On Representations of Semigroups Having Hypercube-like Cayley Graphs

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Abstract

The n-dimensional hypercube, or n-cube, is the Cayley graph of the Abelian group $\mathbb{Z}_2{}^n$. A number of combinatorially-interesting groups and semigroups arise from modified hypercubes. The inherent combinatorial properties of these groups and semigroups make them useful in a number of contexts, including coding theory, graph theory, stochastic processes, and even quantum mechanics. In this paper, particular groups and semigroups whose Cayley graphs are generalizations of hypercubes are described, and their irreducible representations are characterized. Constructions of faithful representations are also presented for each semigroup. The associated semigroup algebras are realized within the context of Clifford algebras.

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1 Introduction

Hypercubes are regular polytopes frequently arising in computer science and combinatorics. They are intricately connected to Gray codes, and their graphs arise as Hasse diagrams of finite boolean algebras. Hamilton cycles in hypercubes correspond to cyclic Gray codes and have received significant attention in light of the "Middle Level Conjecture" originally proposed by I. Havel [23].

On the classical stochastic side, random walks on hypercubes are useful in modeling tree-structured parallel computations [8]. On the quantum side, hypercubes often appear in quantum random walks, e.g. [1, 10]. Random walks on Clifford algebras have also been studied as random walks on directed hypercubes [15].

By considering specific generalizations of hypercubes, combinatorial properties can be obtained for tackling a variety of problems in graph theory and combinatorics [17, 21, 22]. By defining combinatorial raising and lowering operators on the associated semigroup algebras, an operator calculus (OC) on graphs

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is obtained, making graph-theoretic problems accessible to the tools of algebraic (quantum) probability [14, 16, 18, 19].

The goal of the current work is to describe some particular groups and semigroups whose Cayley graphs are generalizations of hypercubes, to characterize their irreducible complex representations, and to provide constructions for faithful representations.

2 Signed, Directed, & Looped Hypercubes

Hypercubes play an important role throughout the operator calculus approach. The n-dimensional cube, or *hypercube* $\mathcal{Q}_n$, is the graph whose vertices are in one-to-one correspondence with the n-tuples of zeros and ones and whose edges are the pairs of n-tuples that differ in exactly one position. This graph has natural applications in computer science, symbolic dynamics, and coding theory. The structure of the hypercube allows one to construct a random walk on the hypercube by "flipping" a randomly selected digit from 0 to 1 or vice versa.

Given two binary strings $a = (a_1a_2\cdots a_n)$ and $b = (b_1b_2\cdots b_n)$, the *Hamming distance* between a and b, denoted $d_{\mathrm{H}}(a,b)$, is defined as the number of positions at which the strings differ. That is,

$$d_{\mathrm{H}}(a,b) = |\{i : 1 \leq i \leq n, a_i \neq b_i\}|.$$

Let b be a block, or *word*, of length n; that is, let b be a sequence of n zeros and ones. The *Hamming weight* of b, denoted $w_{\mathrm{H}}(b)$, is defined as the number of ones in the sequence. The *binary sum* of two such words is the sequence resulting from addition modulo-two of the two sequences. The *Hamming distance* between two binary words is defined as the weight of their binary sum.

With Hamming distance defined, the formal definition of the *hypercube* $\mathcal{Q}_n$ can be given.

Definition 2.1. The *n-dimensional hypercube* $\mathcal{Q}_n$ is the graph whose vertices are the 2^n n-tuples from $\{0,1\}$ and whose edges are defined by the rule

$$\{v_1, v_2\} \in E(\mathcal{Q}_n) \Leftrightarrow w_{\mathrm{H}}(v_1 \oplus v_2) = 1.$$

Here $v_1 \oplus v_2$ is bitwise addition modulo-two, and w_{H} is the Hamming weight. In other words, two vertices of the hypercube are adjacent if and only if their Hamming distance is 1.

Fixing the set $B = \{e_1, \ldots, e_n\}$, the power set of B is in one-to-one correspondence with the vertices of $\mathcal{Q}_n$ via the binary subset representation

$$(a_1a_2\cdots a_n) \leftrightarrow e_I \Leftrightarrow a_i = \begin{cases} 1 & i \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Of particular interest are some variations on the traditional hypercube defined above. First, the *looped hypercube* is the pseudograph obtained from the traditional hypercube $\mathcal{Q}_n$ by appending a loop at each vertex. In particular, ${\mathcal{Q}_n}^\circ = (V_\circ, E_\circ)$, where $V = V(\mathcal{Q}_n)$ and $E = E(\mathcal{Q}_n) \cup \{(v,v) : v \in V(\mathcal{Q}_n)\}$.

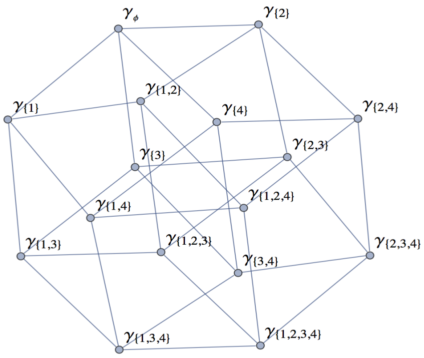

Figure 2.1: Four-dimensional hypercube.

Definition 2.2. Let V denote the vertex set of the n-dimensional hypercube, $\mathcal{Q}_n$. Let αV denote the set obtained from V by appending the symbol α to each vertex in V. The Hamming weight of α is taken to be zero. A *signed hypercube* is a (possibly directed) graph, G, on vertex set $V \cup \alpha V$ such that

$$(u, w) \in E(G) \Rightarrow w_{\mathrm{H}}(u \oplus w) = 1.$$

With various notions of generalized hypercubes in mind, a few relevant examples of finitely-generated semigroups can be given.

Let $\mathcal{S}_4^0$ denote the Abelian group generated by commutative generators $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ along with $\gamma_\emptyset$ satisfying ${\gamma_i}^2 = \gamma_\emptyset$ for each i. The Cayley graph of $\mathcal{S}_4^0$ is readily seen to be the four-dimensional hypercube of Figure 2.1.

Let $\mathcal{J}_4$ denote the Abelian group generated by $\gamma_\emptyset$ along with commutative generators $\{\gamma_1, \ldots, \gamma_4\}$ satisfying ${\gamma_i}^2 = \gamma_i$ for each $i \in \{1, 2, 3, 4\}$. The Cayley graph of $\mathcal{J}_4$ is then readily seen to be the four-dimensional looped hypercube obtained by appending a loop to each vertex of the graph seen in Figure 2.1.

Let $\mathcal{B}_0^3$ denote the non-Abelian group generated by $\{\gamma_1, \gamma_2, \gamma_3\}$ along with $\gamma_\emptyset$ and γ_α satisfying ${\gamma_i}^2 = \gamma_\alpha$ for each i, and $\gamma_i\gamma_j = \gamma_\alpha\gamma_j\gamma_i$ for $i \neq j$. The signed three-dimensional hypercube of Figure 2.2 is the undirected graph underlying the Cayley graph of $\mathcal{B}_0^3$.

Hypercube generalizations appearing in this paper are summarized in Table 2.

2.1 The blade group $\mathcal{B}_p^q$

Let $B = \{e_1, \ldots, e_n\}$, and let p and q be nonnegative integers such that $p + q = n$. Let $\mathcal{B}_p^q$ be the *multiplicative group generated by* B along with the elements

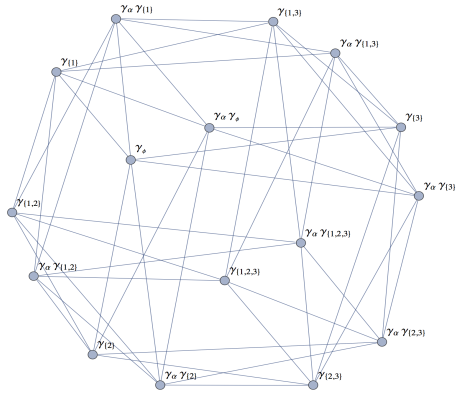

Figure 2.2: Three-dimensional signed hypercube.

(Semi) Group	Generator Commutation	Generator Squares
$\mathcal{B}_p^q$	$\gamma_i\gamma_j = \gamma_\alpha\gamma_j\gamma_i$	$\{\underbrace{\gamma_\emptyset, \ldots, \gamma_\emptyset}_{p}, \underbrace{\gamma_\alpha, \ldots, \gamma_\alpha}_{q}\}$
$\mathcal{S}_p^q$	Abelian	$\{\underbrace{\gamma_\emptyset, \ldots, \gamma_\emptyset}_{p}, \underbrace{\gamma_\alpha, \ldots, \gamma_\alpha}_{q}\}$
$\mathfrak{G}_n$	$\gamma_i\gamma_j = \gamma_\alpha\gamma_j\gamma_i$	${\gamma_i}^2 = 0_\gamma,\ i = 1, \ldots, n$
$\mathfrak{Z}_n$	Abelian	${\gamma_i}^2 = 0_\gamma,\ i = 1, \ldots, n$
$\mathcal{J}_n$	Abelian	${\gamma_i}^2 = \gamma_i,\ i = 1, \ldots, n$

Table 1: Semigroups summarized by generators.

