# CliffMath2018 Users' Guide & Examples

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# 1

# Getting started

## Downloading and installing the package

The **CliffMath** package can be downloaded directly via the Research link at http://www.siue.edu/~sstaple. The source code and package are freely available as separate files. To get up and running, it is sufficient to download the package file **CliffMath2018.m** and place it into a folder easily accessible to *Mathematica*. One can examine the default list of directories searched by Mathematica when attempting to find an external file by executing the $Path command, as seen in the following example.

```
$Path
```

```
{/Users/g.staceystaples/Library/Mathematica/DocumentationIndices,
 /Applications/Mathematica.app/Contents/SystemFiles/Links,
 /Users/g.staceystaples/Library/Mathematica/Kernel,
 /Users/g.staceystaples/Library/Mathematica/Autoload,
 /Users/g.staceystaples/Library/Mathematica/Applications, /Library/Mathematica/Kernel,
 /Library/Mathematica/Autoload, /Library/Mathematica/Applications, .,
 /Users/g.staceystaples, /Applications/Mathematica.app/Contents/AddOns/Packages,
 /Applications/Mathematica.app/Contents/SystemFiles/Autoload,
 /Applications/Mathematica.app/Contents/AddOns/Autoload,
 /Applications/Mathematica.app/Contents/AddOns/Applications,
 /Applications/Mathematica.app/Contents/AddOns/ExtraPackages,
 /Applications/Mathematica.app/Contents/SystemFiles/Kernel/Packages,
 /Applications/Mathematica.app/Contents/Documentation/English/System,
 /Applications/Mathematica.app/Contents/SystemFiles/Data/ICC}
```

The package file can either be placed into one of the folders appearing in the list, or a new folder can be appended to the list. For example, the following code appends a folder called "CliffMath" (assumed to be located in my home directory) to the search path.

```
AppendTo[$Path, FileNameJoin[{$HomeDirectory, "CliffMath"}]]
```

```
{/Users/g.staceystaples/Library/Mathematica/DocumentationIndices,
 /Applications/Mathematica.app/Contents/SystemFiles/Links,
 /Users/g.staceystaples/Library/Mathematica/Kernel,
 /Users/g.staceystaples/Library/Mathematica/Autoload,
 /Users/g.staceystaples/Library/Mathematica/Applications, /Library/Mathematica/Kernel,
 /Library/Mathematica/Autoload, /Library/Mathematica/Applications, .,
 /Users/g.staceystaples, /Applications/Mathematica.app/Contents/AddOns/Packages,
 /Applications/Mathematica.app/Contents/SystemFiles/Autoload,
 /Applications/Mathematica.app/Contents/AddOns/Autoload,
 /Applications/Mathematica.app/Contents/AddOns/Applications,
 /Applications/Mathematica.app/Contents/AddOns/ExtraPackages,
 /Applications/Mathematica.app/Contents/SystemFiles/Kernel/Packages,
 /Applications/Mathematica.app/Contents/Documentation/English/System,
 /Applications/Mathematica.app/Contents/SystemFiles/Data/ICC,
 /Users/g.staceystaples/CliffMath}
```

### Loading the package and initializing the Clifford (sub) algebras

Once the package file has been saved and its location has been included in the search path, it is necessary to load the package. This is accomplished with *Get*.

```
<< CliffMath2018`
```

The **CliffMath** package facilitates computations in Clifford algebras and a number of subalgebras:

- Clifford algebras of arbitrary signature $Cl_{p,q,r}$
- Grassmann (exterior) algebras
- Zeon algebras $Cl_n^{nil}$
- Sym-Clifford algebras $Cl_{p,q,r}{}^{sym}$
- Idem-Clifford algebras $Cl_n{}^{idem}$

In order to get started with any of these algebras, one first must set the signature (or dimension) of the algebra in which one intends to work. The first command to be issued is setSignature*[p,q,r]*.

```
? setSignature
```

setSignature[p,q,r] – Initializes Clifford multiplication in $Cl_{p,q,r}$, as well as `commutative Clifford' multiplication in $Cl_{p,q,r}{}^{sym}$. For n=p+q+r, initializes exterior (Grassmann) multiplication in $\Lambda R^n$, `idempotent–Clifford' multiplication in $Cl_n{}^{idem}$ and `zeon multiplication' in $Cl_n{}^{nil}$

Note: The first argument, *p*, is required. Default values (zero) apply to *q* and *r* if values are omitted.

### Basis elements as context for multiplication

The Clifford multiplication operator (CircleDot[a,b] = a⊙b) acts according to the multiplication rules of the algebra being used, i.e., by context. The context is determined by the basis elements appearing in the arguments. **CliffMath** reserves four specific symbols to represent the Clifford algebra and subalgebras: e, $\zeta$, $\varsigma$, and $\varepsilon$.

```
setSignature[3, 0];
Print["---"];
Print["Clifford Algebras"];
Print["clBasis: ", clBasis];
Print["---"];
Print["Zeon (nil-Clifford) Algebras"];
Print["zeonBasis: ", zeonBasis];
Print["---"];
Print["sym-Clifford Algebras"];
Print["symBasis: ", symBasis];
Print["---"];
Print["idem-Clifford Algebras"];
Print["idemBasis: ", idemBasis];
```

---

Clifford Algebras

clBasis: $\{1, e_{\{1\}}, e_{\{2\}}, e_{\{1,2\}}, e_{\{3\}}, e_{\{1,3\}}, e_{\{2,3\}}, e_{\{1,2,3\}}\}$

---

Zeon (nil-Clifford) Algebras

zeonBasis: $\{1, \zeta_{\{1\}}, \zeta_{\{2\}}, \zeta_{\{1,2\}}, \zeta_{\{3\}}, \zeta_{\{1,3\}}, \zeta_{\{2,3\}}, \zeta_{\{1,2,3\}}\}$

---

sym-Clifford Algebras

symBasis: $\{1, \varsigma_{\{1\}}, \varsigma_{\{2\}}, \varsigma_{\{1,2\}}, \varsigma_{\{3\}}, \varsigma_{\{1,3\}}, \varsigma_{\{2,3\}}, \varsigma_{\{1,2,3\}}\}$

---

idem-Clifford Algebras

idemBasis: $\{1, \varepsilon_{\{1\}}, \varepsilon_{\{2\}}, \varepsilon_{\{1,2\}}, \varepsilon_{\{3\}}, \varepsilon_{\{1,3\}}, \varepsilon_{\{2,3\}}, \varepsilon_{\{1,2,3\}}\}$

## Prememorization vs. Adaptive Learning

Unlike many other software implementations of Clifford algebras that begin by establishing precomputed multiplication tables for basis blades, **CliffMath** has always performed computations “on the fly”. The most obvious advantage of this approach is reduced memory allocation. The newest version of **CliffMath** is designed to recall (or memorize) the results of basic computations as they are carried out. As a result, processing gets faster as computations are performed. Unlike using precomputed tables, the results memorized are only the results being used. It is therefore possible to work efficiently in algebras of high dimension when computations are being carried out in relatively low-dimensional subspaces.

For those who prefer maximum computational speed, the precomputed multiplication tables can be established by executing the following single line of code. For example, when working in the Clifford algebra $Cl_{p,q,r}$, the following command initializes the required multiplication table.

```
clExpand[Total[clBasis] ⊙ Total[clBasis]];
```

In addition to defining the signature or dimensions of the algebras being used, the setSignature command also releases memory that has been allocated to storing previous **CliffMath** results.

```
In[•]:= setSignature[6]
```

```
In[∘]:= α = Table[Random[Integer, {-5, 5}], {j, 2^maxIndex}].clBasis;
β = Table[Random[Integer, {-5, 5}], {j, 2^maxIndex}].clBasis;
Print["α = ", α];
Print["β = ", β];
{time, res} = Timing[clExpand[α ⊙ β]];
Print["αβ = ", res];
Print["Computation time in seconds: ", time];
```

$$\alpha = -2\,e_{\{2\}} + 4\,e_{\{3\}} + 5\,e_{\{4\}} - 2\,e_{\{5\}} + 5\,e_{\{6\}} + 5\,e_{\{1,2\}} - 5\,e_{\{1,3\}} + 3\,e_{\{1,4\}} + 5\,e_{\{1,5\}} + 3\,e_{\{1,6\}} - 3\,e_{\{2,3\}} + e_{\{2,4\}} - 3\,e_{\{2,5\}} + 4\,e_{\{2,6\}} + 2\,e_{\{3,4\}} + 5\,e_{\{3,5\}} - 3\,e_{\{3,6\}} - 4\,e_{\{4,5\}} + 5\,e_{\{4,6\}} + e_{\{5,6\}} + 2\,e_{\{1,2,3\}} + e_{\{1,2,4\}} + 5\,e_{\{1,2,5\}} + e_{\{1,2,6\}} - 5\,e_{\{1,3,4\}} + 5\,e_{\{1,3,6\}} + 4\,e_{\{1,4,5\}} + 5\,e_{\{1,5,6\}} + 2\,e_{\{2,3,4\}} - 2\,e_{\{2,3,5\}} - 2\,e_{\{2,3,6\}} + 3\,e_{\{2,4,5\}} - 3\,e_{\{2,4,6\}} + 3\,e_{\{2,5,6\}} + 2\,e_{\{3,4,5\}} + e_{\{3,4,6\}} + e_{\{3,5,6\}} - 3\,e_{\{4,5,6\}} + 2\,e_{\{1,2,3,4\}} + 4\,e_{\{1,2,3,5\}} - 2\,e_{\{1,2,3,6\}} - 3\,e_{\{1,2,4,5\}} + e_{\{1,2,4,6\}} - 4\,e_{\{1,2,5,6\}} - 4\,e_{\{1,3,4,5\}} + e_{\{1,3,4,6\}} - 5\,e_{\{1,3,5,6\}} - 5\,e_{\{1,4,5,6\}} - 3\,e_{\{2,3,4,5\}} + 4\,e_{\{2,3,4,6\}} - 2\,e_{\{2,3,5,6\}} + e_{\{2,4,5,6\}} + 5\,e_{\{3,4,5,6\}} - e_{\{1,2,3,4,6\}} - 2\,e_{\{1,2,3,5,6\}} - e_{\{1,3,4,5,6\}} + 2\,e_{\{2,3,4,5,6\}} + e_{\{1,2,3,4,5,6\}}$$

$$\beta = 2 + e_{\{1\}} - 2\,e_{\{2\}} + 5\,e_{\{3\}} + 2\,e_{\{4\}} + 5\,e_{\{5\}} + 3\,e_{\{6\}} + 4\,e_{\{1,2\}} + 5\,e_{\{1,3\}} - e_{\{1,4\}} - 3\,e_{\{1,5\}} + 4\,e_{\{1,6\}} + 2\,e_{\{2,3\}} + 4\,e_{\{2,4\}} - 3\,e_{\{2,5\}} + 3\,e_{\{2,6\}} + 3\,e_{\{3,4\}} - 4\,e_{\{3,5\}} + 4\,e_{\{3,6\}} - 3\,e_{\{4,5\}} - 5\,e_{\{4,6\}} - 5\,e_{\{5,6\}} + e_{\{1,2,5\}} - 3\,e_{\{1,2,6\}} - 3\,e_{\{1,3,4\}} - 4\,e_{\{1,3,5\}} + 3\,e_{\{1,3,6\}} - 2\,e_{\{1,4,5\}} + 2\,e_{\{1,4,6\}} - 5\,e_{\{1,5,6\}} - 4\,e_{\{2,3,4\}} - 3\,e_{\{2,3,6\}} + 2\,e_{\{2,4,5\}} - 4\,e_{\{2,4,6\}} - 5\,e_{\{2,5,6\}} - 5\,e_{\{3,4,5\}} - 5\,e_{\{3,4,6\}} - 4\,e_{\{3,5,6\}} - 2\,e_{\{4,5,6\}} - e_{\{1,2,3,4\}} + 5\,e_{\{1,2,3,5\}} + e_{\{1,2,3,6\}} - 3\,e_{\{1,2,4,5\}} + 2\,e_{\{1,2,4,6\}} + 5\,e_{\{1,2,5,6\}} + 4\,e_{\{1,3,4,5\}} - 3\,e_{\{1,3,4,6\}} - 3\,e_{\{1,3,5,6\}} + 5\,e_{\{1,4,5,6\}} - 3\,e_{\{2,3,4,5\}} - 2\,e_{\{2,3,5,6\}} + 4\,e_{\{2,4,5,6\}} - 4\,e_{\{3,4,5,6\}} + 2\,e_{\{1,2,3,4,6\}} - 3\,e_{\{1,2,3,5,6\}} + 3\,e_{\{1,2,4,5,6\}} + 4\,e_{\{1,3,4,5,6\}} - e_{\{2,3,4,5,6\}} + 5\,e_{\{1,2,3,4,5,6\}}$$

