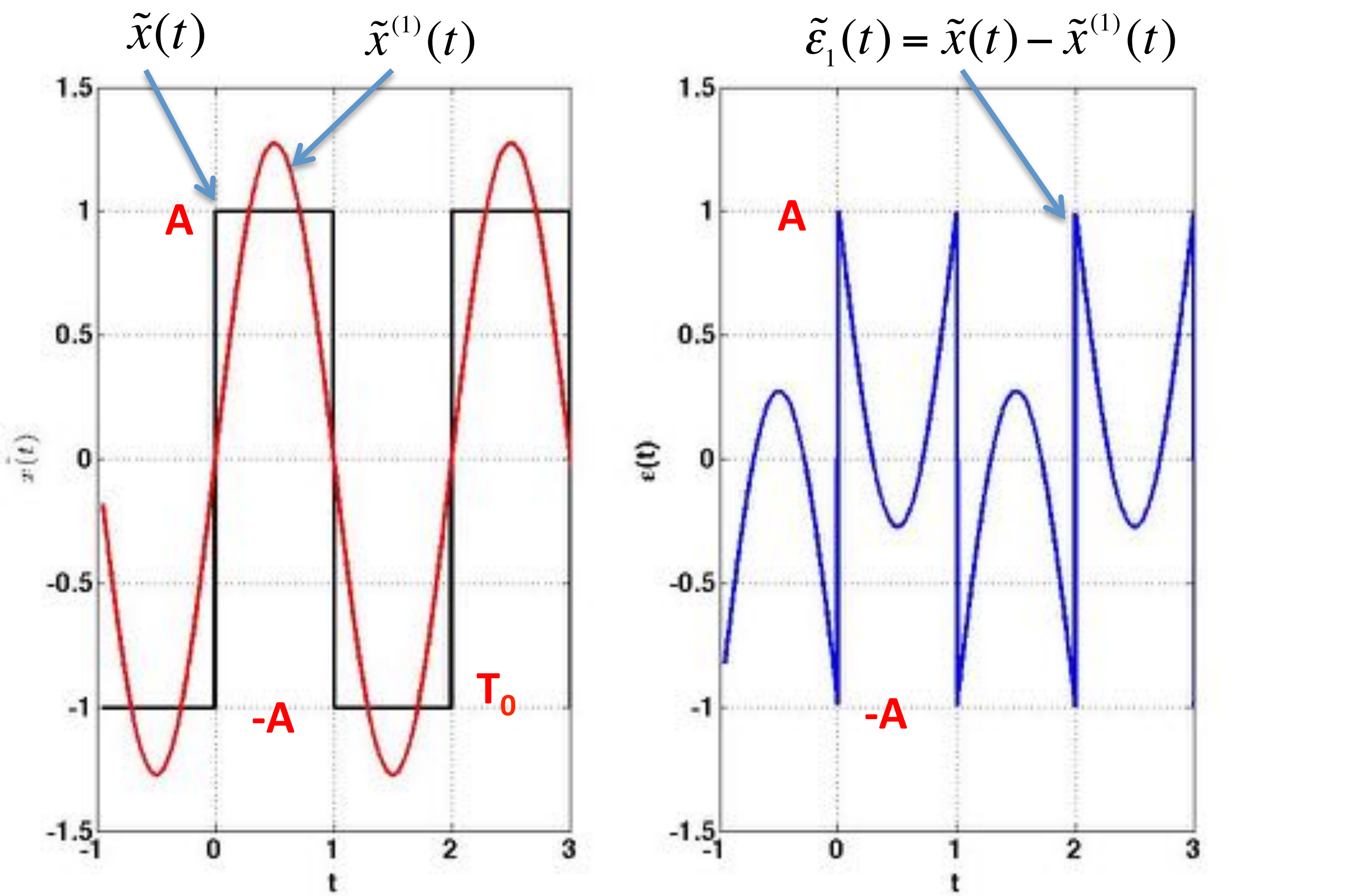
A: find the parameters that can minimize the ***approximation error*** function.

$$\tilde{\varepsilon}_1(t) = \tilde{x}(t) - \tilde{x}^{(1)}(t) = \tilde{x}(t) - b_1 \sin(\omega_0 t)$$

4.2.1 Approximating a periodic signal with trigonometric functions

The best approximation to $\tilde{x}(t)$ using only one trigonometric function is

$$\tilde{x}^{(1)}(t) = b_1 \sin(\omega_0 t) \qquad \omega_0 = \frac{2\pi}{T_0} \qquad b_1 = \frac{4A}{\pi}$$