(Semi) Group	Directed?	Signed?	Looped
$\mathcal{B}_p^q$	Yes	Yes	No
$\mathcal{S}_p^q$	No	Only if $q > 0$.	No
$\mathfrak{G}_n$	Yes	No	No
$\mathfrak{Z}_n$	No	No	No
$\mathcal{J}_n$	No	No	Yes

Table 2: Properties of hypercubes underlying semigroups discussed.

$\{e_\emptyset, e_\alpha\}$, subject to the following generating relations: for all $x \in B \cup \{e_\emptyset, e_\alpha\}$,

$$e_\emptyset\, x = x\, e_\emptyset = x,$$

$$e_\alpha\, x = x\, e_\alpha,$$

$${e_\emptyset}^2 = {e_\alpha}^2 = e_\emptyset,$$

and

$$e_ie_j = \begin{cases} e_\alpha\, e_je_i & \text{if } i \neq j, \\ e_\emptyset & \text{if } i = j \leq p, \\ e_\alpha & \text{if } p+1 \leq i = j \leq n. \end{cases}$$

The group $\mathcal{B}_p^q$ is referred to as the *blade group* [1] of signature (p, q).

Let $2^{[n]}$ denote the power set of the n-set, $[n] = \{1, 2, \ldots, n\}$, used as indices of the generators in B. Elements of $2^{[n]}$ are assumed to be canonically ordered by

$$I \prec J \Leftrightarrow \sum_{i \in I} 2^{i-1} < \sum_{j \in J} 2^{j-1}. \tag{2.1}$$

Note that the ordering is inherited from the binary subset representation of integers.

Remark 2.3. The order of the blade group $\mathcal{B}_p^q$ is 2^{p+q+1} as seen by noting the form of its elements, i.e., $\mathcal{B}_p^q = \{e_I, e_\alpha\, e_I : I \in 2^{[n]}\}$.

To simplify multiplication within $\mathcal{B}_p^q$, some additional mappings will be useful. For fixed positive integer j, define the map $\mu_j : 2^{[n]} \to \mathbb{N}_0$ by

$$\mu_j(I) = |\{i \in I : i > j\}|. \tag{2.2}$$

In other words, $\mu_j(I)$ is the *counting measure* of the set $\{i \in I : i > j\}$.

Definition 2.4. The *product signature map* $\vartheta : 2^{[n]} \times 2^{[n]} \to \{e_\emptyset, e_\alpha\}$ is defined by

$$\vartheta(I, J) = {e_\alpha}^{\left(\mu_p(I \cap J) + \sum_{j \in J} \mu_j(I)\right)}. \tag{2.3}$$

Applying multi-index notation to the generators B according to the ordered product

$$e_I = \prod_{i \in I} e_i \tag{2.4}$$

for arbitrary $I \in 2^{[n]}$, the multiplicative group $\mathcal{B}_p^q$ is now seen to be determined by the multi-indexed set $\{e_I, e_\alpha e_I : I \in 2^{[n]}\}$ along with the associative multiplication defined by

$$e_I\, e_J = \vartheta(I, J) e_{I \triangle J}, \tag{2.5}$$

where $I \triangle J = (I \cup J) \setminus (I \cap J)$ denotes set-symmetric difference. Inverses in $\mathcal{B}_p^q$ are given by

$${e_I}^{-1} = \vartheta(I, I) e_I$$

since

$$e_I \vartheta(I, I) e_I = \vartheta(I, I)^2 e_{I \triangle I} = e_\emptyset.$$

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[1] The elements of this group are analogous to the "basis blades" of a Clifford (Grassmann) algebra.

Elements of the form e_I are called *positive*, while elements of the form $e_\alpha e_I$ are called *negative*. Positive elements of $\mathcal{B}_p^q$ are now canonically ordered by

$$e_I \prec e_J \Leftrightarrow I \prec J$$

using the ordering on $2^{[n]}$ given by (2.1).

An element $e_I \in \mathcal{B}_p^q$ is said to be *even* if $|I| = 2k$ for some nonnegative integer k. Otherwise, e_I is said to be *odd*.

Lemma 2.5. *The collection of even elements of $\mathcal{B}_p^q$ forms a normal subgroup, denoted $\mathcal{B}_p^{q+}$.*

$$\mathcal{B}_p^{q+} \lhd \mathcal{B}_p^q.$$

Proof. First, note that multiplicative identity, $e_\emptyset$ is indexed by a set of size zero so that $\mathcal{B}_p^{q+}$ contains the identity. Secondly, the inverse of any element e_I is indexed by the same subset so that $\mathcal{B}_p^{q+}$ is closed with respect to inverses. Finally, the symmetric difference of two sets of even cardinality is also of even cardinality so that $\mathcal{B}_p^{q+}$ is closed under multiplication. Thus, $\mathcal{B}_p^{q+}$ is a subgroup of $\mathcal{B}_p^q$.

To see that $\mathcal{B}_p^{q+}$ is a normal subgroup, let $e_I \in \mathcal{B}_p^q$ be fixed and consider conjugation of elements of $\mathcal{B}_p^{q+}$. That is, consider $e_I \mathcal{B}_p^{q+} {e_I}^{-1}$. Choosing arbitrary $e_J \in \mathcal{B}_p^{q+}$, one finds

$$\begin{aligned} e_I\, e_J\, {e_I}^{-1} &= \vartheta(I,I) e_I\, e_J\, e_I = \vartheta(I,I) e_I \vartheta(J,I) e_{J \triangle I} \\ &= \vartheta(I,I)\vartheta(J,I)\vartheta(I, J\triangle I) e_{I\triangle(J\triangle I)} \\ &= \vartheta(I,I)\vartheta(J,I)\vartheta(I, J\triangle I) e_J \in \mathcal{B}_p^{q+}. \end{aligned}$$

Hence, the result. ☐

Allowing commutativity of generators leads to another combinatorially interesting group referred to as the “Abelian blade group.”

Definition 2.6. The *Abelian blade group*, $\mathcal{S}_p^q$, is defined as the abelian group of order 2^{n+1} generated by the collection $S = \{\varsigma_i : 1 \le i \le n\}$ along with elements $\{\varsigma_\emptyset, \varsigma_\alpha\}$ satisfying the following generating relations: for all $x \in S \cup \{\varsigma_\emptyset, \varsigma_\alpha\}$,

$$\begin{aligned} \varsigma_\emptyset\, x = x\, \varsigma_\emptyset &= x, \\ \varsigma_\alpha\, x &= x\, \varsigma_\alpha, \\ {\varsigma_\emptyset}^2 = {\varsigma_\alpha}^2 &= \varsigma_\emptyset, \end{aligned}$$

and

$$\varsigma_i\varsigma_j = \begin{cases} \varsigma_j\varsigma_i & \text{if } 1 \le i \ne j \le n, \\ \varsigma_\emptyset & \text{if } 1 \le i = j \le p, \\ \varsigma_\alpha & \text{if } p+1 \le i = j \le n. \end{cases}$$

The quotient group algebra $\mathbb{R}\mathcal{S}_p^q / \langle \varsigma_\alpha + \varsigma_\emptyset \rangle$ is canonically isomorphic to the symmetric-Clifford algebra ${\mathcal{C}\ell_{p,q}}^{\mathrm{sym}}$ appearing in [21], where it is used to induce homogeneous random walks on hypercubes.

3 Group Representations

All group and semigroup representations considered in this paper are complex. A *representation* of a given group, G, is a homomorphism $\rho : G \to \mathrm{GL}_n(\mathbb{C})$. The *degree* of this representation is n, and the *representation space* is the space $\mathbb{C}^n$ on which the elements of $\mathrm{GL}_n(\mathbb{C})$ act.

Given a representation ρ and a subspace W of $\mathbb{C}^n$, we say W is G-*invariant* if $\rho(g)W \subseteq W$ for every $g \in G$. If the only invariant spaces are $\{0\}$ and $\mathbb{C}^n$, the representation is said to be *irreducible.* The *character* of a representation, $\chi : G \to \mathbb{C}$, is defined by $\chi(g) = \mathrm{tr}(\rho(\mathrm{g}))$.