$$\alpha\beta = 56 - e_{\{1\}} - 122\,e_{\{2\}} + 91\,e_{\{3\}} - 163\,e_{\{4\}} - e_{\{5\}} - 73\,e_{\{6\}} + 2\,e_{\{1,2\}} - 12\,e_{\{1,3\}} + 174\,e_{\{1,4\}} + 56\,e_{\{1,5\}} - 71\,e_{\{1,6\}} - 170\,e_{\{2,3\}} + 82\,e_{\{2,4\}} - 7\,e_{\{2,5\}} - 31\,e_{\{2,6\}} - 80\,e_{\{3,4\}} + 19\,e_{\{3,5\}} - 49\,e_{\{3,6\}} - 6\,e_{\{4,5\}} - 58\,e_{\{4,6\}} - 95\,e_{\{5,6\}} + 13\,e_{\{1,2,3\}} + 42\,e_{\{1,2,4\}} - 40\,e_{\{1,2,5\}} + 95\,e_{\{1,2,6\}} + 33\,e_{\{1,3,4\}} + 38\,e_{\{1,3,5\}} + 134\,e_{\{1,3,6\}} - 12\,e_{\{1,4,5\}} + 138\,e_{\{1,4,6\}} - 126\,e_{\{1,5,6\}} - 93\,e_{\{2,3,4\}} - 77\,e_{\{2,3,5\}} - 18\,e_{\{2,3,6\}} + 103\,e_{\{2,4,5\}} - 9\,e_{\{2,4,6\}} + 134\,e_{\{2,5,6\}} + 85\,e_{\{3,4,5\}} - 45\,e_{\{3,4,6\}} + 83\,e_{\{3,5,6\}} - 176\,e_{\{4,5,6\}} + 34\,e_{\{1,2,3,4\}} + 19\,e_{\{1,2,3,5\}} + 6\,e_{\{1,2,3,6\}} - 17\,e_{\{1,2,4,5\}} + 64\,e_{\{1,2,4,6\}} - 118\,e_{\{1,2,5,6\}} + 62\,e_{\{1,3,4,5\}} + 117\,e_{\{1,3,4,6\}} + 97\,e_{\{1,3,5,6\}} + 17\,e_{\{1,4,5,6\}} - 31\,e_{\{2,3,4,5\}} + 75\,e_{\{2,3,4,6\}} - 6\,e_{\{2,3,5,6\}} - 69\,e_{\{2,4,5,6\}} + 50\,e_{\{3,4,5,6\}} + 41\,e_{\{1,2,3,4,5\}} - 163\,e_{\{1,2,3,4,6\}} - 80\,e_{\{1,2,3,5,6\}} - 43\,e_{\{1,2,4,5,6\}} - 9\,e_{\{1,3,4,5,6\}} - 95\,e_{\{2,3,4,5,6\}} + e_{\{1,2,3,4,5,6\}}$$

```
Computation time in seconds: 0.178471
```

After memorizing the blade multiplication table, the experiment is repeated.

```
In[∘]:= setSignature[6];
(* Memorize the basis blade multiplication table *)
clExpand[Total[clBasis] ⊙ Total[clBasis]];
```

```
In[∘]:= α = Table[Random[Integer, {-5, 5}], {j, 2^maxIndex}].clBasis;
β = Table[Random[Integer, {-5, 5}], {j, 2^maxIndex}].clBasis;
Print["α = ", α];
Print["β = ", β];
{time, res} = Timing[clExpand[α ⊙ β]];
Print["αβ = ", res];
Print["Computation time in seconds: ", time];
```

$$\alpha = 4 - 5\,e_{\{1\}} - 5\,e_{\{2\}} + 3\,e_{\{3\}} + 3\,e_{\{5\}} - e_{\{6\}} + 5\,e_{\{1,3\}} + e_{\{1,4\}} - 2\,e_{\{2,3\}} + 5\,e_{\{2,4\}} + 4\,e_{\{2,5\}} - 5\,e_{\{2,6\}} - 3\,e_{\{3,4\}} - 3\,e_{\{3,5\}} + 3\,e_{\{3,6\}} + 5\,e_{\{4,5\}} - 3\,e_{\{4,6\}} - 4\,e_{\{5,6\}} - e_{\{1,2,3\}} - 4\,e_{\{1,2,4\}} + 2\,e_{\{1,2,5\}} + 4\,e_{\{1,2,6\}} + 3\,e_{\{1,3,4\}} + 5\,e_{\{1,3,5\}} + 2\,e_{\{1,3,6\}} + 4\,e_{\{1,4,5\}} - 3\,e_{\{1,4,6\}} - e_{\{1,5,6\}} + 3\,e_{\{2,3,4\}} - 5\,e_{\{2,3,5\}} - 3\,e_{\{2,3,6\}} - 4\,e_{\{2,4,5\}} - 3\,e_{\{2,4,6\}} + 2\,e_{\{2,5,6\}} + 4\,e_{\{3,4,5\}} - 2\,e_{\{3,5,6\}} - 3\,e_{\{4,5,6\}} - 4\,e_{\{1,2,3,4\}} - e_{\{1,2,3,5\}} + 5\,e_{\{1,2,3,6\}} + 2\,e_{\{1,2,4,5\}} - 2\,e_{\{1,2,4,6\}} + 3\,e_{\{1,2,5,6\}} + 2\,e_{\{1,3,4,5\}} - 4\,e_{\{1,3,4,6\}} + 2\,e_{\{1,3,5,6\}} - 2\,e_{\{1,4,5,6\}} - 5\,e_{\{2,3,4,5\}} + e_{\{2,3,4,6\}} - e_{\{2,3,5,6\}} - 4\,e_{\{2,4,5,6\}} - 3\,e_{\{3,4,5,6\}} - 2\,e_{\{1,2,3,4,5\}} + 2\,e_{\{1,2,3,4,6\}} + 3\,e_{\{1,2,3,5,6\}} - 5\,e_{\{1,2,4,5,6\}} + 3\,e_{\{1,3,4,5,6\}} - 5\,e_{\{2,3,4,5,6\}} + 4\,e_{\{1,2,3,4,5,6\}}$$

$$\beta = -4 - 5\,e_{\{1\}} - 4\,e_{\{2\}} - 4\,e_{\{4\}} - 4\,e_{\{5\}} + 2\,e_{\{6\}} - e_{\{1,2\}} - e_{\{1,3\}} + 3\,e_{\{1,4\}} - 3\,e_{\{1,5\}} + 3\,e_{\{1,6\}} - 2\,e_{\{2,3\}} + e_{\{2,4\}} - 4\,e_{\{2,5\}} + 5\,e_{\{3,4\}} - 2\,e_{\{3,5\}} - 4\,e_{\{3,6\}} - e_{\{4,5\}} + 2\,e_{\{4,6\}} - 5\,e_{\{5,6\}} + 4\,e_{\{1,2,3\}} + 2\,e_{\{1,2,4\}} + 2\,e_{\{1,2,5\}} - 3\,e_{\{1,2,6\}} - 3\,e_{\{1,3,4\}} - e_{\{1,3,5\}} + 4\,e_{\{1,4,5\}} + 5\,e_{\{1,4,6\}} + 4\,e_{\{2,3,4\}} - 4\,e_{\{2,3,5\}} + 2\,e_{\{2,3,6\}} - 3\,e_{\{2,4,5\}} + 2\,e_{\{2,4,6\}} + 5\,e_{\{2,5,6\}} + 5\,e_{\{3,4,5\}} + 3\,e_{\{3,4,6\}} + e_{\{3,5,6\}} - e_{\{4,5,6\}} - e_{\{1,2,3,4\}} - 3\,e_{\{1,2,3,5\}} + e_{\{1,2,3,6\}} - 5\,e_{\{1,2,4,5\}} - 2\,e_{\{1,2,4,6\}} - e_{\{1,2,5,6\}} - 5\,e_{\{1,3,4,5\}} - 4\,e_{\{1,3,5,6\}} + 5\,e_{\{2,3,4,5\}} - 4\,e_{\{2,3,4,6\}} - 3\,e_{\{2,3,5,6\}} - 4\,e_{\{2,4,5,6\}} + 4\,e_{\{3,4,5,6\}} + 4\,e_{\{1,2,3,4,5\}} - 3\,e_{\{1,2,3,4,6\}} - 4\,e_{\{1,2,3,5,6\}} - 3\,e_{\{1,2,4,5,6\}} + 2\,e_{\{1,3,4,5,6\}} - 2\,e_{\{2,3,4,5,6\}} - 2\,e_{\{1,2,3,4,5,6\}}$$

$$\alpha\beta = -18 - 28\,e_{\{1\}} + 61\,e_{\{2\}} + 48\,e_{\{3\}} - 133\,e_{\{4\}} - 37\,e_{\{5\}} - 87\,e_{\{6\}} + 66\,e_{\{1,2\}} + 9\,e_{\{1,3\}} - 31\,e_{\{1,4\}} - 23\,e_{\{1,5\}} + 104\,e_{\{1,6\}} - 194\,e_{\{2,3\}} - 109\,e_{\{2,4\}} - 40\,e_{\{2,5\}} + 5\,e_{\{2,6\}} - 151\,e_{\{3,4\}} + 106\,e_{\{3,5\}} - 98\,e_{\{3,6\}} - 5\,e_{\{4,5\}} - 72\,e_{\{4,6\}} + 69\,e_{\{5,6\}} + 46\,e_{\{1,2,3\}} - 86\,e_{\{1,2,4\}} + 26\,e_{\{1,2,5\}} + 71\,e_{\{1,2,6\}} - 69\,e_{\{1,3,4\}} + 2\,e_{\{1,3,5\}} - 51\,e_{\{1,3,6\}} - 2\,e_{\{1,4,5\}} + 54\,e_{\{1,4,6\}} - 82\,e_{\{1,5,6\}} + 45\,e_{\{2,3,4\}} - 75\,e_{\{2,3,5\}} + 111\,e_{\{2,3,6\}} - 40\,e_{\{2,4,5\}} + 4\,e_{\{2,4,6\}} + 85\,e_{\{2,5,6\}} - 83\,e_{\{3,4,5\}} + 38\,e_{\{3,4,6\}} + 51\,e_{\{3,5,6\}} + 48\,e_{\{4,5,6\}} - 46\,e_{\{1,2,3,4\}} + 121\,e_{\{1,2,3,5\}} - 17\,e_{\{1,2,3,6\}} - 3\,e_{\{1,2,4,5\}} + 34\,e_{\{1,2,4,6\}} + 18\,e_{\{1,2,5,6\}} - 102\,e_{\{1,3,4,5\}} - 97\,e_{\{1,3,4,6\}} + 4\,e_{\{1,3,5,6\}} - 153\,e_{\{1,4,5,6\}} - 39\,e_{\{2,3,4,5\}} - 109\,e_{\{2,3,4,6\}} + 5\,e_{\{2,3,5,6\}} - 119\,e_{\{2,4,5,6\}} + 49\,e_{\{3,4,5,6\}} - 29\,e_{\{1,2,3,4,5\}} - 20\,e_{\{1,2,3,4,6\}} + 12\,e_{\{1,2,3,5,6\}} - 7\,e_{\{1,2,4,5,6\}} - 120\,e_{\{1,3,4,5,6\}} + 62\,e_{\{2,3,4,5,6\}} - 73\,e_{\{1,2,3,4,5,6\}}$$

```
Computation time in seconds: 0.091106
```