4.2.1 Approximating a periodic signal with trigonometric functions

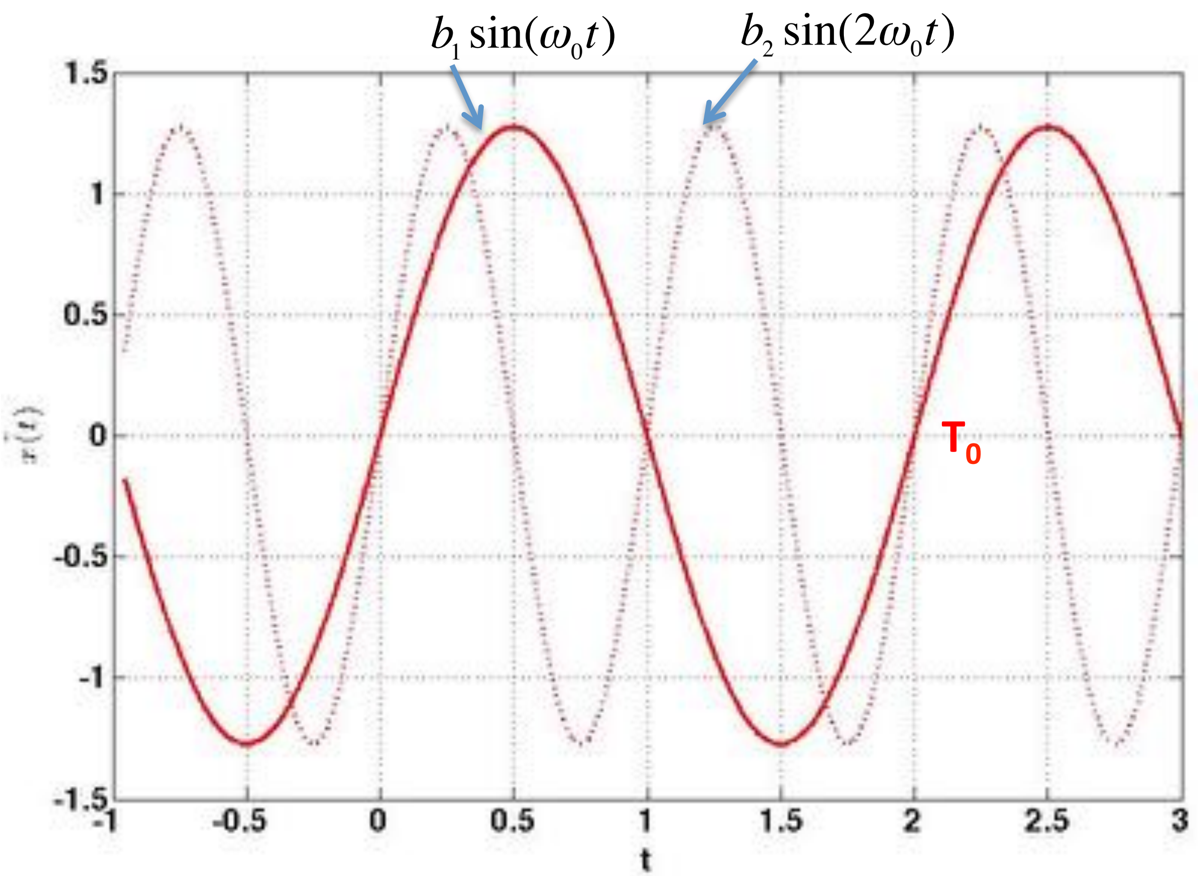
Let's try a two-frequency approximation to $\tilde{x}(t)$ and see if the ***approximate error*** can be reduced.

$$\tilde{x}^{(2)}(t) = b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t)$$

$$\tilde{\varepsilon}_2(t) = \tilde{x}(t) - \tilde{x}^{(2)}(t) = \tilde{x}(t) - b_1 \sin(\omega_0 t) - b_2 \sin(2\omega_0 t)$$

$$b_1 = \frac{4A}{\pi}$$

$$b_2 = 0$$



4.2.1 Approximating a periodic signal with trigonometric functions

Let's try a three-frequency approximation to $\tilde{x}(t)$ and see if the ***approximate error*** can be reduced.