A fundamental result in group representation theory [20] is that a representation ρ with character χ is irreducible if and only if χ satisfies

$$(\chi|\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\chi(g)} = 1.$$

Two representations ρ and r of a group G are said to be *isomorphic* if there exists an invertible mapping $f : \mathbb{C}^n \to \mathbb{C}^n$ such that

$$f \circ \rho = r \circ f.$$

Lemma 3.1. *The group* $\mathcal{B}_p^q$ *has at least* 2^{p+q} *distinct degree-1 irreducible representations.*

Proof. First, any representation ρ of degree 1 must satisfy $\rho(e_\emptyset) = 1$. We claim that in the case $p + q > 1$, this implies $\rho(e_\alpha) = \rho(e_\emptyset) = 1$. To see this, note that ${e_\alpha}^2 = e_\emptyset$ clearly implies $\rho(e_\alpha) = \pm 1$. Suppose $\rho(e_\alpha) = -1$, and consider the following cases:

1. The case $p = 0$ or $q = 0$. In this case, either ${e_i}^2 = e_\emptyset$ for all $1 \le i \le p+q$ or ${e_i}^2 = e_\alpha$ for all $1 \le i \le p+q$. In either case, since $p+q > 1$, one considers the product $e_i\, e_j$ for $1 \le i \ne j \le p+q$. By anticommutativity, $(e_i\, e_j)^2 = e_\alpha$, but $\rho(e_\alpha) = -1$ guarantees a contradiction.
2. The case $p, q \ge 1$. In this case, one considers a pair, i, j, satisfying $1 \le i \le p$ and $p+1 \le j \le p+q$. Then ${e_i}^2 = e_\emptyset$, ${e_j}^2 = e_\alpha$, and $(e_i\, e_j)^2 = e_\emptyset$, again leading to a contradiction.

Let $J \in 2^{[p+q]}$ denote a multi-index. A degree-1 representation ρ_J is defined by setting $\rho_J(e_\emptyset) = \rho_J(e_\alpha) = 1$, and for $1 \le i \le p+q$, setting

$$\rho_J(e_i) = \begin{cases} 1 & i \in J, \\ -1 & i \notin J \end{cases}$$

By considering the number of distinct subsets J, it follows immediately that the total number of representations created this way is 2^{p+q}. These representations are clearly irreducible and distinct, i.e., pairwise non-isomorphic. $\square$

Remark 3.2. The groups $\mathcal{B}_0^1$ and $\mathcal{B}_1^0$ are instances of *Abelian blade groups* considered in Section 3.1. Irreducible representations of $\mathcal{B}_0^1$ are characterized in Example 3.4.

Recall that given a group G, the *conjugacy class* of an element $g \in G$ is the set

$$\mathfrak{Cl}(g) = \{hgh^{-1} : h \in G\}.$$

A well-known result in representation theory says that the number of irreducible representations of a group G is equal to the number of its conjugacy classes [20]. This provides a useful tool for establishing the following result.

Theorem 3.3. *Given the group $\mathcal{B}_p^q$ the number of conjugacy classes and subsequently the number of irreducible representations is given by the formula*

$$\kappa = 2^{p+q} + 1 + c,$$

where $c = p + q \pmod 2$.

Proof. If $p + q = 1$ the formula trivially works.

Suppose $p + q \neq 1$ and denote the center of $\mathcal{B}_p^q$ by $Z(\mathcal{B}_p^q)$. If $g \in \mathcal{B}_p^q \backslash Z(\mathcal{B}_p^q)$ then it is easily seen that

$$\begin{aligned} \mathfrak{Cl}(g) &= \{hgh^{-1} : h \in \mathcal{B}_p^q\} \\ &= \{e_I g e_I^{-1} : I \in 2^{[p+q]}\}. \end{aligned}$$

Combining the following facts: ${e_I}^{-1} \in \{e_I, e_\alpha e_I\}$, $e_I g \in \{g e_I, e_\alpha g e_I\}$, and ${e_I}^2 \in \{e_\alpha, e_\emptyset\}$, one sees that $\mathfrak{Cl}(g) = \{g, e_\alpha g\}$.

Further, if $g \in Z(\mathcal{B}_p^q)$, then $e_\alpha g \in Z(\mathcal{B}_p^q)$ and $\mathfrak{Cl}(g) = \{g\}$. Hence,

$$\kappa = \frac{|\mathcal{B}_p^q \backslash Z(\mathcal{B}_p^q)|}{2} + |Z(\mathcal{B}_p^q)|.$$

Therefore to finish the proof we need to understand the order of the center. Let $e_I \in \mathcal{B}_p^q$ be arbitrary. If $I = \emptyset$ one can see that $e_\emptyset \in Z(\mathcal{B}_p^q)$.

Now suppose $I \neq \emptyset$. Let $I = \{i_1, \ldots, i_h\}$ for h even. Then,

$$\begin{aligned} e_{i_h} e_I &= (e_\alpha)^{h-1} e_I e_{i_h} \\ &= e_\alpha e_I e_{i_h}. \end{aligned}$$

Whence, $e_I \notin Z(\mathcal{B}_p^q)$.

Assume now that $I = \{i_1, ...i_h\} \neq \{1, 2, ..., p+q\}$ for h odd. Then there is a natural number ℓ such that $\ell \notin I$ and

$$e_\ell e_I = e_\alpha e_I e_\ell.$$

Thus, $e_I \notin Z(\mathcal{B}_p^q)$.

Finally, suppose $I = \{1, 2, ..., p+q\} = [p+q]$ for $p+q$ odd. It is claimed that

$$e_J e_{[p+q]} = e_{[p+q]} e_J$$

for every indexing set $J \subseteq I$. To see this, note that if $J = \emptyset$ the result trivially holds. By way of induction on the cardinality of J, assume $J = \{j\}$, set $\mu^+ = |\{i \in [p+q] : i < j\}|$, and set $\mu^- = |\{i \in [p+q] : i > j\}|$. Then,

$$\begin{aligned} e_{[p+q]}e_j &= (e_\alpha)^{\mu^- + \mu^+} e_j e_{[p+q]} \\ &= (e_\alpha)^{p+q-1} e_j e_{[p+q]} \\ &= e_j e_{[p+q]}. \end{aligned}$$

Suppose $e_{[p+q]}e_J = e_J e_{[p+q]}$ for some multi-index cardinality $|J|$. The task now is to show $e_{[p+q]}e_J e_h = e_J e_h e_{[p+q]}$ for some natural number $h \notin J$. From the inductive hypothesis we know

$$(e_{[p+q]}e_J)e_h = (e_J e_{[p+q]})e_h$$

and from the basis step, we know $e_{[p+q]}e_h = e_h e_{[p+q]}$.

Combining these two facts and using associativity of the group operation,

$$e_{[p+q]}e_J e_h = e_J e_{[p+q]} e_h = e_J e_h e_{[p+q]}.$$

Hence, by induction, $e_J e_{[p+q]} = e_{[p+q]} e_J$ for all $J \in 2^{[p+q]}$.

If $p+q$ is even, then $Z(\mathcal{B}_p^q) = \{e_\alpha, e_\emptyset\}$. In this case,

$$\begin{aligned} \kappa &= \frac{|\mathcal{B}_p^q \backslash Z(\mathcal{B}_p^q)|}{2} + |Z(\mathcal{B}_p^q)| \\ &= (2^{p+q} - 1) + 2 \\ &= 2^{p+q} + 1. \end{aligned}$$

If $p+q$ is odd, then $Z(\mathcal{B}_p^q) = \{e_\alpha, e_\emptyset, e_{[p+q]}, e_\alpha e_{[p+q]}\}$, which gives

$$\begin{aligned} \kappa &= \frac{|\mathcal{B}_p^q \backslash Z(\mathcal{B}_p^q)|}{2} + |Z(\mathcal{B}_p^q)| \\ &= (2^{p+q} - 2) + 4 \\ &= 2^{p+q} + 2. \end{aligned}$$

□

Example 3.4. Consider the group $\mathcal{B}_0^1$. In this case, two non-faithful irreducible representations are constructed as in the proof of Lemma 3.1. Two more faithful irreducible representations are found in agreement with Theorem 3.3. The four representations are listed in the left table of Figure 3.1. Four irreducible representations of $\mathcal{B}_1^0$ are similarly constructed in the right table of Figure 3.1. No faithful irreducible representations exist in this case. It is not difficult to verify that the representations are distinct for each group.

The above example could have been completed using real representations, although none would be faithful. When $p+q > 1$, $\mathcal{B}_p^q$ is non-Abelian, and hence has no faithful degree-1 representation regardless of representation space.

Another well known result in group representation theory is the following, found in [20].