Similar commands are used when working in nil-Clifford (zeon), sym-Clifford, and idem-Clifford algebras.

```
clExpand[Total[zeonBasis] ⊙ Total[zeonBasis]];
clExpand[Total[symBasis] ⊙ Total[symBasis]];
clExpand[Total[idemBasis] ⊙ Total[idemBasis]];
```

**Note**: All memorized values are cleared by the **setSignature** command.

# 2

# Clifford algebras of arbitrary signature

Canonical generators for $\mathrm{Cl}_{p,q,r}$ are denoted $e_{\{i\}}$ for $i$=1,..., $n$=$p$+$q$+$r$. The generators anti-commute and their squares satisfy the following:

$e_{\{i\}}{}^2 = 1\ (1 \le i \le p)$

$e_{\{i\}}{}^2 = -1\ (p+1 \le i \le p+q)$

$e_{\{i\}}{}^2 = 0\ (p+q+1 \le i \le p+q+r)$.

In[∘]:=
```
(* We'll work in Cl_{2,1,2} first. *)
setSignature[2, 1, 2]
```

Since $n$=$p$+$q$+$4$=5, the algebra is 32-dimensional. The canonical basis can be accessed directly in **CliffMath** via clBasis.

In[∘]:=
```
clBasis
```

Out[∘]= $\{1, e_{\{1\}}, e_{\{2\}}, e_{\{1,2\}}, e_{\{3\}}, e_{\{1,3\}}, e_{\{2,3\}}, e_{\{1,2,3\}}, e_{\{4\}}, e_{\{1,4\}}, e_{\{2,4\}}, e_{\{1,2,4\}}, e_{\{3,4\}}, e_{\{1,3,4\}}, e_{\{2,3,4\}}, e_{\{1,2,3,4\}}, e_{\{5\}}, e_{\{1,5\}}, e_{\{2,5\}}, e_{\{1,2,5\}}, e_{\{3,5\}}, e_{\{1,3,5\}}, e_{\{2,3,5\}}, e_{\{1,2,3,5\}}, e_{\{4,5\}}, e_{\{1,4,5\}}, e_{\{2,4,5\}}, e_{\{1,2,4,5\}}, e_{\{3,4,5\}}, e_{\{1,3,4,5\}}, e_{\{2,3,4,5\}}, e_{\{1,2,3,4,5\}}\}$

In[∘]:=
```
(* Pseudorandomly generate two elements *)
α = Table[Random[Integer, {-5, 5}], {j, 2^maxIndex}].clBasis

β = Table[Random[Integer, {-5, 5}], {j, 2^maxIndex}].clBasis
```

Out[∘]= $-1 + 5\,e_{\{1\}} - 4\,e_{\{2\}} - 2\,e_{\{3\}} - 4\,e_{\{4\}} - 5\,e_{\{5\}} - 2\,e_{\{1,2\}} + 3\,e_{\{1,3\}} - 2\,e_{\{1,4\}} + 3\,e_{\{1,5\}} + 4\,e_{\{2,3\}} + 3\,e_{\{2,4\}} + 5\,e_{\{2,5\}} + 4\,e_{\{3,4\}} - 5\,e_{\{3,5\}} - e_{\{4,5\}} + 4\,e_{\{1,2,3\}} - 5\,e_{\{1,2,4\}} - 2\,e_{\{1,2,5\}} - 5\,e_{\{1,3,4\}} - e_{\{1,3,5\}} - 3\,e_{\{1,4,5\}} - e_{\{2,3,4\}} + e_{\{2,3,5\}} + 4\,e_{\{2,4,5\}} + 4\,e_{\{3,4,5\}} - 5\,e_{\{1,2,3,4\}} - e_{\{1,2,3,5\}} - 2\,e_{\{1,2,4,5\}} + e_{\{1,3,4,5\}} + 4\,e_{\{1,2,3,4,5\}}$

Out[∘]= $1 + 4\,e_{\{1\}} + 5\,e_{\{2\}} - 3\,e_{\{3\}} - 2\,e_{\{4\}} + 2\,e_{\{5\}} - e_{\{1,2\}} - 5\,e_{\{1,4\}} + 3\,e_{\{1,5\}} + 3\,e_{\{2,3\}} - e_{\{2,4\}} + 5\,e_{\{2,5\}} + 2\,e_{\{3,4\}} - 3\,e_{\{3,5\}} + 4\,e_{\{4,5\}} + 2\,e_{\{1,2,3\}} - e_{\{1,2,4\}} + 2\,e_{\{1,2,5\}} + 4\,e_{\{1,3,4\}} - 2\,e_{\{1,3,5\}} - 5\,e_{\{1,4,5\}} - 3\,e_{\{2,3,4\}} - 4\,e_{\{2,3,5\}} + 2\,e_{\{2,4,5\}} + 3\,e_{\{3,4,5\}} - 5\,e_{\{1,2,3,4\}} + e_{\{1,2,3,5\}} - 4\,e_{\{1,2,4,5\}} + 3\,e_{\{1,3,4,5\}} + 4\,e_{\{2,3,4,5\}} - 4\,e_{\{1,2,3,4,5\}}$

In[∘]:=
```
(*Compute their product *)
α⊙β // clExpand
```

Out[∘]= $11 + 16\,e_{\{1\}} - 6\,e_{\{2\}} - 35\,e_{\{3\}} - 58\,e_{\{4\}} - 51\,e_{\{5\}} + 65\,e_{\{1,2\}} - 18\,e_{\{1,3\}} - 9\,e_{\{1,4\}} + 42\,e_{\{1,5\}} + 46\,e_{\{2,3\}} + e_{\{2,4\}} + 27\,e_{\{2,5\}} + 8\,e_{\{3,4\}} - 39\,e_{\{3,5\}} - 11\,e_{\{4,5\}} + 26\,e_{\{1,2,3\}} - 34\,e_{\{1,2,4\}} + 39\,e_{\{1,2,5\}} - 46\,e_{\{1,3,4\}} - 29\,e_{\{1,3,5\}} + 89\,e_{\{1,4,5\}} - 10\,e_{\{2,3,4\}} + 28\,e_{\{2,3,5\}} - 87\,e_{\{2,4,5\}} - 31\,e_{\{3,4,5\}} - 64\,e_{\{1,2,3,4\}} - 14\,e_{\{1,2,3,5\}} - 30\,e_{\{1,2,4,5\}} - 22\,e_{\{1,3,4,5\}} - 55\,e_{\{2,3,4,5\}} + 34\,e_{\{1,2,3,4,5\}}$

Note: the CircleDot operator that represents multiplication in CliffMath can be typed directly as ESCc.ESC. So, $\alpha \odot \beta$ is obtained by ESCaESC ESCc.ESC ESCbESC.

There are multiple ways of performing this computation in *Mathematica*.

In[∘]:=
```
clExpand[α⊙β]
```

Out[∘]=
$$11 + 16\, e_{\{1\}} - 6\, e_{\{2\}} - 35\, e_{\{3\}} - 58\, e_{\{4\}} - 51\, e_{\{5\}} + 65\, e_{\{1,2\}} - 18\, e_{\{1,3\}} - 9\, e_{\{1,4\}} + 42\, e_{\{1,5\}} + 46\, e_{\{2,3\}} + e_{\{2,4\}} + 27\, e_{\{2,5\}} + 8\, e_{\{3,4\}} - 39\, e_{\{3,5\}} - 11\, e_{\{4,5\}} + 26\, e_{\{1,2,3\}} - 34\, e_{\{1,2,4\}} + 39\, e_{\{1,2,5\}} - 46\, e_{\{1,3,4\}} - 29\, e_{\{1,3,5\}} + 89\, e_{\{1,4,5\}} - 10\, e_{\{2,3,4\}} + 28\, e_{\{2,3,5\}} - 87\, e_{\{2,4,5\}} - 31\, e_{\{3,4,5\}} - 64\, e_{\{1,2,3,4\}} - 14\, e_{\{1,2,3,5\}} - 30\, e_{\{1,2,4,5\}} - 22\, e_{\{1,3,4,5\}} - 55\, e_{\{2,3,4,5\}} + 34\, e_{\{1,2,3,4,5\}}$$

In[∘]:=
```
clExpand[CircleDot[α, β]]
```

Out[∘]=
$$11 + 16\, e_{\{1\}} - 6\, e_{\{2\}} - 35\, e_{\{3\}} - 58\, e_{\{4\}} - 51\, e_{\{5\}} + 65\, e_{\{1,2\}} - 18\, e_{\{1,3\}} - 9\, e_{\{1,4\}} + 42\, e_{\{1,5\}} + 46\, e_{\{2,3\}} + e_{\{2,4\}} + 27\, e_{\{2,5\}} + 8\, e_{\{3,4\}} - 39\, e_{\{3,5\}} - 11\, e_{\{4,5\}} + 26\, e_{\{1,2,3\}} - 34\, e_{\{1,2,4\}} + 39\, e_{\{1,2,5\}} - 46\, e_{\{1,3,4\}} - 29\, e_{\{1,3,5\}} + 89\, e_{\{1,4,5\}} - 10\, e_{\{2,3,4\}} + 28\, e_{\{2,3,5\}} - 87\, e_{\{2,4,5\}} - 31\, e_{\{3,4,5\}} - 64\, e_{\{1,2,3,4\}} - 14\, e_{\{1,2,3,5\}} - 30\, e_{\{1,2,4,5\}} - 22\, e_{\{1,3,4,5\}} - 55\, e_{\{2,3,4,5\}} + 34\, e_{\{1,2,3,4,5\}}$$

In[∘]:=
```
(* Squaring rules for the generators *)
Table[clExpand[e{i}⊙e{i}], {i, 1, maxIndex}]
```

Out[∘]=
```
{1, 1, -1, 0, 0}
```

In[∘]:=
```
gradeKPart[α, 3]
```

Out[∘]=
$$4\, e_{\{1,2,3\}} - 5\, e_{\{1,2,4\}} - 2\, e_{\{1,2,5\}} - 5\, e_{\{1,3,4\}} - e_{\{1,3,5\}} - 3\, e_{\{1,4,5\}} - e_{\{2,3,4\}} + e_{\{2,3,5\}} + 4\, e_{\{2,4,5\}} + 4\, e_{\{3,4,5\}}$$

In[∘]:=
```
cliffordDecomposableQ[β]
```

Out[∘]=
```
False
```

In[∘]:=
```
(* Generate a random vector in current Clifford
  algebra with random integer coefficients between-ell and ell.*)
intrv[ell_] := Total[Table[Random[Integer, {-ell, ell}] e{j}, {j, 1, maxIndex}]]
```

In[∘]:=
```
setSignature[3, 2];
grade = 4;
L = Table[intrv[5], {t, 1, grade}];
B = Chop[Expand[cliffordListProduct[L]]]
```