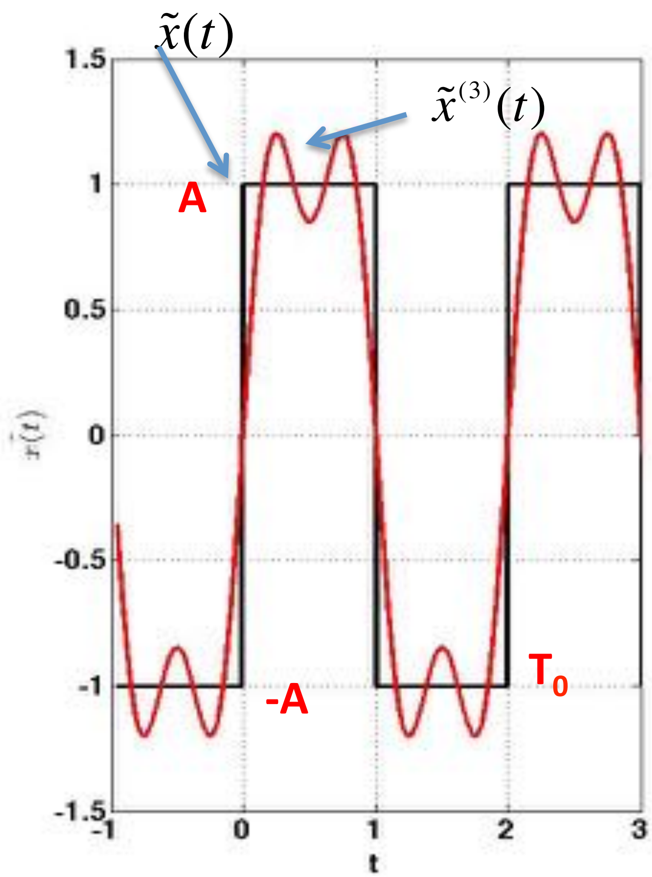
$$\tilde{x}^{(3)}(t) = b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + b_3 \sin(3\omega_0 t)$$

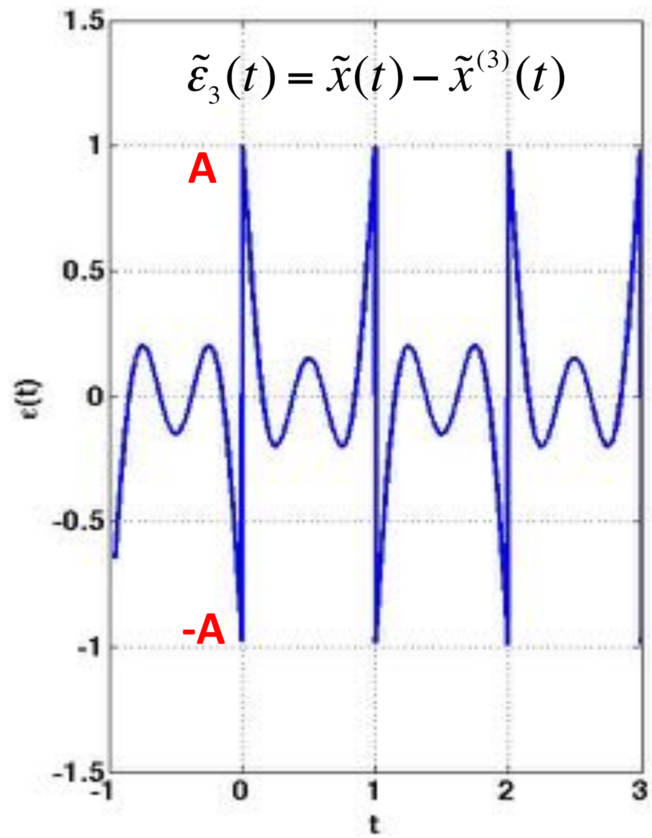
$$\tilde{\varepsilon}_3(t) = \tilde{x}(t) - \tilde{x}^{(3)}(t) = \tilde{x}(t) - b_1 \sin(\omega_0 t) - b_2 \sin(2\omega_0 t) - b_3 \sin(3\omega_0 t)$$