$\mathcal{B}_0^1$	$e_\emptyset$	e_α	e_1	$e_\alpha\, e_1$
$\rho_\emptyset$	1	1	1	1
$\rho_{\{1\}}$	1	1	-1	-1
δ_1	1	-1	$\imath$	$-\imath$
δ_2	1	-1	$-\imath$	$\imath$

$\mathcal{B}_1^0$	$e_\emptyset$	e_α	e_1	$e_\alpha\, e_1$
$\rho_\emptyset$	1	1	1	1
$\rho_{\{1\}}$	1	1	-1	-1
δ_1	1	-1	1	-1
δ_2	1	-1	-1	1

Figure 3.1: Irreducible degree-1 representations of $\mathcal{B}_0^1$ (left) and $\mathcal{B}_1^0$ (right).

Lemma 3.5. *Let G be a finite group having κ irreducible representations. For each $i = 1, \ldots, \kappa$, let n_i denote the degree of the i^{th} irreducible representation of G. Then,*

$$|G| = \sum_{i=1}^{\kappa} {n_i}^2.$$

Given the group $\mathcal{B}_p^q$ there are always 2^{p+q} distinct irreducible representations of degree 1. The remaining irreducible (complex) representations are now enumerated in the next theorem.

Theorem 3.6. *If $p + q = 2k > 1$, then $\mathcal{B}_p^q$ has one irreducible representation of degree 2^k. If $p + q = 2k + 1$, then $\mathcal{B}_p^q$ has two irreducible representations of degree 2^k. Moreover, all of these irreducible representations are faithful except when p is odd and q is even.*

Proof. This will be treated in two cases. First is the case of $p + q$ even. Since $p+q$ is even, one can write $p+q = 2k$ for some $k \in \mathbb{N}$. Define $\tau : \mathcal{B}_p^q \to \mathrm{GL}_{2^k}(\mathbb{C})$ by

$$\tau(e_j) = \begin{cases} {\sigma_{\mathrm{x}}}^{\otimes(j-1)} \otimes \sigma_{\mathrm{z}} \otimes {\sigma_0}^{\otimes(k-j)} & 1 \le j \le k,\ j \le p \\ \imath\left({\sigma_{\mathrm{x}}}^{\otimes(j-1)} \otimes \sigma_{\mathrm{z}} \otimes {\sigma_0}^{\otimes(k-j)}\right) & 1 \le j \le k,\ j > p \\ {\sigma_{\mathrm{x}}}^{\otimes(j-k-1)} \otimes \sigma_{\mathrm{y}} \otimes {\sigma_0}^{\otimes(2k-j)} & k+1 \le j \le 2k,\ j \le p \\ \imath\left({\sigma_{\mathrm{x}}}^{\otimes(j-k-1)} \otimes \sigma_{\mathrm{y}} \otimes {\sigma_0}^{\otimes(2k-j)}\right) & k+1 \le j \le 2k,\ j > p. \end{cases} \tag{3.1}$$

Setting $\tau(e_\emptyset) = {\sigma_0}^{\otimes k}$ and $\tau(e_\alpha) = -{\sigma_0}^{\otimes k}$, this extends by multiplication to all of $\mathcal{B}_p^q$. More specifically, for any multi-index I, $\tau(e_I) = \prod_{\ell \in I} \tau(e_\ell)$, and $\tau(e_\alpha e_I) = -\tau(e_I)$.

This representation is clearly well defined. To verify that this representation is irreducible, let ξ be the character of τ. Let $e_I \in \mathcal{B}_p^q$ be arbitrary. Letting $\sigma_{(\ell)} \in \{\sigma_0, \sigma_{\mathrm{x}}, \sigma_{\mathrm{y}}, \sigma_{\mathrm{z}}\}$ for each $\ell = 1, \ldots, k$, one writes

$$\begin{aligned} \xi(e_I) &= \mathrm{tr}(\tau(e_I)) \\ &= \mathrm{tr}\left(u \bigotimes_{\ell=1}^{k} \sigma_{(\ell)}\right) \\ &= u \prod_{\ell \in I} \mathrm{tr}(\sigma_{(\ell)}) \end{aligned}$$

for some unit $u \in \{\pm 1, \pm \imath\}$. Since $\mathrm{tr}(\sigma_{\mathrm{x}}) = \mathrm{tr}(\sigma_{\mathrm{y}}) = \mathrm{tr}(\sigma_{\mathrm{z}}) = 0$, and $\mathrm{tr}(\sigma_0) = 2$, it follows that $\xi(e_I) = \xi(e_\alpha e_I) = 0$ unless $I = \emptyset$, in which case $\xi(e_\emptyset) = 2^k$ and $\xi(e_a e_\emptyset) = -2^k$. Now,

$$\begin{aligned}(\xi|\xi) &= \frac{1}{|\mathcal{B}_p^q|} \sum_{g \in \mathcal{B}_p^q} \xi(g)\overline{\xi(g)} \\ &= \frac{1}{2^{2k+1}}((\xi(e_\emptyset))^2 + (\xi(e_\alpha e_\emptyset))^2) \\ &= \frac{1}{2^{2k+1}}(2)(2^{2k}) = 1.\end{aligned}$$

Thus, τ is irreducible.

To see that the representation (3.1) is faithful, consider the kernel:

$$\ker(\tau) = \{e_I : \tau(e_I) = {\sigma_0}^{\otimes k}\}.$$

Noting that $\tau(\mathcal{B}_p^q)$ is a subgroup of $\mathrm{GL}_{2^k}(\mathbb{C})$, we begin by showing that the center of this subgroup consists only of elements having the form $\pm u {\sigma_0}^{\otimes k}$, where $u \in \{\pm 1, \pm \imath\}$. To begin, the center of $\tau(\mathcal{B}_p^q)$ is

$$Z(\tau(\mathcal{B}_p^q)) = \{\tau(e_E) : \tau(e_E)\tau(e_J) = \tau(e_J)\tau(e_E),\ \forall J \in 2^{[p+q]}\}.$$

Suppose an element of $Z(\tau(\mathcal{B}_p^q))$ is of the form $M = u\, \sigma_{(1)} \otimes \ldots \otimes \sigma_{(k)}$, where for some index h, $\sigma_{(h)} \neq \sigma_0$ but $\sigma_{(h+1)} = \cdots = \sigma_{(k)} = \sigma_0$. If $\sigma_{(h)} = \sigma_{\mathrm{z}}$ or σ_{y} then we can see

$$\tau(e_{\{h,h+k\}}) = u({\sigma_0}^{\otimes(h-1)} \otimes \sigma_{\mathrm{x}} \otimes {\sigma_0}^{(k-h)}),$$

which will anti-commute with M.

If $\sigma_{(h)} = \sigma_{\mathrm{x}}$, an element anti-commuting with M is given by

$$\begin{aligned}&\tau(e_{\{1,k+1,2,k+2,\ldots,h-1,k+h-1,h\}}) = \\ &\quad\left(\prod_{j=1}^{h-1}(-\imath)({\sigma_0}^{\otimes(j-1)} \otimes \sigma_{\mathrm{x}} \otimes {\sigma_0}^{\otimes(k-j)})\right)\left({\sigma_{\mathrm{x}}}^{\otimes(h-1)} \otimes \sigma_{\mathrm{z}} \otimes {\sigma_0}^{\otimes(k-h)}\right) \\ &\qquad = u\left({\sigma_{\mathrm{x}}}^{\otimes(h-1)} \otimes {\sigma_0}^{\otimes(k-h+1)}\right)\left({\sigma_{\mathrm{x}}}^{\otimes(h-1)} \otimes \sigma_{\mathrm{z}} \otimes {\sigma_0}^{\otimes(k-h)}\right) \\ &\qquad\qquad = u\, {\sigma_0}^{\otimes(h-1)} \otimes \sigma_{\mathrm{z}} \otimes {\sigma_0}^{\otimes(k-h)}.\end{aligned}$$

This proves that every element of $Z(\tau(\mathcal{B}_p^q))$ is of the form $u\, {\sigma_0}^{\otimes k}$ for $u \in \{\pm 1, \pm i\}$.