Out[∘]=
$$-595 - 533\, e_{\{1,2\}} + 1056\, e_{\{1,3\}} + 268\, e_{\{1,4\}} - 519\, e_{\{1,5\}} + 712\, e_{\{2,3\}} - 304\, e_{\{2,4\}} + 4\, e_{\{2,5\}} + 1015\, e_{\{3,4\}} - 939\, e_{\{3,5\}} + 198\, e_{\{4,5\}} + 49\, e_{\{1,2,3,4\}} - 213\, e_{\{1,2,3,5\}} - 86\, e_{\{1,2,4,5\}} + 111\, e_{\{1,3,4,5\}} + 236\, e_{\{2,3,4,5\}}$$

In[∘]:=
```
Timing[cliffordDecomposableQ[B]]
```

Out[∘]=
```
{0.048301, True}
```

In[•]:= `f = cliffordDecomp[B]`

Out[•]=
$$\left\{-\frac{6966773806592004837885806048513633151\, e_{\{1\}}}{90159989516032615615181865240211463360}-\frac{4417671612675516934763867521136434111737\, e_{\{2\}}}{66177432304767939861543489086315214106240}+\frac{2377372090090846919812029057948416579943\, e_{\{3\}}}{33088716152383969930771744543157607053120}+\frac{245806610364\, e_{\{4\}}}{2515485575635}-\frac{1028754347207038270386351592525400216799\, e_{\{5\}}}{13235486460953587972308697817263042821248},\right.$$
$$-\frac{1159886909462955271036281172\, e_{\{1\}}}{300397237047134841775455}-\frac{8789122432484639369560803274\, e_{\{2\}}}{2703575133424213575979095}+\frac{9486799603690826214731990572\, e_{\{3\}}}{2703575133424213575979095}+\frac{13154007509675810866904933312\, e_{\{4\}}}{2703575133424213575979095}-\frac{2032373991632712793975459430\, e_{\{5\}}}{540715026684842715195819},$$
$$\frac{63683179130\, e_{\{1\}}}{9299073}+\frac{515207186906\, e_{\{2\}}}{83691657}-\frac{556250464328\, e_{\{3\}}}{83691657}-\frac{730891999582\, e_{\{4\}}}{83691657}+\frac{607497576596\, e_{\{5\}}}{83691657},$$
$$\left.-\frac{12877\, e_{\{1\}}}{147}-\frac{101989\, e_{\{2\}}}{1323}+\frac{108667\, e_{\{3\}}}{1323}+\frac{146723\, e_{\{4\}}}{1323}-\frac{119932\, e_{\{5\}}}{1323}\right\}$$

In[•]:= `B2 = cliffordListProduct[f]`

Out[•]=
$$-595-533\, e_{\{1,2\}}+1056\, e_{\{1,3\}}+268\, e_{\{1,4\}}-519\, e_{\{1,5\}}+712\, e_{\{2,3\}}-304\, e_{\{2,4\}}+4\, e_{\{2,5\}}+1015\, e_{\{3,4\}}-939\, e_{\{3,5\}}+198\, e_{\{4,5\}}+49\, e_{\{1,2,3,4\}}-213\, e_{\{1,2,3,5\}}-86\, e_{\{1,2,4,5\}}+111\, e_{\{1,3,4,5\}}+236\, e_{\{2,3,4,5\}}$$

In[•]:= `B2 - B // clExpand`

Out[•]= 0

In[•]:= `Timing[cliffordInvertibleQ[α]]`

Out[•]= {0.182032, False}

In[•]:= `getSignature`

Out[•]= $Cl_{3,2}$

In[•]:= `signedSeminormSquare[B]`

Out[•]= −18522

In[•]:= `Timing[clExpand[B ⊙ cliffordReversion[B]]]`

Out[•]= {0.033248, −18522}

In[•]:= `Timing[bi = cliffordInverse[B]]`

Out[•]=
$$\left\{0.058384,\ \frac{85}{2646}-\frac{533\, e_{\{1,2\}}}{18522}+\frac{176\, e_{\{1,3\}}}{3087}+\frac{134\, e_{\{1,4\}}}{9261}-\frac{173\, e_{\{1,5\}}}{6174}+\frac{356\, e_{\{2,3\}}}{9261}-\frac{152\, e_{\{2,4\}}}{9261}+\frac{2\, e_{\{2,5\}}}{9261}+\frac{145\, e_{\{3,4\}}}{2646}-\frac{313\, e_{\{3,5\}}}{6174}+\frac{11\, e_{\{4,5\}}}{1029}-\frac{1}{378}\, e_{\{1,2,3,4\}}+\frac{71\, e_{\{1,2,3,5\}}}{6174}+\frac{43\, e_{\{1,2,4,5\}}}{9261}-\frac{37\, e_{\{1,3,4,5\}}}{6174}-\frac{118\, e_{\{2,3,4,5\}}}{9261}\right\}$$

In[•]:= `bi ⊙ B // clExpand`

Out[•]= 1

In[∘]:= `Timing[binv = cliffordInverse[B]]`

Out[∘]= $\left\{0.056979, \frac{85}{2646} - \frac{533\, e_{\{1,2\}}}{18522} + \frac{176\, e_{\{1,3\}}}{3087} + \frac{134\, e_{\{1,4\}}}{9261} - \frac{173\, e_{\{1,5\}}}{6174} + \frac{356\, e_{\{2,3\}}}{9261} - \frac{152\, e_{\{2,4\}}}{9261} + \frac{2\, e_{\{2,5\}}}{9261} + \frac{145\, e_{\{3,4\}}}{2646} - \frac{313\, e_{\{3,5\}}}{6174} + \frac{11\, e_{\{4,5\}}}{1029} - \frac{1}{378} e_{\{1,2,3,4\}} + \frac{71\, e_{\{1,2,3,5\}}}{6174} + \frac{43\, e_{\{1,2,4,5\}}}{9261} - \frac{37\, e_{\{1,3,4,5\}}}{6174} - \frac{118\, e_{\{2,3,4,5\}}}{9261}\right\}$

In[∘]:= `binv ⊙ B // clExpand`

Out[∘]= 1

## Blades and decomposable elements

### Fontijne's blade factorization algorithm

`? fastBladeFactor`

fastBladeFactor[B] – Returns scalar $\alpha$ and vectors {b_i: 1≤i≤clGrade(B)} such that $B=\alpha \prod_{i=1}^{\text{clGrade}(B)} b_i$, provided $B \in Cl_{p,q,r}$ is a blade.

In[∘]:= `setSignature[6]`

In[∘]:= `BList = Table[Table[Random[Integer, {-3, 3}], {maxIndex}].gradeBasis[1, e], {j, 1, 3}]`

Out[∘]= $\{e_{\{1\}} - e_{\{2\}} - e_{\{3\}} - 2\, e_{\{4\}} + 2\, e_{\{5\}} + 3\, e_{\{6\}},\ -2\, e_{\{1\}} + 3\, e_{\{3\}} - 3\, e_{\{4\}} + e_{\{5\}},\ -3\, e_{\{1\}} - 2\, e_{\{2\}} + 2\, e_{\{3\}} + e_{\{4\}} + e_{\{6\}}\}$

In[∘]:= `? wedgeListProduct`

wedgeListProduct[{$u_1$, $u_2$, ..., $u_k$}] computes wedge product $u_1 \wedge u_2 \wedge \ldots \wedge u_k$

In[∘]:= `blade = wedgeListProduct[BList]`

Out[∘]= $7\, e_{\{1,2,3\}} - 25\, e_{\{1,2,4\}} + 13\, e_{\{1,2,5\}} + 10\, e_{\{1,2,6\}} - 12\, e_{\{1,3,4\}} + 11\, e_{\{1,3,5\}} + 16\, e_{\{1,3,6\}} - 17\, e_{\{1,4,5\}} - 40\, e_{\{1,4,6\}} + 14\, e_{\{1,5,6\}} - 27\, e_{\{2,3,4\}} + 16\, e_{\{2,3,5\}} + 15\, e_{\{2,3,6\}} - 7\, e_{\{2,4,5\}} - 15\, e_{\{2,4,6\}} + 5\, e_{\{2,5,6\}} + 15\, e_{\{3,4,5\}} + 36\, e_{\{3,4,6\}} - 13\, e_{\{3,5,6\}} + e_{\{4,5,6\}}$

In[∘]:= `factors = fastBladeFactor[blade]`

Out[∘]= $\left\{e_{\{3\}} - \frac{25\, e_{\{4\}}}{7} + \frac{13\, e_{\{5\}}}{7} + \frac{10\, e_{\{6\}}}{7},\ -e_{\{2\}} - \frac{12\, e_{\{4\}}}{7} + \frac{11\, e_{\{5\}}}{7} + \frac{16\, e_{\{6\}}}{7},\ e_{\{1\}} - \frac{27\, e_{\{4\}}}{7} + \frac{16\, e_{\{5\}}}{7} + \frac{15\, e_{\{6\}}}{7},\ 7\right\}$

In[∘]:= `wedgeListProduct[factors]`

Out[∘]= $7\, e_{\{1,2,3\}} - 25\, e_{\{1,2,4\}} + 13\, e_{\{1,2,5\}} + 10\, e_{\{1,2,6\}} - 12\, e_{\{1,3,4\}} + 11\, e_{\{1,3,5\}} + 16\, e_{\{1,3,6\}} - 17\, e_{\{1,4,5\}} - 40\, e_{\{1,4,6\}} + 14\, e_{\{1,5,6\}} - 27\, e_{\{2,3,4\}} + 16\, e_{\{2,3,5\}} + 15\, e_{\{2,3,6\}} - 7\, e_{\{2,4,5\}} - 15\, e_{\{2,4,6\}} + 5\, e_{\{2,5,6\}} + 15\, e_{\{3,4,5\}} + 36\, e_{\{3,4,6\}} - 13\, e_{\{3,5,6\}} + e_{\{4,5,6\}}$

### Decompositions of conformal orthogonal group elements

In[•]:=
```
? cliffordDecomposableQ
```

cliffordDecomposableQ[u] – Decomposability
test for Clifford algebra element. Necessary in indefinite signatures

In[•]:=
```
(* Generate a random vector in current Clifford
  algebra with random integer coefficients between-ell and ell.*)
Intrv[ell_] := Total[Table[Random[Integer, {-ell, ell}] e{j}, {j, 1, maxIndex}]];
```