$$b_1 = \frac{4A}{\pi}$$

$$b_2 = 0$$

$$b_3 = \frac{4A}{3\pi}$$



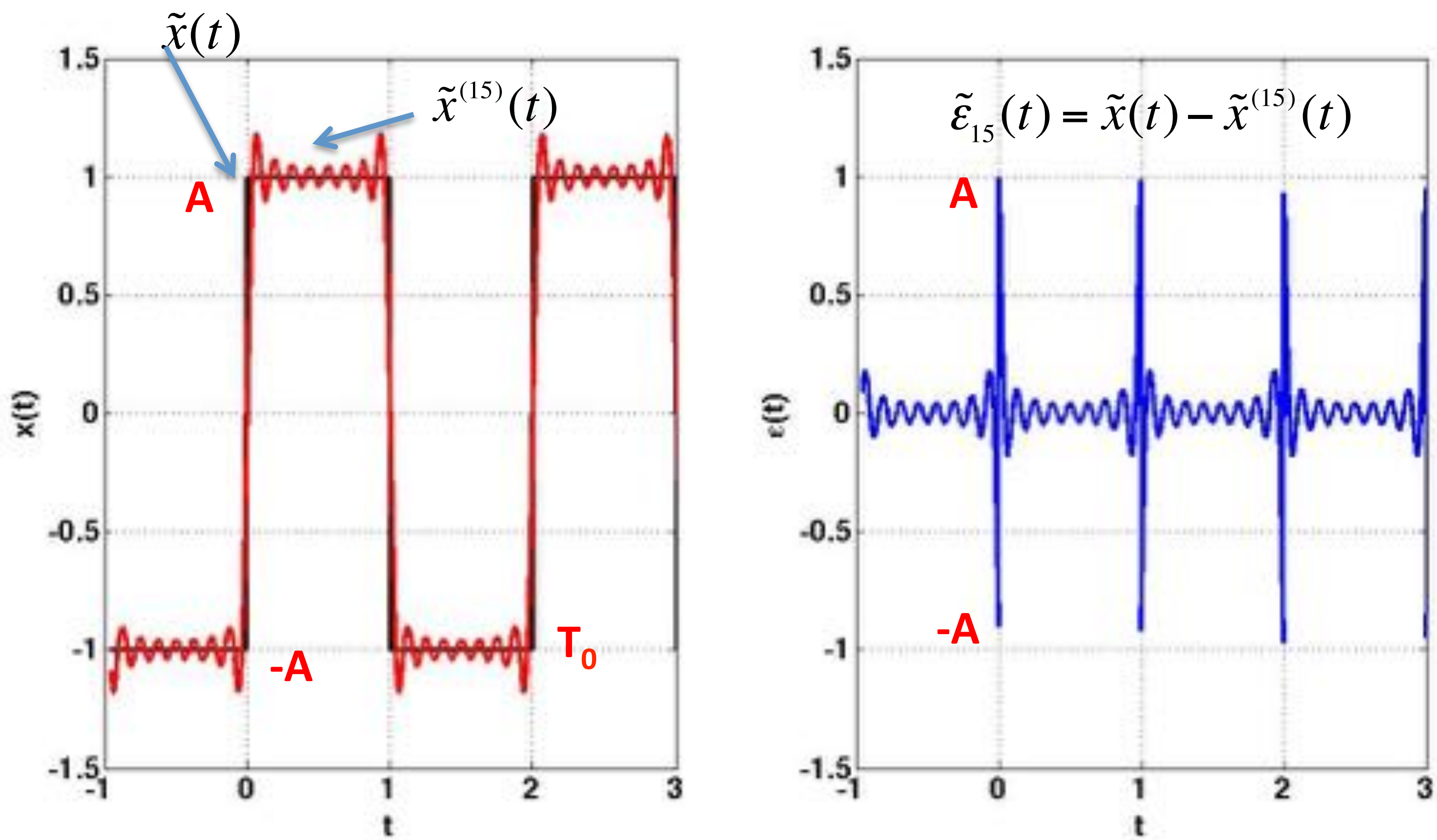


4.2.1 Approximating a periodic signal with trigonometric functions

Let's try a **15-frequency** approximation to $\tilde{x}(t)$ and see if the ***approximate error*** can be reduced.

$$\tilde{x}^{(15)}(t) = b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + \quad . \quad . \quad . \quad + b_{15} \sin(15\omega_0 t)$$

$$\tilde{\varepsilon}_{15}(t) = \tilde{x}(t) - \tilde{x}^{(15)}(t)$$



4.2.1 Approximating a periodic signal with trigonometric functions

We can draw some conclusion based on previous observations:

✧ The normalized average power of the error signal $\tilde{\varepsilon}_3(t)$ seems to be less than that of the error $\tilde{\varepsilon}_1(t)$

✧ Consequently, $\tilde{x}^{(3)}(t)$ is a better approximation to the signal than $\tilde{x}^{(1)}(t)$

✧ On the other hand, the peak value of the approximate error seems to be $\pm A$ for both $\tilde{\varepsilon}_1(t)$ and $\tilde{\varepsilon}_3(t)$. If we try higher-order approximations using more trigonometric basic function, the peak approximation error would still be $\pm A$.