By construction, $\tau(e_\emptyset)$ and $\tau(e_\alpha)$ are elements of $Z(\tau(\mathcal{B}_p^q))$. Suppose E is a non-empty indexing set, and to the contrary suppose $\tau(e_E) \in Z(\tau(\mathcal{B}_p^q))$, so that $\tau(e_E) = u\, {\sigma_0}^{\otimes k}$. It is not difficult to see that $e_E \notin Z(\mathcal{B}_p^q)$, so there exists an integer m such that

$$e_E e_m = e_\alpha e_m e_E.$$

Applying τ reveals

$$\tau(e_E e_m) = u\,{\sigma_0}^{\otimes k}\tau(e_m) \neq -u\,\tau(e_m){\sigma_0}^{\otimes k} = \tau(e_\alpha e_m e_E).$$

This contradicts the homomorphism property of τ. We conclude then that $e_E \notin Z(\tau(\mathcal{B}_p^q))$, which means

$$Z(\tau(\mathcal{B}_p^q)) = \{\tau(e_\emptyset), \tau(e_\alpha)\}.$$

However, only one of these, $\tau(e_\emptyset)$, is ${\sigma_0}^{\otimes k}$. Thus, $\ker(\tau) = \{e_\emptyset\}$. Since the kernel is trivial, τ is faithful.

Now suppose $p+q = 2k+1$ is odd; more specifically, suppose p is even and q is odd. Let $\tau : \mathcal{B}_p^q \to \mathrm{GL}_{2^k}(\mathbb{C})$ be defined by

$$\tau(e_j) = \begin{cases} {\sigma_\mathrm{x}}^{\otimes(j-1)} \otimes \sigma_\mathrm{z} \otimes {\sigma_0}^{\otimes(k-j)} & 1 \leq j \leq k,\ j \leq p \\ \imath\left({\sigma_\mathrm{x}}^{\otimes(j-1)} \otimes \sigma_\mathrm{z} \otimes {\sigma_0}^{\otimes(k-j)}\right) & 1 \leq j \leq k,\ j > p \\ {\sigma_\mathrm{x}}^{\otimes(j-k-1)} \otimes \sigma_\mathrm{y} \otimes {\sigma_0}^{\otimes(2k-j)} & k+1 \leq j \leq 2k,\ j \leq p \\ \imath\left({\sigma_\mathrm{x}}^{\otimes(j-k-1)} \otimes \sigma_\mathrm{y} \otimes {\sigma_0}^{\otimes(2k-j)}\right) & k+1 \leq j \leq 2k,\ j > p \\ {\sigma_\mathrm{x}}^{\otimes k} & j = 2k+1,\ j \leq p \\ \imath\left({\sigma_\mathrm{x}}^{\otimes k}\right) & j = 2k+1,\ j > p. \end{cases} \tag{3.2}$$

Setting $\tau(e_\emptyset) = {\sigma_0}^{\otimes k}$ and $\tau(e_\alpha e_i) = -\tau(e_i)$, this extends by multiplication to all of $\mathcal{B}_p^q$.

To check irreducibility of τ, let ξ denote the character of τ. It is already known that $\xi(e_I) = 0$ in all but the extreme case $I = \emptyset$. Further, since $e_{[p+q]}$ is in the center of the group, its image under τ must be of the for $u\,{\sigma_0}^{\otimes k}$ for some scalar $u \in \{\pm 1, \pm\imath\}$. It thereby follows that

$$\begin{aligned} (\xi|\xi) &= \frac{1}{|\mathcal{B}_p^q|}\sum_{g\in\mathcal{B}_p^q}\xi(g)\overline{\xi(g)} \\ &= \frac{1}{2^{2k+2}}\left((\xi(e_\emptyset))^2 + (\xi(e_\alpha))^2\right) \\ &\quad + \frac{1}{2^{2k+2}}\left(\xi(e_{[2k+1]})\overline{\xi(e_{[2k+1]})} + \xi(e_\alpha e_{[2k+1]})\overline{\xi(e_\alpha e_{[2k+1]})}\right) \\ &= \frac{1}{(2^{2k+2})}\left(2^{2k} + 2^{2k} + 2^{2k} + 2^{2k}\right) \\ &= 1. \end{aligned}$$

Hence, τ is irreducible.

Recall that when $p+q$ is odd, $Z(\mathcal{B}_p^q) = \{e_\alpha, e_\emptyset, e_{[p+q]}, e_\alpha e_{[p+q]}\}$. In light of the proof that τ was faithful for $p+q$ even, showing that $e_{[p+q]} \notin \ker(\tau)$ is sufficient to show that τ is faithful. Computing $e_{[p+q]}$, one finds

$$e_{[p+q]} \mapsto \imath^q(\sigma_\mathrm{z}\sigma_\mathrm{y}\sigma_\mathrm{x})^{\otimes k} = \imath^q({\sigma_0}^{\otimes k}), \tag{3.3}$$

so that

$$\tau(e_{[p+q]}) = \begin{cases} \imath({\sigma_0}^{\otimes k}) & \text{if } q \equiv 1 \pmod 4, \\ -\imath({\sigma_0}^{\otimes k}) & \text{if } q \equiv 3 \pmod 4. \end{cases} \tag{3.4}$$

It follows that $\ker(\tau)$ is trivial.

Finally, in the case p is odd and q is even, the construction of (3.2) is again used. This representation is again irreducible, and $\tau(e_{[p+q]}) = \imath^q({\sigma_0}^{\otimes k})$, as seen in Equation 3.3. In this case, however, one has

$$\tau(e_{[p+q]}) = \begin{cases} {\sigma_0}^{\otimes k} = \tau(e_\emptyset) & \text{when } q \equiv 0 \pmod 4, \\ -{\sigma_0}^{\otimes k} = \tau(e_\alpha\, e_\emptyset) & \text{when } q \equiv 2 \pmod 4, \end{cases} \tag{3.5}$$

so that the representation is *not* faithful.

Recalling that the order of $\mathcal{B}_p^q$ is equal to the sum of the squares of degrees of irreducible representations, there remains one irreducible representation of $\mathcal{B}_p^q$ in the case $p+q$ is odd: the complex conjugate of τ. This representation is given explicitly by

$$\overline{\tau}(e_j) = \begin{cases} {\sigma_{\mathrm{x}}}^{\otimes(j-1)} \otimes \sigma_{\mathrm{z}} \otimes {\sigma_0}^{\otimes(k-j)} & 1 \leq j \leq k,\ j \leq p \\ -\imath\left({\sigma_{\mathrm{x}}}^{\otimes(j-1)} \otimes \sigma_{\mathrm{z}} \otimes {\sigma_0}^{\otimes(k-j)}\right) & 1 \leq j \leq k,\ j > p \\ -{\sigma_{\mathrm{x}}}^{\otimes(j-k-1)} \otimes \sigma_{\mathrm{y}} \otimes {\sigma_0}^{\otimes(2k-j)} & k+1 \leq j \leq 2k,\ j \leq p \\ \imath\left({\sigma_{\mathrm{x}}}^{\otimes(j-k-1)} \otimes \sigma_{\mathrm{y}} \otimes {\sigma_0}^{\otimes(2k-j)}\right) & k+1 \leq j \leq 2k,\ j > p \\ {\sigma_{\mathrm{x}}}^{\otimes k} & j = 2k+1,\ j \leq p \\ -\imath\left({\sigma_{\mathrm{x}}}^{\otimes k}\right) & j = 2k+1,\ j > p, \end{cases}$$

where, $\overline{\tau}(e_\emptyset) = {\sigma_0}^{\otimes k}$ and $\overline{\tau}(e_\alpha e_i) = -\tau(e_i)$. This extends by multiplication to all of $\mathcal{B}_p^q$.

To see that $\overline{\tau}$ is not isomorphic to τ, one considers the action of $\overline{\tau}$ on $e_{[p+q]}$. In particular, (3.4) and (3.5) imply

$$\overline{\tau}(e_{[p+q]}) = \begin{cases} {\sigma_0}^{\otimes k} = \tau(e_{[p+q]}) & \text{when } q \equiv 0 \pmod 4, \\ -\imath({\sigma_0}^{\otimes k}) = -\tau(e_{[p+q]}) & \text{when } q \equiv 1 \pmod 4, \\ -{\sigma_0}^{\otimes k} = \tau(e_{[p+q]}) & \text{when } q \equiv 2 \pmod 4, \\ \imath({\sigma_0}^{\otimes k}) = -\tau(e_{[p+q]}) & \text{when } q \equiv 3 \pmod 4. \end{cases}$$

Suppose there exists an invertible linear transformation $f \in \mathrm{GL}(\mathbb{C}^{2^k})$ satisfying $f \circ \tau = \overline{\tau} \circ f$. Then, the cases $q \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$ imply

$$f \circ \left(\imath\left({\sigma_0}^{\otimes k}\right)\right) = -\imath\left({\sigma_0}^{\otimes k}\right) \circ f \Rightarrow f(\mathbf{v}) = -\mathbf{v},\ \forall \mathbf{v} \in \mathbb{C}^{2^k},$$

which contradicts $f \circ \overline{\tau}(e_\emptyset) = {\sigma_0}^{\otimes k}$. Similarly, in the cases $q \equiv 0 \pmod 4$ and $q \equiv 2 \pmod 4$,

$$f \circ \left({\sigma_0}^{\otimes k}\right) = \left({\sigma_0}^{\otimes k}\right) \circ f \Rightarrow f(\mathbf{v}) = \mathbf{v},\ \forall \mathbf{v} \in \mathbb{C}^{2^k},$$

contradicting $f \circ \tau = \overline{\tau} \circ f$, since $\tau \neq \overline{\tau}$. □