In[•]:=
```
setSignature[6, 0];
grade = 4;
MaxTrials = 100;
coeffrange = 5;

correct = 0;
trials = 0;
Timelist46 = {};
For[m = 1, m ≤ MaxTrials, m++,
  bfacts = {};
  buildtable = Table[Intrv[coeffrange], {i, 1, grade}];
  u = Chop[Expand[N[cliffordListProduct[buildtable]]]];
  If[clGrade[u] == grade,
   trials = trials + 1;
   facts = Timing[bfacts = cliffordDecomp[u]];
   prodcheck = Chop[N[cliffordListProduct[bfacts]]];
   If[Chop[signedSeminormSquare[clExpand[prodcheck - u]]] == 0,
    PrintTemporary["Trial ", m, " verified!"];
    correct = correct + 1;,
    Print["Trial ", m, "Not verified!  u=", u ];
   ];
   Timelist46 = Append[Timelist46, facts[[1]]];
  ];
 ];
Print[correct, " correct = ", N[correct / (trials) * 100], "%"];
setSignature[6, 0];

correct = 0;
trials = 0;
Timelist46B = {};
For[m = 1, m ≤ MaxTrials, m++,
  ClearSystemCache[];
  bfacts = {};
  buildtable = Table[Intrv[coeffrange], {i, 1, grade}];
  u = Expand[N[wedgeListProduct[buildtable]]];
  If[clGrade[u] == grade,
```

```
    trials = trials + 1;
    facts = Timing[bfacts = cliffordDecomp[u]];
    prodcheck = Chop[N[cliffordListProduct[bfacts]]];
    If[Chop[signedSeminormSquare[clExpand[prodcheck - u]]] == 0,
     PrintTemporary["Trial ", m, " verified!"];
     correct = correct + 1;,
     Print["Trial ", m, "Not verified!  u=", u ];
     Print["Product check: ", prodcheck]];
    Timelist46B = Append[Timelist46B, facts[[1]]];
   ];
  ];
Print[correct, " correct = ", N[correct / (trials) * 100], "%"];
setSignature[6, 0];

correct = 0;
trials = 0;
Timelist46FBF = {};
For[m = 1, m ≤ MaxTrials, m++,
  ClearSystemCache[];
  bfacts = {};
  buildtable = Table[Intrv[coeffrange], {i, 1, grade}];
  u = Expand[N[wedgeListProduct[buildtable]]];
  If[clGrade[u] == grade,
   trials = trials + 1;
   facts = Timing[bfacts = fastBladeFactor[u]];
   prodcheck = Chop[N[wedgeListProduct[bfacts]]];
   If[Chop[clExpand[prodcheck - u]] == 0,
    {PrintTemporary["Trial ", m, " verified!"];
     correct = correct + 1;},
    {Print["Trial ", m, "Not verified!  u=", u ];
     Print["Product check: ", prodcheck]}];
   Timelist46FBF = Append[Timelist46FBF, facts[[1]]];
  ];
 ];
Print[correct, " correct = ", N[correct / (trials) * 100], "%"];

ListPlot[{Timelist46, Timelist46B, Timelist46FBF}, AxesLabel -> {"Trial #", "Time(s)"}]

Print["General element min time: ", Min[Timelist46], " seconds."];

Print["General element max time: ", Max[Timelist46], " seconds."];

Print["General element Mean time: ", Mean[Timelist46], " seconds."];

Print["Blade min time: ", Min[Timelist46B], " seconds."];
```

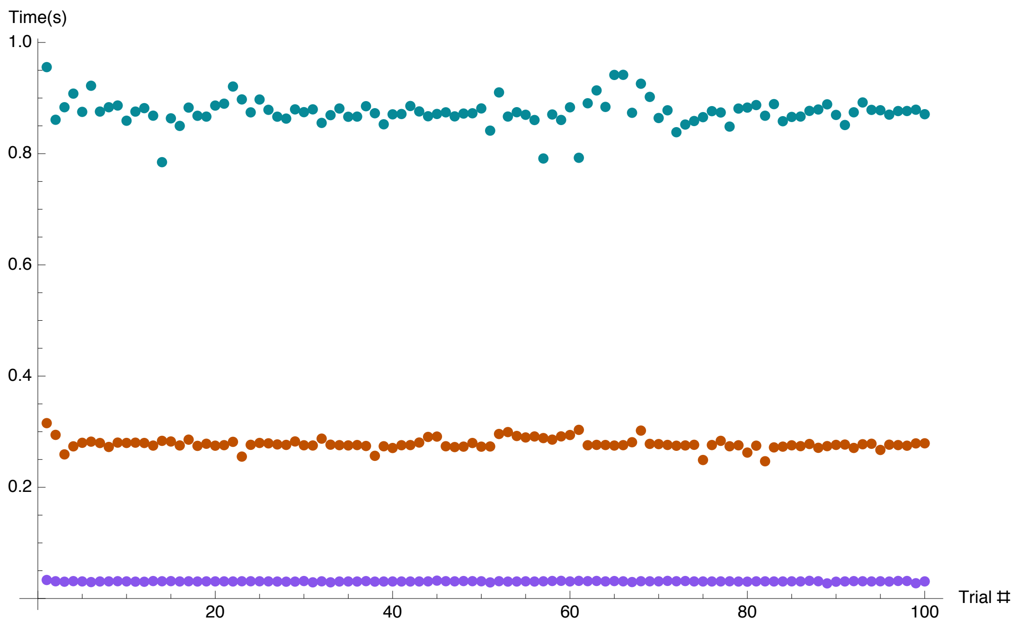
```
Print["Blade max time: ", Max[Timelist46B], " seconds."];

Print["Blade mean time: ", Mean[Timelist46B], " seconds."];

Print["Fast Blade Factor min time: ", Min[Timelist46FBF], " seconds."];

Print["Fast Blade Factor max time: ", Max[Timelist46FBF], " seconds."];

Print["Fast Blade Factor mean time: ", Mean[Timelist46FBF], " seconds."];
```

100 correct = 100.%

100 correct = 100.%

100 correct = 100.%

Out[∘]=




General element min time: 0.78471 seconds.

General element max time: 0.955646 seconds.

General element Mean time: 0.875348 seconds.

Blade min time: 0.246837 seconds.

Blade max time: 0.315404 seconds.

Blade mean time: 0.278117 seconds.

Fast Blade Factor min time: 0.027234 seconds.

Fast Blade Factor max time: 0.03332 seconds.

Fast Blade Factor mean time: 0.0307919 seconds.

In[∘]:=

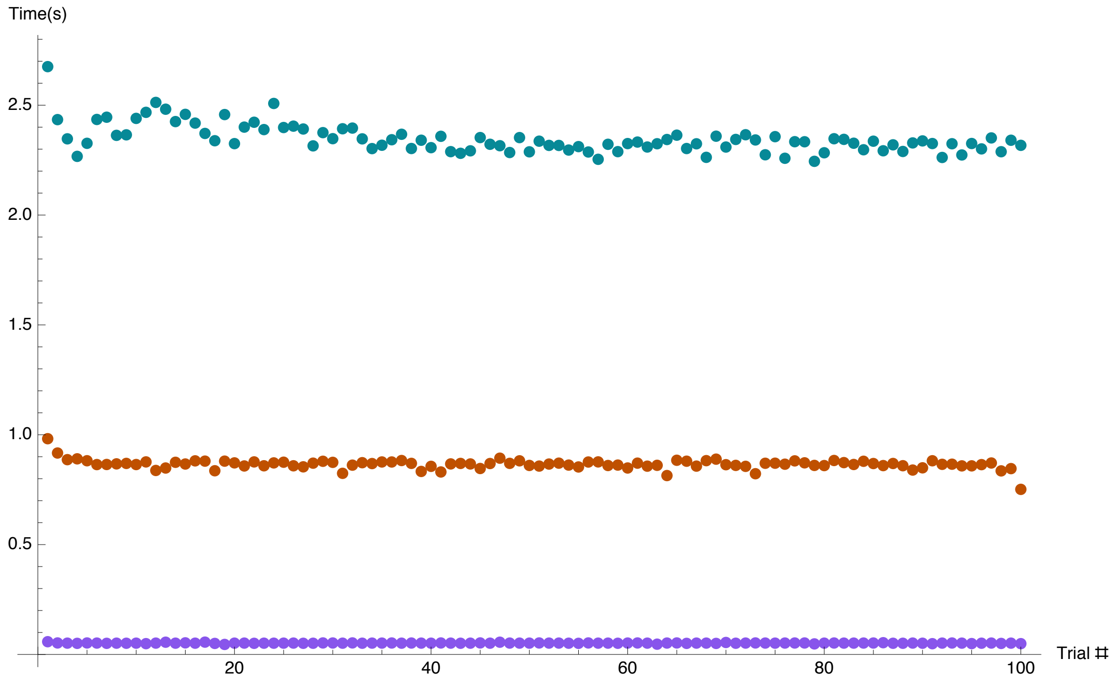
```
(* Apply cliffordDecomp and fastBladeFactor to elements of particular grade
 in Cl7,0.  Compile runtimes over MaxTrials trials for analysis and comparison.*)
setSignature[7, 0];
grade = 4;
```

```
MaxTrials = 100;

correct = 0;
trials = 0;
Timelist47 = {};
For[m = 1, m ≤ MaxTrials, m++,
  ClearSystemCache[];
  bfacts = {};
  buildtable = Table[Intrv[coeffrange], {i, 1, grade}];
  u = Chop[Expand[N[cliffordListProduct[buildtable]]]];
  If[clGrade[u] == grade,
   trials = trials + 1;
   facts = Timing[bfacts = cliffordDecomp[u]];
   prodcheck = Chop[N[cliffordListProduct[bfacts]]];
   If[Chop[clExpand[prodcheck - u]] == 0,
    {PrintTemporary["Trial ", m, " verified!"];
     correct = correct + 1;},
    {Print["Trial ", m, "Not verified!  u=", u ];
     Print["Product check: ", prodcheck]}];
   Timelist47 = Append[Timelist47, facts[[1]]];
  ];
 ];
Print[correct, " correct = ", N[correct / (trials) * 100], "%"];

setSignature[7, 0];

correct = 0;
trials = 0;
Timelist47B = {};
For[m = 1, m ≤ MaxTrials, m++,
  ClearSystemCache[];
  bfacts = {};
  buildtable = Table[Intrv[coeffrange], {i, 1, grade}];
  u = Expand[N[wedgeListProduct[buildtable]]];
  If[clGrade[u] == grade,
   trials = trials + 1;
   facts = Timing[bfacts = cliffordDecomp[u]];
   prodcheck = Chop[N[cliffordListProduct[bfacts]]];
   If[Chop[clExpand[prodcheck - u]] == 0,
    {PrintTemporary["Trial ", m, " verified!"];
     correct = correct + 1;},
    {Print["Trial ", m, "Not verified!  u=", u ];
     Print["Product check: ", prodcheck]}];
   Timelist47B = Append[Timelist47B, facts[[1]]];
  ];];
Print[correct, " correct = ", N[correct / (trials) * 100], "%"];
```

```
setSignature[7, 0];

correct = 0;
trials = 0;
Timelist47FBF = {};
For[m = 1, m ≤ MaxTrials, m++,
  ClearSystemCache[];
  bfacts = {};
  buildtable = Table[Intrv[coeffrange], {i, 1, grade}];
  u = Expand[N[wedgeListProduct[buildtable]]];
  If[clGrade[u] == grade,
   trials = trials + 1;
   facts = Timing[bfacts = fastBladeFactor[u]];
   prodcheck = Chop[N[wedgeListProduct[bfacts]]];
   If[Chop[clExpand[prodcheck - u]] == 0,
    {PrintTemporary["Trial ", m, " verified!"];
     correct = correct + 1;},
    {Print["Trial ", m, "Not verified!  u=", u ];
     Print["Product check: ", prodcheck]}];
   Timelist47FBF = Append[Timelist47FBF, facts[[1]]];
  ];];
Print[correct, " correct = ", N[correct / (trials) * 100], "%"];

ListPlot[{Timelist47, Timelist47B, Timelist47FBF}, AxesLabel -> {"Trial #", "Time(s)"}]

Print["General element min time: ", Min[Timelist47], " seconds."];
Print["General element Mean time: ", Mean[Timelist47], " seconds."];
Print["General element max time: ", Max[Timelist47], " seconds."];

Print["Blade min time: ", Min[Timelist47B], " seconds."];
Print["Blade mean time: ", Mean[Timelist47B], " seconds."];
Print["Blade max time: ", Max[Timelist47B], " seconds."];

Print["Fast Blade Factor min time: ", Min[Timelist47FBF], " seconds."];
Print["Fast Blade Factor mean time: ", Mean[Timelist47FBF], " seconds."];
Print["Fast Blade Factor max time: ", Max[Timelist47FBF], " seconds."];


ListLinePlot[
 {Timelist46, Timelist46B, Timelist46FBF, Timelist47, Timelist47B, Timelist47FBF},
 PlotStyle → {RGBColor[0, 0, 1], RGBColor[1, 0, 0], RGBColor[0, 1, 0], RGBColor[.7, .7, 0],
   RGBColor[0, .7, .7], RGBColor[.7, 0, .7]}, AxesLabel -> {"Trial #", "Time(s)"},
 PlotLegends → {"Cl6", "Cl6Blade", "Cl6FBF", "Cl7", "Cl7Blade", "Cl7FBF"}]
```