4.2.2 Trigonometric Fourier Series (TFS)

We may want to represent this signal using a linear combination of sinusoidal functions in the form:

$$\tilde{x}(t) = a_0 + a_1 \cos(\omega_0 t) + a_2 \cos(2\omega_0 t) + \ldots + a_k \cos(k\omega_0 t) + \ldots$$
$$+ b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + \ldots + b_k \sin(k\omega_0 t) + \ldots$$

In a compact notation (trigonometric Fourier Series):

$$\tilde{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kw_0 t) + \sum_{k=1}^{\infty} b_k \sin(kw_0 t)$$

Where $\omega_0 = 2\pi f_0$ is the **fundamental frequency** in rad/s.

and the sinusoidal functions with radian frequency $\omega_0, 2\omega_0, \ldots k\omega_0$ are referred to as the basis functions ($\cos(k\omega_0 t)$, $\sin(k\omega_0 t)$)

4.2.2 Trigonometric Fourier Series (TFS)

Synthesis equation:

$$\tilde{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kw_0 t) + \sum_{k=1}^{\infty} b_k \sin(kw_0 t)$$

We need to determine the coefficients: $\boldsymbol{a_0}$, $\boldsymbol{a_k}$, and $\boldsymbol{b_k}$

4.2.2 Trigonometric Fourier Series (TFS)

Useful orthogonal sets:

$$\int_{t_0}^{t_0+T_0} \cos(m\omega_0 t)\cos(k\omega_0 t)dt = \begin{cases} \frac{T_0}{2}, & m = k \\ 0, & m \neq k \end{cases}$$

$$\int_{t_0}^{t_0+T_0} \sin(m\omega_0 t)\sin(k\omega_0 t)dt = \begin{cases} \frac{T_0}{2}, & m = k \\ 0, & m \neq k \end{cases}$$

$$\int_{t_0}^{t_0+T_0} \cos(m\omega_0 t)\sin(k\omega_0 t)dt = 0$$

4.2.2 Trigonometric Fourier Series (TFS)

Synthesis equation:

$$\tilde{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kw_0 t) + \sum_{k=1}^{\infty} b_k \sin(kw_0 t)$$

Analysis equation:

$$a_k = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} \tilde{x}(t) \cos(kw_0 t)\, dt, \quad for \quad k = 1, \ldots, \infty$$

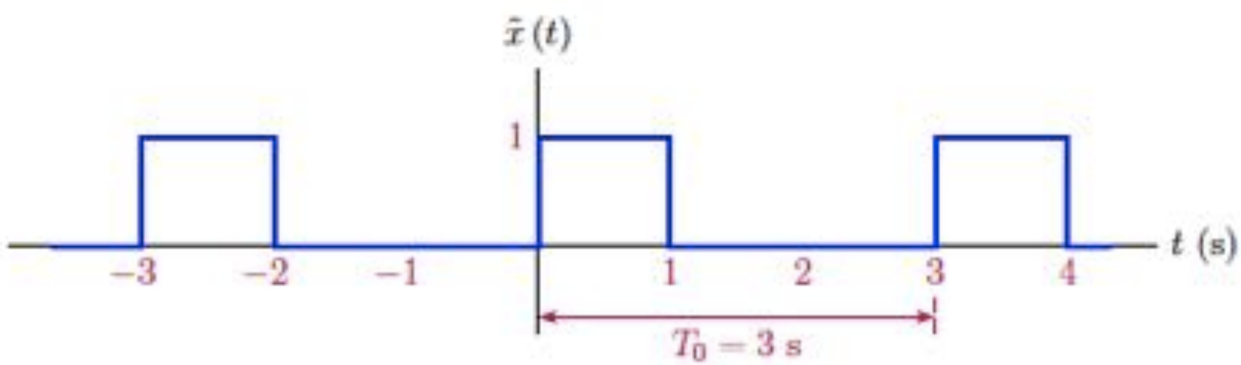
$$b_k = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} \tilde{x}(t) \sin(kw_0 t)\, dt, \quad for \quad k = 1, \ldots, \infty$$

$$a_0 = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} \tilde{x}(t)\, dt$$ (dc component)