It becomes evident in the case $p \equiv 1 \pmod 2$ and $q \equiv 0 \pmod 2$ that in order to obtain a faithful representation of $\mathcal{B}_p^q$, one must pass to a larger representation space. It is not difficult to show that a faithful representation is given by defining $\tau : \mathcal{B}_p^q \to \mathrm{GL}_{2^{k+1}}(\mathbb{C})$ by

$$\tau(e_j) = \begin{cases} {\sigma_{\mathrm{x}}}^{\otimes(j-1)} \otimes \sigma_{\mathrm{z}} \otimes {\sigma_0}^{\otimes(k-j+1)} & 1 \le j \le k,\, j \le p \\ \imath \left({\sigma_{\mathrm{x}}}^{\otimes(j-1)} \otimes \sigma_{\mathrm{z}} \otimes {\sigma_0}^{\otimes(k-j+1)}\right) & 1 \le j \le k,\, j > p \\ {\sigma_{\mathrm{x}}}^{\otimes(j-k-1)} \otimes \sigma_{\mathrm{y}} \otimes {\sigma_0}^{\otimes(2k-j)} & k+1 \le j \le 2k,\, j \le p \\ \imath \left({\sigma_{\mathrm{x}}}^{\otimes(j-k-1)} \otimes \sigma_{\mathrm{y}} \otimes {\sigma_0}^{\otimes(2k-j)}\right) & k+1 \le j \le 2k,\, j > p \\ {\sigma_{\mathrm{x}}}^{\otimes(k+1)} & j = 2k+1,\, j \le p \\ \imath \left({\sigma_{\mathrm{x}}}^{\otimes(k+1)}\right) & j = 2k+1,\, j > p. \end{cases}$$

Setting $\tau(e_\emptyset) = {\sigma_0}^{\otimes(k+1)}$ and $\tau(e_\alpha e_i) = -\tau(e_i)$, this is again extended by multiplication to all of $\mathcal{B}_p^q$.

3.1 The Abelian blade group $\mathcal{S}_p^q$

In the case of the Abelian blade group, $\mathcal{S}_p^q$, commutativity removes any hope of finding an irreducible faithful representation except for the case of $\mathcal{S}_0^1 \cong \mathcal{B}_0^1$, as noted in Example 3.4. The order of $\mathcal{S}_p^q$ is 2^{p+q+1}, and its irreducible representations are found as follows.

As in Lemma 3.1, let $J \in 2^{[p+q]}$ denote a multi-index. A degree-1 representation ρ_J is defined by setting $\rho_J(\varsigma_\emptyset) = \rho_J(\varsigma_\alpha) = 1$, and for $1 \le i \le p+q$, setting

$$\rho_J(\varsigma_i) = \begin{cases} 1 & i \in J, \\ -1 & i \notin J \end{cases}$$

Similarly, a degree-1 representation, δ_J, is obtained for each multi index J by setting $\delta_J(\varsigma_\emptyset) = 1$, $\delta_J(\varsigma_\alpha) = -1$, and

$$\delta_J(\varsigma_\ell) = \begin{cases} 1 & 1 \le \ell \le p \text{ and } \ell \in J \\ -1 & 1 \le \ell \le p \text{ and } \ell \notin J \\ \imath & p+1 \le \ell \le p+q \text{ and } \ell \in J, \\ -\imath & p+1 \le \ell \le p+q \text{ and } \ell \notin J. \end{cases}$$

Hence, all 2^{p+q+1} degree-1 irreducible representations are obtained.

One can find a faithful representation of order 2^{p+q}. Let φ be given by multiplicative extension of

$$\varphi(\varsigma_j) = u\, \sigma_0^{\otimes(j-1)} \otimes \sigma_{\mathrm{z}} \otimes \sigma_0^{\otimes(p+q-j)},$$

where $u = 1$ or $u = \imath$ depending on j. This representation is clearly faithful by construction. A meaningful question to ask is whether a smaller faithful representation exists. This question is answered in the affirmative by defining the degree-$2(p+q)$ faithful representation, $r : \mathcal{S}_p^q \to \mathrm{GL}_{2(p+q)}(\mathbb{C})$ as follows:

$$r(\varsigma_I) = u \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & A_n \end{pmatrix}.$$

Here, each A_j is a 2×2 matrix given by

$$A_j = \begin{cases} \sigma_{\mathrm{x}} & \text{if } j \in I, \\ \sigma_0 & \text{otherwise,} \end{cases}$$

and u is a complex unit determined by

$$u = \begin{cases} 1 & \text{if } {\varsigma_I}^2 = \varsigma_\emptyset, \\ \imath & \text{if } {\varsigma_I}^2 = \varsigma_\alpha. \end{cases}$$

4 Semigroup Representations

Essential definitions and notational conventions for semigroup representation theory follow the formalism of Izhakian, Rhodes, and Steinberg [7]. As previously noted, all semigroup representation spaces are complex.

Given a semigroup S, two elements $a, b \in S$ are said to be $\mathfrak{J}$-*equivalent* (written $a \,\mathfrak{J}\, b$) if $SaS = SbS$. The set of all things $\mathfrak{J}$-equivalent to $a \in S$ forms a $\mathfrak{J}$-class. The $\mathfrak{J}$-classes partition the semigroup S. A $\mathfrak{J}$-class is said to be *regular* if it contains an idempotent.

For every idempotent e of a semigroup S, we call G_e the *maximal subgroup of S at e* where $G_e = \{\text{invertible elements of } eSe\}$. Two idempotent elements $e, f \in S$ are said to be *isomorphic* if there exists an $x \in eSf$ and $x^* \in fSe$ such that $xx^* = e$ and $x^*x = f$.

The regular $\mathfrak{J}$-classes will play a large role in determining the number of irreducible representations of a semigroup S. Before we give the exact number to expect, we need a few more results. The following useful lemma can be found in [4].

Lemma 4.1. *If $e, f \in S$ are isomorphic idempotents, then $G_e \simeq G_f$. Moreover, e and f are isomorphic if and only if $e \,\mathfrak{J}\, f$.*

For a semigroup S we define a representation to be a homomorphism to the set of endomorphisms of $\mathbb{C}^n$, which can be realized as $n \times n$ matrices with entries in $\mathbb{C}$. In other words, a representation ρ of S is a homomorphism

$$\rho : S \to \mathrm{End}(\mathbb{C}^n).$$

The familiar terminology of a faithful representation, trivial representation and character of a representation follows. The idea of an irreducible semigroup representation is again the same, except we require that the representation is not constantly 0.

The next theorem, based on results of Clifford-Suchkewitch [4] and Munn (as found in Rhodes and Zalcstein [13]), will be useful for determining the number of irreducible representations.

Theorem 4.2. *Let* $G_1, ..., G_m$ *be a choice of exactly one maximal subgroup from each regular* $\mathfrak{J}$*-class of* S*. Then, letting* k_i *denote the number of conjugacy classes of* G_i*, the number of irreducible representations of* S *is* $\sum_{i=1}^{m} k_i$*.*

4.1 Null blade semigroups $\mathfrak{G}_n$ and $\mathfrak{Z}_n$

By modifying the multiplication in $\mathcal{B}_p^q$ such that generators square to zero, one obtains a non-Abelian *semigroup* generated by *null squares.* The principal difference from this point forward is a lack of multiplicative inverses for elements in the algebraic structures.

Definition 4.3. Let $\mathfrak{G}_n$ denote the *null blade semigroup* defined as the semigroup generated by the collection $G = \{\gamma_i : 1 \leq i \leq n\}$ along with $\{\gamma_\emptyset, \gamma_\alpha, 0_\gamma\}$ satisfying the following generating relations: for all $x \in G \cup \{\gamma_\emptyset, \gamma_\alpha, 0_\gamma\}$,

$$\begin{aligned} \gamma_\emptyset \, x = x \, \gamma_\emptyset &= x, \\ \gamma_\alpha \, x &= x \, \gamma_\alpha, \\ 0_\gamma \, x = x \, 0_\gamma &= 0_\gamma, \\ {\gamma_\emptyset}^2 = {\gamma_\alpha}^2 &= \gamma_\emptyset, \end{aligned}$$

and

$$\gamma_i \gamma_j = \begin{cases} 0_\gamma & \text{if and only if } i = j, \\ \gamma_\alpha \, \gamma_j \gamma_i & i \neq j. \end{cases}$$

Define the *antisymmetric product signature map* $\phi : 2^{[n]} \times 2^{[n]} \to \{\gamma_\emptyset, \gamma_\alpha\}$ by

$$\phi(I, J) = {\gamma_\alpha}^{\sum_{j \in J} \mu_j(I)}.$$

Remark 4.4. Note that the product signature map defined by (2.3) can be extended to $\mathcal{G} \times \mathcal{G}$ and written in terms of ϕ as

$$\vartheta(I, J) = {\gamma_\alpha}^{\mu_p(I \cap J) + \phi(I,J)}.$$

Hence, ϑ has a decomposition into *signature-dependent* and *signature-independent* parts.