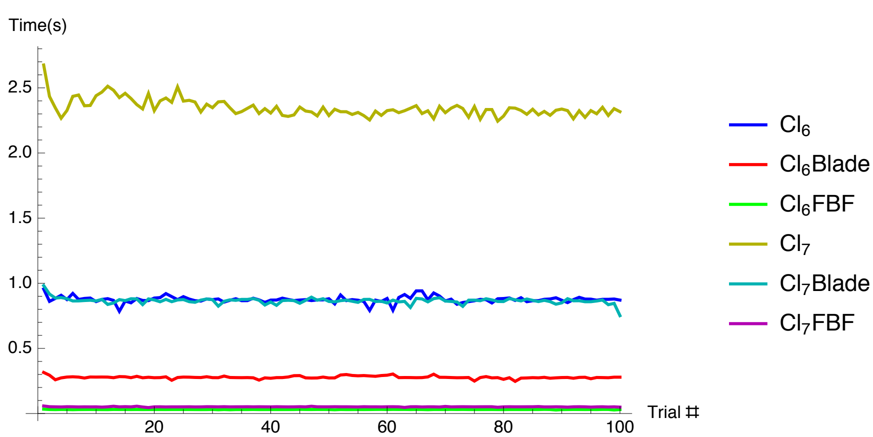
```
100 correct = 100.%
100 correct = 100.%
100 correct = 100.%
```

*Out[∘]=*




```
General element min time: 2.24445 seconds.
General element Mean time: 2.34606 seconds.
General element max time: 2.6755 seconds.
Blade min time: 0.751426 seconds.
Blade mean time: 0.865457 seconds.
Blade max time: 0.981953 seconds.
Fast Blade Factor min time: 0.045574 seconds.
Fast Blade Factor mean time: 0.0515244 seconds.
Fast Blade Factor max time: 0.057993 seconds.
```

*Out[∘]=*



## Matrices with Clifford elements

In[•]:=
```
? cliffordMatrixProduct
```

cliffordMatrixProduct[A,B] Compute matrix product AB
using multiplication in appropriate algebra $Cl_{p,q,r}$, $Cl_n^{nil}$, $Cl_n^{idem}$, or $Cl_{p,q,r}^{sym}$.

In[•]:=
```
? cliffordMatrixPower
```

cliffordMatrixPower[A,m] – Computes $A^{\odot m}$ for square matrix $A$ having entries from $Cl_{p,q,r}$ or $Cl_*^*$.

In[•]:=
```
(* Some Clifford matrices *)
setSignature[1, 1];
For[lp = 1, lp ≤ 5, lp++,
  rows = Random[Integer, {2, 4}];
  cols = Random[Integer, {2, 4}];
  Print[Subscript["A", lp], " = ",
   MatrixForm[A_lp = Table[Table[Random[Integer] Random[Integer, {-1, 2}], {j, 2^maxIndex}].
      clBasis, {i, rows}, {j, cols}]]]];
```

$$A_1 = \begin{pmatrix} -1 & 1 & -e_{\{1\}} + e_{\{1,2\}} & 0 \\ -1 + e_{\{1,2\}} & -e_{\{1\}} - e_{\{1,2\}} & 1 + e_{\{1,2\}} & 1 + 2\, e_{\{1\}} \\ -1 & -e_{\{1\}} - e_{\{2\}} & 2\, e_{\{2\}} & -1 + 2\, e_{\{1,2\}} \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & -e_{\{1,2\}} & 2\, e_{\{1\}} + e_{\{2\}} \\ -e_{\{1,2\}} & -e_{\{1\}} + e_{\{2\}} & e_{\{1\}} \\ e_{\{1\}} - e_{\{2\}} & 1 + 2\, e_{\{2\}} & e_{\{1\}} + 2\, e_{\{1,2\}} \\ 2\, e_{\{1\}} - e_{\{2\}} & 2\, e_{\{1\}} & e_{\{2\}} \end{pmatrix}$$

$$A_3 = \begin{pmatrix} -1 - e_{\{1\}} - e_{\{2\}} & e_{\{1,2\}} & 0 \\ 1 + 2\, e_{\{1\}} & -1 + 2\, e_{\{1\}} + e_{\{2\}} & 2\, e_{\{1\}} \\ 2 & -e_{\{1\}} + e_{\{1,2\}} & 2\, e_{\{1\}} \\ -1 - e_{\{1,2\}} & -1 - e_{\{1\}} + e_{\{1,2\}} & e_{\{1\}} - e_{\{1,2\}} \end{pmatrix}$$

$$A_4 = \begin{pmatrix} e_{\{1\}} + 2\, e_{\{1,2\}} & -e_{\{2\}} & -1 + e_{\{1\}} & e_{\{1\}} - e_{\{2\}} \\ -1 & 2 - e_{\{1,2\}} & e_{\{1\}} - e_{\{2\}} & 2\, e_{\{1\}} \\ 1 & 2 + 2\, e_{\{2\}} - e_{\{1,2\}} & -1 - e_{\{2\}} & e_{\{1,2\}} \\ -e_{\{1\}} & 2 & 2\, e_{\{1\}} - e_{\{1,2\}} & e_{\{1\}} \end{pmatrix}$$

$$A_5 = \begin{pmatrix} -e_{\{1\}} - e_{\{1,2\}} & 0 \\ 1 + 2\, e_{\{1\}} + 2\, e_{\{1,2\}} & 0 \end{pmatrix}$$

In[•]:=
```
(* Product *)
clExpand[cliffordMatrixProduct[A₄, A₂]] // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 2 + e_{\{2\}} & e_{\{1\}} - 3\, e_{\{2\}} + 3\, e_{\{1,2\}} & 4 - 3\, e_{\{1\}} - 2\, e_{\{2\}} + e_{\{1,2\}} \\ 5 - 4\, e_{\{1,2\}} & 6 + 3\, e_{\{1,2\}} & 1 - 2\, e_{\{1\}} + 2\, e_{\{2\}} + 3\, e_{\{1,2\}} \\ -2\, e_{\{1\}} - e_{\{2\}} - e_{\{1,2\}} & -1 - e_{\{1\}} - 4\, e_{\{2\}} + e_{\{1,2\}} & 2\, e_{\{2\}} - 3\, e_{\{1,2\}} \\ 4 - e_{\{1\}} + e_{\{2\}} - 5\, e_{\{1,2\}} & 2 + 2\, e_{\{1\}} + 3\, e_{\{2\}} + 3\, e_{\{1,2\}} & -2 + 2\, e_{\{1\}} + 5\, e_{\{2\}} \end{pmatrix}$$

In[•]:=
```
(* Matrix power *)
clExpand[cliffordMatrixPower[A₄, 5]] // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} -166 - 52\, e_{\{1\}} + 103\, e_{\{2\}} - 139\, e_{\{1,2\}} & 249 + 545\, e_{\{1\}} - 531\, e_{\{2\}} + 243\, e_{\{1,2\}} & -20 + 158\, e_{\{1\}} - 181\, e_{\{2\}} - 33\, e_{\{1} \\ -233 - 208\, e_{\{1\}} - 121\, e_{\{2\}} - 2\, e_{\{1,2\}} & 527 + 461\, e_{\{1\}} + 25\, e_{\{2\}} - 299\, e_{\{1,2\}} & 185 - 4\, e_{\{1\}} - 95\, e_{\{2\}} - 154\, e_{\{1,2} \\ -37 - 102\, e_{\{1\}} - 131\, e_{\{2\}} + 27\, e_{\{1,2\}} & 343 + 156\, e_{\{1\}} + 134\, e_{\{2\}} - 179\, e_{\{1,2\}} & 48 + 43\, e_{\{1\}} - 95\, e_{\{2\}} - 189\, e_{\{1,2} \\ -248 - 133\, e_{\{1\}} - 56\, e_{\{2\}} - 10\, e_{\{1,2\}} & 426 + 381\, e_{\{1\}} - 145\, e_{\{2\}} - 34\, e_{\{1,2\}} & 32 + 100\, e_{\{1\}} - 122\, e_{\{2\}} - 119\, e_{\{1} \end{pmatrix}$$

In[•]:=
```
(* Some zeon matrices *)
For[lp = 1, lp ≤ 5, lp++,
  rows = Random[Integer, {2, 4}];
  cols = Random[Integer, {2, 4}];
  Print[Subscript["B", lp], " = ",
   MatrixForm[B_lp = Table[Table[Random[Integer] Random[Integer, {-1, 2}], {j, 2^maxIndex}].
      zeonBasis, {i, rows}, {j, cols}]]]];
```

$$B_1 = \begin{pmatrix} 2\zeta_{\{1\}} + \zeta_{\{2\}} & 2\zeta_{\{1\}} - \zeta_{\{2\}} + 2\zeta_{\{1,2\}} \\ \zeta_{\{1\}} & -\zeta_{\{2\}} \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 2 + \zeta_{\{2\}} & -1 & \zeta_{\{1\}} + 2\zeta_{\{2\}} + 2\zeta_{\{1,2\}} & -\zeta_{\{1\}} + \zeta_{\{2\}} \\ 0 & 2 - \zeta_{\{1,2\}} & -\zeta_{\{1\}} + 2\zeta_{\{1,2\}} & 2\zeta_{\{2\}} \\ 0 & 0 & 1 + 2\zeta_{\{2\}} + \zeta_{\{1,2\}} & -1 + 2\zeta_{\{1\}} \\ \zeta_{\{1\}} + \zeta_{\{2\}} & 0 & -\zeta_{\{2\}} & 0 \end{pmatrix}$$

$$B_3 = \begin{pmatrix} -\zeta_{\{2\}} & 2\zeta_{\{1\}} \\ \zeta_{\{1\}} & \zeta_{\{2\}} - \zeta_{\{1,2\}} \end{pmatrix}$$

$$B_4 = \begin{pmatrix} 2\zeta_{\{1\}} - \zeta_{\{2\}} - \zeta_{\{1,2\}} & -1 + 2\zeta_{\{1\}} + 2\zeta_{\{1,2\}} \\ 1 - \zeta_{\{1,2\}} & 2\zeta_{\{1\}} \\ 0 & \zeta_{\{2\}} \\ 2 & -\zeta_{\{2\}} \end{pmatrix}$$

$$B_5 = \begin{pmatrix} 2\zeta_{\{1,2\}} & 1 + \zeta_{\{1\}} - \zeta_{\{2\}} & 0 & 0 \\ 2 + \zeta_{\{1\}} + \zeta_{\{1,2\}} & 1 - \zeta_{\{2\}} & 2\zeta_{\{1\}} & 0 \\ 2 + 2\zeta_{\{1\}} + 2\zeta_{\{2\}} & -1 - \zeta_{\{1,2\}} & 0 & 0 \end{pmatrix}$$