4.2.2 Trigonometric Fourier Series (TFS)

<u>**Example 4.1** Trigonometric Fourier Series of a Periodic Pulse Train</u>

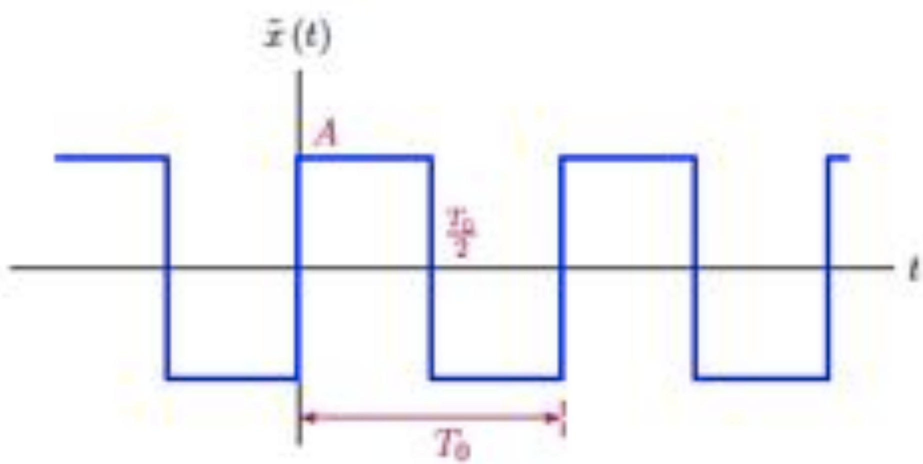
A pulse-train signal $\tilde{x}(t)$ with a period of T_0 = 3 seconds is shown as below. Determine the coefficients of the TFS representation of this signal.



4.2.2 Trigonometric Fourier Series (TFS)

Example 4.4 Trigonometric Fourier Series of a Square Wave

Determine the TFS for the periodic square wave shown as below.



4.2.3 Exponential Fourier Series (EFS)

TFS $$\tilde{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kw_0 t) + \sum_{k=1}^{\infty} b_k \sin(kw_0 t)$$

$$\mathbf{cos}(k\omega_0 t) = \frac{1}{2}\left(e^{jkw_0 t} + e^{-jkw_0 t}\right)$$

$$\mathbf{sin}(k\omega_0 t) = \frac{1}{2j}\left(e^{jkw_0 t} - e^{-jkw_0 t}\right)$$

EFS $$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

4.2.3 Exponential Fourier Series (EFS)

General Format of Exponential Fourier Series

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

EXAMPLES

Single-tone Signals

$$\begin{aligned}\tilde{x}(t) &= A\cos(\omega_0 t + \theta) \\ &= \frac{A}{2}\left(e^{j(\omega_0 t+\theta)} + e^{-j(\omega_0 t+\theta)}\right) \\ &= \frac{A}{2}\left(e^{j\theta}e^{j\omega_0 t} + e^{-j\theta}e^{-j\omega_0 t}\right)\end{aligned}$$

Comparing with general format

$$c_1 = \frac{A}{2}e^{j\theta} \qquad c_{-1} = \frac{A}{2}e^{-j\theta}$$

4.2.3 Exponential Fourier Series (EFS)

General Format of Exponential Fourier Series

EFS

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

TFS

$$\tilde{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kw_0 t) + \sum_{k=1}^{\infty} b_k \sin(kw_0 t)$$

$$c_k = \frac{1}{2}\left(a_k - jb_k\right)$$

4.2.3 Exponential Fourier Series (EFS)

General Format of Exponential Fourier Series

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

We need to find out coefficient *C_k*

4.2.3 Exponential Fourier Series (EFS)

Useful orthogonal set

$$\int_{t_0}^{t_0+T_0} e^{jm\omega_0 t} e^{-jk\omega_0 t}\, dt = \begin{cases} T_0, & m = k \\ 0, & m \neq k \end{cases}$$

4.2.3 Exponential Fourier Series (EFS)

Synthesis equation:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

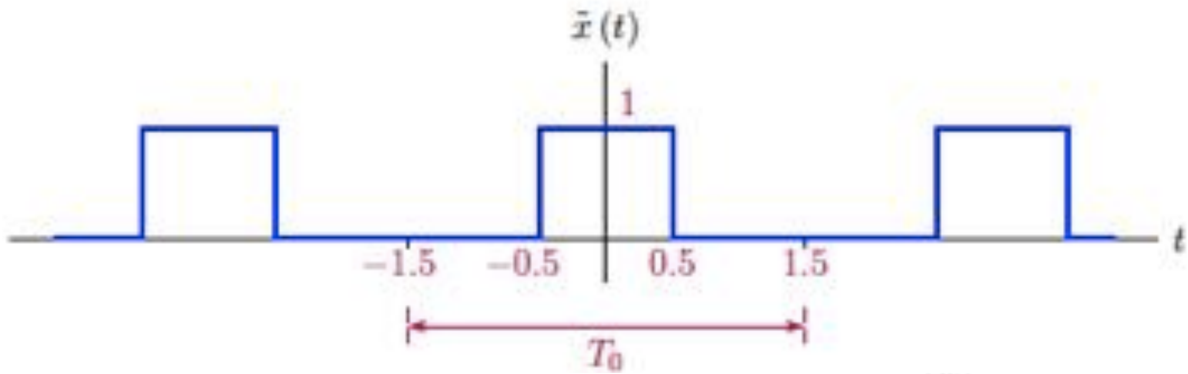
Analysis equation:

$$c_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} \tilde{x}(t) e^{-jk\omega_0 t}\, dt$$

4.2.3 Exponential Fourier Series (EFS)

Example 4.5 Exponential Fourier Series For a Periodic Pulse Train

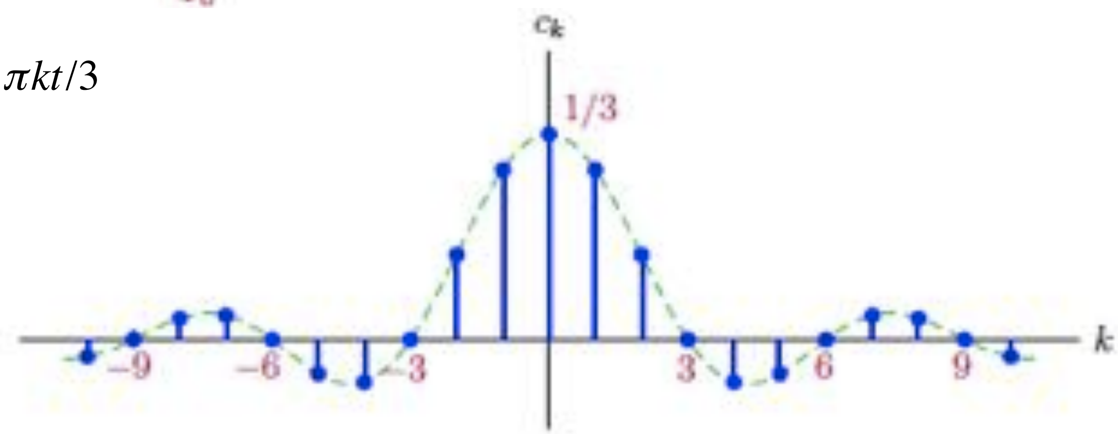
Determine the EFS for the periodic square wave shown as below.



$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \left(\frac{\sin(k\pi/3)}{\pi k} \right) e^{j2\pi kt/3}$$