Applying multi-index notation to the generators $G = \{\gamma_i : 1 \leq i \leq n\}$ according to the ordered product

$$\gamma_I = \prod_{i \in I} \gamma_i$$

for arbitrary $I \in 2^{[n]}$, the multiplicative semigroup $\mathfrak{G}_n$ is now seen to be determined by the multi-indexed set $\{0_\gamma\} \cup \{\gamma_\alpha\gamma_I, \gamma_I : I \in 2^{[n]}\}$ along with the associative multiplication defined by

$$\gamma_I\,\gamma_J = \begin{cases} {\gamma_\alpha}^{\sum_{i\in I}\mu_i(J)}\gamma_{I\cup J} & I \cap J = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the order of the null blade semigroup is $|\mathfrak{G}_n| = 2^{n+1} + 1$. The next combinatorial algebra can now be defined.

Definition 4.5. For fixed positive integer n, the *null blade algebra* is defined as the real semigroup algebra $\mathbb{R}\mathfrak{G}_n / \langle 0_\gamma, \gamma_\alpha + \gamma_\emptyset\rangle$, denoted $\mathcal{B}\ell_{\wedge n}$ for convenience.

It now becomes clear that the null blade algebra $\mathcal{B}\ell_{\wedge n}$ is canonically isomorphic to the Grassmann (exterior) algebra $\bigwedge \mathbb{R}^n$.

Theorem 4.6. *For any natural number n, there are three irreducible representations of $\mathfrak{G}_n$.*

Proof. First, we will classify every $\mathfrak{J}$-class of $\mathfrak{G}_n$, then identify which are regular. From there we will compute the maximal subgroups of a choice of distinct idempotents. Then, using the formula above we will find the number of irreducible representations.

Let $\gamma_I \in \mathfrak{G}_n$ be such that $\gamma_I \neq 0_\gamma$ but otherwise arbitrary. Then,

$$\begin{aligned} \mathfrak{G}_n\gamma_I\mathfrak{G}_n &= \{s_1\,\gamma_I\,s_2 : s_1, s_2 \in \mathfrak{G}_n\} \\ &= \{\gamma_E, \gamma_\alpha\gamma_E, 0_\gamma : I \subseteq E\}. \end{aligned}$$

Similarly,

$$\mathfrak{G}_n(0_\gamma)\mathfrak{G}_n = \{0_\gamma\}.$$

It follows that for every $w \in \mathfrak{G}_n$, the set of all things $\mathfrak{J}$ -equivalent to w is simply $\{w, \gamma_\alpha w\}$. The number of $\mathfrak{J}$-classes is thus $2^n + 1$. However we are only concerned with the regular $\mathfrak{J}$-classes. The only idempotent elements of $\mathfrak{G}_n$ are 0_γ and $\gamma_\emptyset$. Thus, the regular $\mathfrak{J}$-classes are $\{\gamma_\emptyset, \gamma_\alpha\}$ and $\{0_\gamma\}$. The two maximal subgroups are

$$G_{\gamma_\emptyset} = \{\text{invertible elements of } \gamma_\emptyset\mathfrak{G}_n\gamma_\emptyset\} = \{\gamma_\emptyset, \gamma_\alpha\},$$

and

$$G_{0_\gamma} = \{\text{invertible elemenets of } 0_\gamma\mathfrak{G}_n 0_\gamma\} = \{0_\gamma\}.$$

The trivial group, G_{0_γ}, has one conjugacy class, while $G_{\gamma_\emptyset}$ is an Abelian group of order 2, consequently having two conjugacy classes. Thus, the number of irreducible representations of $\mathfrak{G}_n$ is three. □

Definition 4.7. Let $\mathfrak{Z}_n$ denote the *Abelian null blade semigroup* defined as the semigroup generated by the collection $C = \{\zeta_i : 1 \leq i \leq n\}$ along with $\{\zeta_\emptyset, 0_\zeta\}$

satisfying the following generating relations: for all $x \in C \cup \{\zeta_\emptyset, 0_\zeta\}$,

$$\begin{aligned} \zeta_\emptyset\, x = x\, \zeta_\emptyset &= x, \\ 0_\zeta\, x = x\, 0_\zeta &= 0_\zeta, \\ {\zeta_\emptyset}^2 &= 0_\zeta, \end{aligned}$$

and

$$\zeta_i\zeta_j = \begin{cases} 0_\zeta & \text{if and only if } i = j, \\ \zeta_j\zeta_i & i \neq j. \end{cases}$$

The Abelian null blade semigroup is of particular interest, as its associated semigroup algebra is canonically isomorphic to the *zeon algebra.* Properties of this algebra have been considered and applied in a number of works in recent years, including [5, 6, 14, 16, 17, 18, 22].

Using nearly the same proof as above, it becomes apparent that $\mathfrak{Z}_n$ has two copies of the trivial group as maximal subgroups, and thus has two irreducible representations, regardless of n.

The irreducible representations of both $\mathfrak{Z}_n$ and $\mathfrak{G}_n$ are almost immediately obvious. For arbitrary n, define degree-1 representations θ, ρ_0, and ρ_1 of $\mathfrak{G}_n$ by

$$\theta(s) = 1,\ \forall s \in \mathfrak{G}_n.$$

$$\rho_0(s) = \begin{cases} 1 & s = \gamma_\emptyset, \\ -1 & s = \gamma_\alpha, \\ 0 & \text{otherwise.} \end{cases} \qquad \rho_1(s) = \begin{cases} 1 & s = \gamma_\emptyset, \\ 1 & s = \gamma_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Note that these representations are clearly not faithful.

In $\mathfrak{Z}_n$, the irreducible representations are simply θ and the degree-1 representation ρ given by

$$\rho(s) = \begin{cases} 1 & s = \zeta_\emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Given an arbitrary natural number n, there is a faithful representation τ of $\mathfrak{G}_n$ of order 2^n given by

$$\tau(\gamma_i) = {\sigma_{\mathrm{x}}}^{\otimes(i-1)} \otimes \eta \otimes {\sigma_0}^{\otimes(n-i)},$$

where $\eta = \sigma_{\mathrm{z}} + \imath\sigma_{\mathrm{y}} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$.

Similarly, a faithful representation ψ of $\mathfrak{Z}_n$ is given by

$$\psi(\zeta_i) = {\sigma_0}^{\otimes(i-1)} \otimes \eta \otimes {\sigma_0}^{\otimes(n-i)}.$$

4.2 The idempotent blade semigroup $\mathscr{J}_n$

In the idempotent blade semigroup, generators are idempotent. The resulting semigroup algebra is isomorphic to the “idem-Clifford” algebra $\mathcal{C}\ell_n{}^{\mathrm{idem}}$ used to define column-idempotent adjacency matrices of graphs [18]. These operators can be used to symbolically represent collections of cycles as products of algebraic elements. In such a product, a graph’s edges are associated with idempotents to avoid “double counting” in enumeration problems.

Definition 4.8. Let $\mathscr{J}_n$ denote the Abelian semigroup of order 2^n generated by the collection $E = \{\varepsilon_i : 1 \leq i \leq n\}$ along with $\varepsilon_\emptyset$ satisfying the following generating relations: for all $x \in E \cup \{\varepsilon_\emptyset\}$ and for all $i, j \in \{1, \ldots, n\}$,

$$\begin{aligned} \varepsilon_\emptyset\, x = x\, \varepsilon_\emptyset &= x, \\ {\varepsilon_i}^2 &= \varepsilon_i, \text{ and} \\ \varepsilon_i \varepsilon_j &= \varepsilon_j \varepsilon_i. \end{aligned}$$

Theorem 4.9. *For any natural number n, there are 2^n irreducible representations of $\mathscr{J}_n$.*

Proof. The proof method follows the same format as the previous one. Every $\mathfrak{J}$-class of $\mathscr{J}_n$ is classified, and then the regular classes are identified. From each regular $\mathfrak{J}$-class one idempotent element is chosen and the maximal subgroup at e is computed.