In[•]:=
```
clExpand[cliffordMatrixProduct[B2, B4]] // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} -1 + 2\zeta_{\{1\}} + \zeta_{\{1,2\}} & -2 + 2\zeta_{\{1\}} - \zeta_{\{2\}} + 8\zeta_{\{1,2\}} \\ 2 + 4\zeta_{\{2\}} - 3\zeta_{\{1,2\}} & 4\zeta_{\{1\}} - \zeta_{\{1,2\}} \\ -2 + 4\zeta_{\{1\}} & 2\zeta_{\{2\}} - 2\zeta_{\{1,2\}} \\ \zeta_{\{1,2\}} & -\zeta_{\{1\}} - \zeta_{\{2\}} + 2\zeta_{\{1,2\}} \end{pmatrix}$$

### Matrices with Grassmann elements - applying exterior products within matrices

In[•]:=
```
? wedgeMatrixProduct
```

wedgeMatrixProduct[A,B] Compute wedge(exterior) matrix product AB.

In[•]:=
```
? wedgeMatrixPower
```

wedgeMatrixPower[A,m] – Computes $A\wedge^{m}$ for square matrix *A* having entries from $Cl_{p,q,r}$.

Recall Clifford matrices from above.

In[•]:=
```
MatrixForm[A1]
```

Out[•]//MatrixForm=

$$\begin{pmatrix} -1 & 1 & -e_{\{1\}} + e_{\{1,2\}} & 0 \\ -1 + e_{\{1,2\}} & -e_{\{1\}} - e_{\{1,2\}} & 1 + e_{\{1,2\}} & 1 + 2e_{\{1\}} \\ -1 & -e_{\{1\}} - e_{\{2\}} & 2e_{\{2\}} & -1 + 2e_{\{1,2\}} \end{pmatrix}$$

In[∘]:=
```
MatrixForm[A4]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} e_{\{1\}}+2e_{\{1,2\}} & -e_{\{2\}} & -1+e_{\{1\}} & e_{\{1\}}-e_{\{2\}} \\ -1 & 2-e_{\{1,2\}} & e_{\{1\}}-e_{\{2\}} & 2e_{\{1\}} \\ 1 & 2+2e_{\{2\}}-e_{\{1,2\}} & -1-e_{\{2\}} & e_{\{1,2\}} \\ -e_{\{1\}} & 2 & 2e_{\{1\}}-e_{\{1,2\}} & e_{\{1\}} \end{pmatrix}$$

In[∘]:=
```
wedgeMatrixProduct[A1, A4] // clExpand // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} -1-2e_{\{1\}}-e_{\{1,2\}} & 2-2e_{\{1\}}+e_{\{2\}}-e_{\{1,2\}} & 1+e_{\{1\}}-e_{\{2\}} & e_{\{1\}}+e_{\{2\}} \\ 1-e_{\{1\}} & 4+2e_{\{1\}}+3e_{\{2\}}-e_{\{1,2\}} & e_{\{1\}}-e_{\{2\}}-2e_{\{1,2\}} & e_{\{2\}}+e_{\{1,2\}} \\ e_{\{1\}}+3e_{\{2\}}-2e_{\{1,2\}} & -2-2e_{\{1\}}+3e_{\{2\}}+4e_{\{1,2\}} & 1-3e_{\{1\}}-2e_{\{2\}}+3e_{\{1,2\}} & -2e_{\{1\}}+e_{\{2\}}+2e_{\{1,2\}} \end{pmatrix}$$

Compare with Clifford matrix product.

In[∘]:=
```
cliffordMatrixProduct[A1, A4] // clExpand // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} -1-2e_{\{1\}}-e_{\{1,2\}} & 1-4e_{\{1\}}+2e_{\{2\}}-e_{\{1,2\}} & 1+2e_{\{1\}}-e_{\{2\}} & 1+e_{\{1\}} \\ 1-e_{\{1\}}-e_{\{2\}} & 4+e_{\{1\}}+4e_{\{2\}}-e_{\{1,2\}} & 3+e_{\{1\}}-3e_{\{2\}}-2e_{\{1,2\}} & 1+e_{\{1\}}+2e_{\{2\}}+e_{\{1,2\}} \\ e_{\{1\}}+5e_{\{2\}}-2e_{\{1,2\}} & -6-3e_{\{1\}}+4e_{\{2\}}+4e_{\{1,2\}} & -1-3e_{\{1\}}-6e_{\{2\}}+3e_{\{1,2\}} & -2-e_{\{2\}}+2e_{\{1,2\}} \end{pmatrix}$$

Consider wedge power of $A_4$.

In[∘]:=
```
clExpand[wedgeMatrixPower[A4, 3]] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 3-7e_{\{1\}}+7e_{\{2\}}-5e_{\{1,2\}} & -2+4e_{\{1\}}-7e_{\{2\}}+2e_{\{1,2\}} & -2e_{\{1\}}-e_{\{2\}}+4e_{\{1,2\}} & -5e_{\{1\}}+e_{\{2\}}+11\,e \\ -3-8e_{\{1\}}-e_{\{1,2\}} & 10+18e_{\{1\}}+2e_{\{2\}}-11e_{\{1,2\}} & 1+2e_{\{1\}}-3e_{\{2\}}-4e_{\{1,2\}} & 6e_{\{1\}}+2e_{\{2\}}+3\,e \\ -2+e_{\{2\}}-5e_{\{1,2\}} & 4+16e_{\{1\}}-e_{\{2\}}-13e_{\{1,2\}} & 3-5e_{\{1\}}+e_{\{2\}}+2e_{\{1,2\}} & e_{\{1\}}+3e_{\{2\}}+5\,e_{\{ \\ -4-7e_{\{1\}}-2e_{\{2\}}-6e_{\{1,2\}} & 8+22e_{\{1\}}-2e_{\{2\}}-2e_{\{1,2\}} & 2-e_{\{1\}}-2e_{\{2\}}-5e_{\{1,2\}} & 6e_{\{1\}}+2e_{\{2\}}-2\,e \end{pmatrix}$$

We can make $A_4$nilpotent by removing the scalar terms from the entries.

In[∘]:=
```
(scalars = gradeKPart[A4, 0]) // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & 2 & -1 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$

In[∘]:=
```
(nilA = A4 - scalars) // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} e_{\{1\}}+2e_{\{1,2\}} & -e_{\{2\}} & e_{\{1\}} & e_{\{1\}}-e_{\{2\}} \\ 0 & -e_{\{1,2\}} & e_{\{1\}}-e_{\{2\}} & 2e_{\{1\}} \\ 0 & 2e_{\{2\}}-e_{\{1,2\}} & -e_{\{2\}} & e_{\{1,2\}} \\ -e_{\{1\}} & 0 & 2e_{\{1\}}-e_{\{1,2\}} & e_{\{1\}} \end{pmatrix}$$

In[∘]:=
```
clExpand[wedgeMatrixPower[nilA, 3]] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[∘]:=
```
clExpand[wedgeMatrixPower[nilA, 2]] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} -e_{\{1,2\}} & e_{\{1,2\}} & 2e_{\{1,2\}} & 2e_{\{1,2\}} \\ 0 & 2e_{\{1,2\}} & -e_{\{1,2\}} & 0 \\ 0 & 0 & -2e_{\{1,2\}} & -4e_{\{1,2\}} \\ 0 & 5e_{\{1,2\}} & -2e_{\{1,2\}} & e_{\{1,2\}} \end{pmatrix}$$

## Matrix representations of Clifford algebras

In[•]:=
```
? cliffordRep
```

cliffordRep[u] – Returns $2^n \times 2^n$ (real) matrix representation of $u \in Cl_{p,q,r}$.

In[•]:=
```
setSignature[1, 1, 1];
Print["e1 → ρ(e1) = ", MatrixForm[a1 = cliffordRep[e{1}]]];
Print["e2 → ρ(e2) = ", MatrixForm[a2 = cliffordRep[e{2}]]];
Print["e3 → ρ(e3) = ", MatrixForm[a3 = cliffordRep[e{3}]]];
Print["Squares: "];
Print["ρ(e1)^2 = ", MatrixForm[a1.a1]];
Print["ρ(e2)^2 = ", MatrixForm[a2.a2]];
Print["ρ(e3)^2 = ", MatrixForm[a3.a3]];
Print["CAR: "];
Print["ρ(e1)ρ(e2) + ρ(e2)ρ(e1) = ", MatrixForm[a1.a2], " + ", MatrixForm[a2.a1]];
Print["                         = ", MatrixForm[a1.a2 + a2.a1]];
Print["ρ(e1)ρ(e3) + ρ(e3)ρ(e1) = ", MatrixForm[a1.a3], " + ", MatrixForm[a3.a1]];
Print["                         = ", MatrixForm[a1.a3 + a3.a1]];
Print["ρ(e2)ρ(e3) + ρ(e3)ρ(e2) = ", MatrixForm[a2.a3], " + ", MatrixForm[a3.a2]];
Print["                         = ", MatrixForm[a2.a3 + a3.a2]];
```

$$e_1 \to \rho(e_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$e_2 \to \rho(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e_3 \to \rho(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Squares:

$$\rho(e_1)^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\rho(e_2)^2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\rho(e_3)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

CAR:

$$\rho(e_1)\rho(e_2) + \rho(e_2)\rho(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho(e_1)\rho(e_3) + \rho(e_3)\rho(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho(e_2)\rho(e_3) + \rho(e_3)\rho(e_2) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} + \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
? generalCliffordElement
```

generalCliffordElement[$\alpha$] – Canonical expansion $\sum_{I \in 2^{[n]}} \alpha_I e_I$, where $\alpha$ is a symbol.

In[∘]:=
```
setSignature[1, 2];
Print["Let u = ", u = generalCliffordElement[a]];
Print["Matrix representation: "];
Print["ρ(u) = ", MatrixForm[cliffordRep[u]]];
```

Let u $= a_0 + a_{\{1\}} e_{\{1\}} + a_{\{2\}} e_{\{2\}} + a_{\{3\}} e_{\{3\}} + a_{\{1,2\}} e_{\{1,2\}} + a_{\{1,3\}} e_{\{1,3\}} + a_{\{2,3\}} e_{\{2,3\}} + a_{\{1,2,3\}} e_{\{1,2,3\}}$

Matrix representation:

$$\rho(u) = \begin{pmatrix} a_0 + a_{\{1\}} & a_{\{2,3\}} + a_{\{1,2,3\}} & 0 & 0 & 0 & 0 \\ -a_{\{2,3\}} - a_{\{1,2,3\}} & a_0 + a_{\{1\}} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_0 + a_{\{1\}} & -a_{\{2,3\}} - a_{\{1,2,3\}} & -a_{\{2\}} - a_{\{1,2\}} & a_{\{3\}} + a_{\{1,} \\ 0 & 0 & a_{\{2,3\}} + a_{\{1,2,3\}} & a_0 + a_{\{1\}} & -a_{\{3\}} - a_{\{1,3\}} & -a_{\{2\}} - a_{\{1} \\ 0 & 0 & a_{\{2\}} - a_{\{1,2\}} & a_{\{3\}} - a_{\{1,3\}} & a_0 - a_{\{1\}} & a_{\{2,3\}} - a_{\{1,} \\ 0 & 0 & -a_{\{3\}} + a_{\{1,3\}} & a_{\{2\}} - a_{\{1,2\}} & -a_{\{2,3\}} + a_{\{1,2,3\}} & a_0 - a_{\{1\}} \\ -a_{\{2\}} + a_{\{1,2\}} & a_{\{3\}} - a_{\{1,3\}} & 0 & 0 & 0 & 0 \\ -a_{\{3\}} + a_{\{1,3\}} & -a_{\{2\}} + a_{\{1,2\}} & 0 & 0 & 0 & 0 \end{pmatrix}$$