$$c_k = \frac{\sin(k\pi/3)}{\pi k}$$

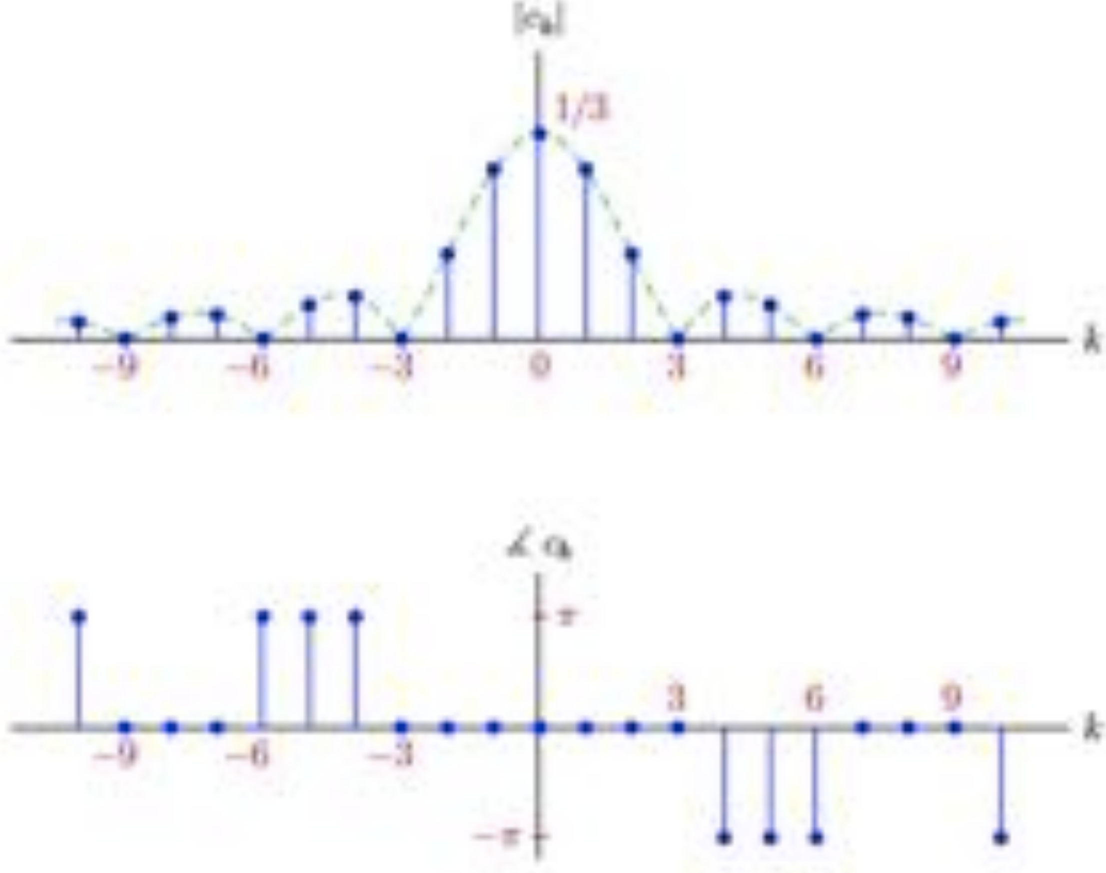
$$c_0 = 1/3$$



4.2.3 Exponential Fourier Series (EFS)

Example 4.5 Exponential Fourier Series For a Periodic Pulse Train

Determine the EFS for the periodic square wave shown as below.

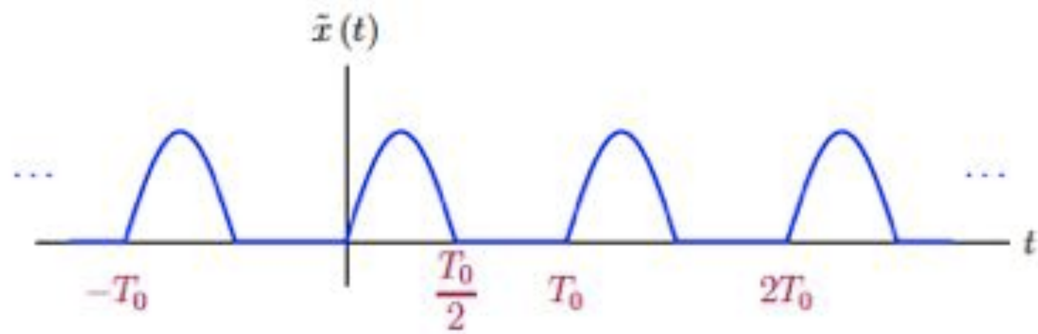


4.2.3 Exponential Fourier Series (EFS)

Example 4.10 Spectrum of half-wave rectified sinusoidal signal

Determine the EFS coefficients and graph the line spectrum for the half-wave periodic signal $\tilde{x}(t)$ defined by .

$$\tilde{x}(t) = \begin{cases} \sin(w_0 t), & 0 \le t \le T_0/2 \\ 0, & T_0/2 \le t < T_0 \end{cases} \quad \text{and} \quad \tilde{x}(t+T_0) = \tilde{x}(t)$$



4.2.5 Existence of Fourier Series

Is it always possible to determine the Fourier series coefficients?

Dirichlet Condition

3F

- ✧ Finite absolute value: $\int_0^{T_0} \left|\tilde{x}(t)\right| dt < \infty$
- ✧ Finite number of discontinuities in $\tilde{x}(t)$
- ✧ Finite number of minima and maxima in one period

4.2.7 Properties of Fourier Series

Linearity:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jkw_0 t} \quad \text{and} \quad \tilde{y}(t) = \sum_{k=-\infty}^{\infty} d_k e^{jkw_0 t}$$

$$a_1\tilde{x}(t) + a_2\tilde{y}(t) = \sum_{k=-\infty}^{\infty} [a_1 c_k + a_2 d_k] e^{jkw_0 t}$$

Where a_1 and a_2 are any two constants

Time shifting:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jkw_0 t}$$

$$\tilde{x}(t-\tau) = \sum_{k=-\infty}^{\infty} [c_k e^{-jkw_0 \tau}] e^{jkw_0 t}$$