Each element is in its own $\mathfrak{J}$-class with no equivalent idempotent elements, giving $|\mathscr{J}_n| = 2^n$ unique idempotents. The maximal subgroups are found to be $G_{\varepsilon_\emptyset} = \{\varepsilon_\emptyset\}$ and $G_{\varepsilon_I} = \{\varepsilon_I\}$ for arbitrary non-trivial idempotent ε_I.

Enumerating the idempotent elements $\{f_1, ..., f_{2^n}\}$ and letting k_i be the number of conjugacy classes in G_{f_i}, the number of irreducible representations is thus

$$\sum_{i=1}^{2^n} k_i = \sum_{i=1}^{2^n} 1 = 2^n.$$

□

It would be nice if we were able to supply faithful representations of $\mathscr{J}_n$, even if they are reducible. This isn't too difficult, let $\tau : \mathscr{J}_n \to \mathrm{End}(\mathbb{C}^{n+1})$ be defined on the set $\{\varepsilon_i\}$ by

$$\tau(\varepsilon_i) = (a^i_{jk}),$$

where

$$a^i_{jk} = \begin{cases} 1 & j = k \neq i, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

In other words, (a^i_{jk}) is the matrix with ones on the diagonal except in the i^{th} position, and zeros elsewhere. These matrices are all idempotent and commute pairwise. This is extended by multiplication to all of $\mathscr{J}_n$ so that

$$\tau(\varepsilon_I) = (a^I_{jk}),$$

where

$$a^I_{jk} = \begin{cases} 1 & j = k \notin I, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 4.10. For each $i = 1, \ldots, n$, the matrix defined in (4.1) represents a *hyperplane projection* in $\mathbb{C}^{n+1}$. In particular, the i^{th} matrix represents a projection onto the hyperplane orthogonal to the i^{th} unit coordinate vector of $\mathbb{C}^{n+1}$.

5 Combinatorial Graded Semigroup Algebras

Beginning with a finite multiplicative semigroup, S, the *semigroup algebra* of S over $\mathbb{R}$ is the algebra $\mathbb{R}S$ whose additive group is the Abelian group of formal $\mathbb{R}$-linear combinations of elements of S, i.e.,

$$\mathbb{R}S = \left\{ \sum_{s\in S} \alpha_s s : \alpha_s \in \mathbb{R} \right\}$$

and whose multiplication operation is defined by linear extension of the group multiplication operation of S. This definition restricts in a natural way to group algebras.

Given a (complex) representation ρ of a finite semigroup S, let $\widetilde{\rho}$ denote the representation of the $|S|$-dimensional semigroup algebra $\mathbb{C}S$ given by

$$x = \sum_{s\in G} \alpha_s s \Rightarrow \widetilde{\rho}(x) = \sum_{s\in S} \alpha_s \rho(s),$$

where $\alpha_s \in \mathbb{C}$ for each $s \in S$.

It is known that ρ is irreducible if and only if $\widetilde{\rho}$ is irreducible [13], so that the irreducible representations of S are in one-to-one correspondence with the irreducible representations of $\mathbb{C}S$. In particular, if $\widetilde{\rho}$ is an irreducible representation of $\mathbb{C}S$, an irreducible representation of S is obtained by restricting $\widetilde{\rho}$ to the elements of S.

Classifying the irreducible representations of $\mathcal{B}_p^q$ and $\mathcal{S}_p^q$ thereby classifies the irreducible representations of the group algebras $\mathbb{R}\mathcal{B}_p^q$ and $\mathbb{R}\mathcal{S}_p^q$. Similarly, classifying the irreducible representations for $\mathfrak{G}_n, \mathfrak{Z}_n$ and $\mathcal{J}_n$ classifies the irreducible representations of the semigroup algebras $\mathbb{R}\mathfrak{G}_n$, $\mathbb{R}\mathfrak{Z}_n$, and $\mathbb{R}\mathcal{J}_n$. Taking quotients reveals the algebras introduced in column 4 of Table 3.

To summarize:

- The Clifford algebra $\mathcal{C}\ell_{p,q}$ $(p+q>1)$ is canonically isomorphic to the blade group quotient algebra $\mathbb{R}\mathcal{B}_p^q/\langle e_\alpha + e_\emptyset\rangle$. Considering the degree-1 representations, $\rho_J(e_\emptyset) = \rho_J(e_\alpha) = 1$ for all $J \in 2^{[p+q]}$. It then becomes clear that passing to the quotient has no effect on the number of irreducible representations. On the other hand, the higher-dimensional irreducible representations satisfy $\tilde{\tau}(e_\emptyset + e_\alpha) = 0$ *a priori*, so that representations of the group algebra are precisely the representations of the quotient algebra [2]

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[2] While results are stated here within the context of complex representation spaces, particular representations are, in fact, real. For example, the construction given in (3.1) for $\mathcal{B}_p^q$ when $p = q$ yields elements of $\mathrm{GL}_{2^k}(\mathbb{R})$. Degrees of faithful representations then vary by group signature. A detailed treatment of smallest fields for representation spaces and minimal degrees of faithful representations is outside the scope of this work, as the goal is to enumerate irreducible complex representations for combinatorial semigroups. Such details for the quotient group algebra $\mathbb{R}\mathcal{B}_p^q/\langle e_\alpha + e_\emptyset\rangle$ are covered by known results on matrix representations of Clifford algebras (e.g., Bott periodicity) [2, 3, 9, 11, 12].

- The *symmetric Clifford algebra*, ${\mathcal{C}\ell_{p,q}}^{\mathrm{sym}}$ [19, 21], is canonically isomorphic to the Abelian blade group algebra $\mathbb{R}\mathcal{S}_p^q/\langle\varsigma_\alpha+\varsigma_\emptyset\rangle$. By similar reasoning to that for the blade group quotient algebra, the number of irreducible representations is unchanged by considering the quotient.

- The Grassmann exterior algebra, $\bigwedge\mathbb{R}^n$, is canonically isomorphic to the null blade semigroup algebra $\mathcal{B}\ell_{\wedge n}=\mathbb{R}\mathfrak{G}_n/\left\langle 0_\gamma,\gamma_\alpha+\gamma_\emptyset\right\rangle$. This algebra is isomorphic to the algebra of fermion creation (or annihilation) operators.

- The n-particle *zeon algebra* [5, 6, 16, 22] is canonically isomorphic to the Abelian null blade semigroup algebra $\mathbb{R}\mathfrak{Z}_n/\left\langle 0_\zeta\right\rangle$. This algebra is isomorphic to an algebra of *commuting* lowering or raising (annihilation or creation) operators.

- The *idem-Clifford algebra*, ${\mathcal{C}\ell_n}^{\mathrm{idem}}$ [17, 19], is canonically isomorphic to the idempotent-generated semigroup algebra $\mathbb{R}\mathcal{J}_n$.

Example 5.1. Regarding $\gamma_\emptyset$ and γ_α as 1 and -1, respectively, the signed hypercube seen in Figure 2.2 is the undirected graph underlying $\mathcal{C}\ell_{0,3}$. Similarly, Figure 2.1 underlies the symmetric Clifford algebra ${\mathcal{C}\ell_{4,0}}^{\mathrm{sym}}$.

Group or Semigroup	Algebra	Quotient Algebra	Isomorphic Algebra
$\mathcal{B}_p^q$	$\mathbb{R}\mathcal{B}_p^q$	$\mathbb{R}\mathcal{B}_p^q/\langle e_\alpha+e_\emptyset\rangle$	$\mathcal{C}\ell_{p,q}$
$\mathcal{S}_p^q$	$\mathbb{R}\mathcal{S}_p^q$	$\mathbb{R}\mathcal{S}_p^q/\langle\varsigma_\alpha+\varsigma_\emptyset\rangle$	${\mathcal{C}\ell_{p,q}}^{\mathrm{sym}}$
$\mathfrak{G}_n$	$\mathbb{R}\mathfrak{G}_n$	$\mathbb{R}\mathfrak{G}_n/\left\langle 0_\gamma,\gamma_\alpha+\gamma_\emptyset\right\rangle$	$\bigwedge\mathbb{R}^n$
$\mathfrak{Z}_n$	$\mathbb{R}\mathfrak{Z}_n$	$\mathbb{R}\mathfrak{Z}_n/\left\langle 0_\zeta\right\rangle$	${\mathcal{C}\ell_n}^{\mathrm{nil}}$
$\mathcal{J}_n$	$\mathbb{R}\mathcal{J}_n$	$\mathbb{R}\mathcal{J}_n$	${\mathcal{C}\ell_n}^{\mathrm{idem}}$

Table 3: Semigroup algebras.

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