# 3

# Zeon algebras

Canonical generators for $Cl_n^{nil}$ are denoted $\zeta_{\{i\}}$ for $i$=1,..., $n$=$p$+$q$+$r$. The generators commute and square to zero:

$$\zeta_{\{i\}}^2 = 0 \ (1 \leq i \leq p+q+r)$$

In[∘]:=
```
setSignature[4]
```

In[∘]:=
```
zeonBasis
```

Out[∘]= $\{1, \zeta_{\{1\}}, \zeta_{\{2\}}, \zeta_{\{1,2\}}, \zeta_{\{3\}}, \zeta_{\{1,3\}}, \zeta_{\{2,3\}}, \zeta_{\{1,2,3\}}, \zeta_{\{4\}}, \zeta_{\{1,4\}}, \zeta_{\{2,4\}}, \zeta_{\{1,2,4\}}, \zeta_{\{3,4\}}, \zeta_{\{1,3,4\}}, \zeta_{\{2,3,4\}}, \zeta_{\{1,2,3,4\}}\}$

In[∘]:=
```
(* Pseudorandomly generate two elements *)
α = Table[Random[Integer, {-5, 5}], {j, 2^maxIndex}].zeonBasis
```

Out[∘]= $1 - 5\zeta_{\{1\}} + 4\zeta_{\{2\}} - 3\zeta_{\{3\}} - 3\zeta_{\{4\}} - 2\zeta_{\{1,2\}} + \zeta_{\{1,3\}} + 5\zeta_{\{1,4\}} + 3\zeta_{\{2,3\}} - 4\zeta_{\{2,4\}} + 5\zeta_{\{3,4\}} + 2\zeta_{\{1,2,3\}} - 2\zeta_{\{1,2,4\}} - 5\zeta_{\{1,3,4\}} + 3\zeta_{\{2,3,4\}} + 5\zeta_{\{1,2,3,4\}}$

In[∘]:=
```
β = Table[Random[Integer, {-5, 5}], {j, 2^maxIndex}].zeonBasis
```

Out[∘]= $5 + \zeta_{\{1\}} - 3\zeta_{\{2\}} - 5\zeta_{\{3\}} - 4\zeta_{\{4\}} + 5\zeta_{\{1,2\}} - 2\zeta_{\{1,3\}} - 5\zeta_{\{1,4\}} + 4\zeta_{\{2,3\}} - 3\zeta_{\{2,4\}} + \zeta_{\{3,4\}} - \zeta_{\{1,2,3\}} + 4\zeta_{\{1,2,4\}} + \zeta_{\{1,3,4\}} + 4\zeta_{\{2,3,4\}} - 4\zeta_{\{1,2,3,4\}}$

In[∘]:=
```
clExpand[α ⊙ β]
```

Out[∘]= $5 - 24\zeta_{\{1\}} + 17\zeta_{\{2\}} - 20\zeta_{\{3\}} - 19\zeta_{\{4\}} + 14\zeta_{\{1,2\}} + 25\zeta_{\{1,3\}} + 37\zeta_{\{1,4\}} + 8\zeta_{\{2,3\}} - 30\zeta_{\{2,4\}} + 53\zeta_{\{3,4\}} - 24\zeta_{\{1,2,3\}} - 37\zeta_{\{1,2,4\}} - 32\zeta_{\{1,3,4\}} + 13\zeta_{\{2,3,4\}} + 49\zeta_{\{1,2,3,4\}}$

In[∘]:=
```
zeonMinGrade[α]
```

Out[∘]= 1

In[∘]:=
```
clGrade[α]
```

Out[∘]= 4

In[∘]:=
```
zeonMaxGenerator[α]
```

Out[∘]= 4

In[∘]:=
```
? zeonMaxGenDecomp
```

zeonMaxGenDecomp[u] – Given zeon u with maximum generator index m, return $\{\omega,\psi\}$ such that u = $\omega + \psi\zeta_{\{m\}}$.

In[•]:= 
```
{ω, ψ} = zeonMaxGenDecomp[α]
```

Out[•]= $\{1 - 5\zeta_{\{1\}} + 4\zeta_{\{2\}} - 3\zeta_{\{3\}} - 2\zeta_{\{1,2\}} + \zeta_{\{1,3\}} + 3\zeta_{\{2,3\}} + 2\zeta_{\{1,2,3\}},\ -3 + 5\zeta_{\{1\}} - 4\zeta_{\{2\}} + 5\zeta_{\{3\}} - 2\zeta_{\{1,2\}} - 5\zeta_{\{1,3\}} + 3\zeta_{\{2,3\}} + 5\zeta_{\{1,2,3\}}\}$

In[•]:= 
```
ω + ψ ⊙ ζ{4} // clExpand
```

Out[•]= $1 - 5\zeta_{\{1\}} + 4\zeta_{\{2\}} - 3\zeta_{\{3\}} - 3\zeta_{\{4\}} - 2\zeta_{\{1,2\}} + \zeta_{\{1,3\}} + 5\zeta_{\{1,4\}} + 3\zeta_{\{2,3\}} - 4\zeta_{\{2,4\}} + 5\zeta_{\{3,4\}} + 2\zeta_{\{1,2,3\}} - 2\zeta_{\{1,2,4\}} - 5\zeta_{\{1,3,4\}} + 3\zeta_{\{2,3,4\}} + 5\zeta_{\{1,2,3,4\}}$

In[•]:= 
```
α
```

Out[•]= $1 - 5\zeta_{\{1\}} + 4\zeta_{\{2\}} - 3\zeta_{\{3\}} - 3\zeta_{\{4\}} - 2\zeta_{\{1,2\}} + \zeta_{\{1,3\}} + 5\zeta_{\{1,4\}} + 3\zeta_{\{2,3\}} - 4\zeta_{\{2,4\}} + 5\zeta_{\{3,4\}} + 2\zeta_{\{1,2,3\}} - 2\zeta_{\{1,2,4\}} - 5\zeta_{\{1,3,4\}} + 3\zeta_{\{2,3,4\}} + 5\zeta_{\{1,2,3,4\}}$

In[•]:= 
```
? zeonPowerZeon
```

zeonPowerZeon[u,v] returns zeon to zeon power $u^v$, where $u,v \in Cl_n^{nil}$ and Re u > 0.

In[•]:= 
```
zeonPowerZeon[β, α]
```

Out[•]= $5 + \zeta_{\{1\}} - 25\,\text{Log}[5]\,\zeta_{\{1\}} - 3\zeta_{\{2\}} + 20\,\text{Log}[5]\,\zeta_{\{2\}} - 5\zeta_{\{3\}} - 15\,\text{Log}[5]\,\zeta_{\{3\}} - 4\zeta_{\{4\}} - 15\,\text{Log}[5]\,\zeta_{\{4\}} + 24\zeta_{\{1,2\}} + 9\,\text{Log}[5]\,\zeta_{\{1,2\}} - 100\,\text{Log}[5]^2\,\zeta_{\{1,2\}} + 20\zeta_{\{1,3\}} + 27\,\text{Log}[5]\,\zeta_{\{1,3\}} + 75\,\text{Log}[5]^2\,\zeta_{\{1,3\}} + 12\zeta_{\{1,4\}} + 42\,\text{Log}[5]\,\zeta_{\{1,4\}} + 75\,\text{Log}[5]^2\,\zeta_{\{1,4\}} - 7\zeta_{\{2,3\}} + 4\,\text{Log}[5]\,\zeta_{\{2,3\}} - 60\,\text{Log}[5]^2\,\zeta_{\{2,3\}} - 10\zeta_{\{2,4\}} - 27\,\text{Log}[5]\,\zeta_{\{2,4\}} - 60\,\text{Log}[5]^2\,\zeta_{\{2,4\}} + 28\zeta_{\{3,4\}} + 52\,\text{Log}[5]\,\zeta_{\{3,4\}} + 45\,\text{Log}[5]^2\,\zeta_{\{3,4\}} - \frac{256}{5}\zeta_{\{1,2,3\}} + 63\,\text{Log}[5]\,\zeta_{\{1,2,3\}} + 18\,\text{Log}[5]^2\,\zeta_{\{1,2,3\}} + 300\,\text{Log}[5]^3\,\zeta_{\{1,2,3\}} - \frac{202}{5}\zeta_{\{1,2,4\}} + 5\,\text{Log}[5]\,\zeta_{\{1,2,4\}} + 253\,\text{Log}[5]^2\,\zeta_{\{1,2,4\}} + 300\,\text{Log}[5]^3\,\zeta_{\{1,2,4\}} - \frac{108}{5}\zeta_{\{1,3,4\}} - 285\,\text{Log}[5]\,\zeta_{\{1,3,4\}} - 341\,\text{Log}[5]^2\,\zeta_{\{1,3,4\}} - 225\,\text{Log}[5]^3\,\zeta_{\{1,3,4\}} - \frac{11}{5}\zeta_{\{2,3,4\}} + 171\,\text{Log}[5]\,\zeta_{\{2,3,4\}} + 196\,\text{Log}[5]^2\,\zeta_{\{2,3,4\}} + 180\,\text{Log}[5]^3\,\zeta_{\{2,3,4\}} + \frac{2486}{25}\zeta_{\{1,2,3,4\}} + \frac{1097}{5}\text{Log}[5]\,\zeta_{\{1,2,3,4\}} - 1211\,\text{Log}[5]^2\,\zeta_{\{1,2,3,4\}} - 1394\,\text{Log}[5]^3\,\zeta_{\{1,2,3,4\}} - 900\,\text{Log}[5]^4\,\zeta_{\{1,2,3,4\}}$

## Algebraic Properties of Zeons

### Zeon factorizations and decompositions

### Elementary factorizations

As established in [K.], every invertible zeon element u has a unique elementary factorization of the form $u = (\Re u) \prod_{I\in 2^{[n]}} (1 + a_I\,\zeta_I)$. The following example shows how the factorization can be obtained and verified. First, an invertible element is generated.

In[•]:= 
```
setSignature[3];
α = Table[Random[Integer, {-5, 5}], {j, 2^maxIndex}].zeonBasis;
Print["u = ", (u = If[nilpotentZeonQ[α], α + Random[Integer, {1, 4}], zeonAbs[α]])];
```

u = $5 + \zeta_{\{1\}} + \zeta_{\{2\}} - \zeta_{\{3\}} + \zeta_{\{1,2\}} + 5\zeta_{\{1,3\}} - 3\zeta_{\{2,3\}} + 5\zeta_{\{1,2,3\}}$

In[•]:= 
```
? zeonElementaryFactorization
```

zeonElementaryFactorization[u] returns factors a, $\{(1+a_I\zeta_I): I\in 2^{[n]}, I\neq\emptyset\}$ such that $u = a\prod(1+a_I\zeta_I)$

In[•]:=
```
factorlist = zeonElementaryFactorization[u];
Print[MatrixForm[factorlist]];
```

$$\begin{pmatrix} 5 \\ 1+\frac{\zeta_{\{1\}}}{5} \\ 1+\frac{\zeta_{\{2\}}}{5} \\ 1-\frac{\zeta_{\{3\}}}{5} \\ 1+\frac{4}{25}\zeta_{\{1,2\}} \\ 1+\frac{26}{25}\zeta_{\{1,3\}} \\ 1-\frac{14}{25}\zeta_{\{2,3\}} \\ 1+\frac{118}{125}\zeta_{\{1,2,3\}} \end{pmatrix}$$

To verify the factorization, the product is computed using *cliffordListProduct*.

In[•]:=
```
Print["Product of factors in factorlist:"];
Print[cliffordListProduct[factorlist]];
Print["Recall that u = ", u];
```

Product of factors in factorlist:

$5+\zeta_{\{1\}}+\zeta_{\{2\}}-\zeta_{\{3\}}+\zeta_{\{1,2\}}+5\,\zeta_{\{1,3\}}-3\,\zeta_{\{2,3\}}+5\,\zeta_{\{1,2,3\}}$

Recall that u $=5+\zeta_{\{1\}}+\zeta_{\{2\}}-\zeta_{\{3\}}+\zeta_{\{1,2\}}+5\,\zeta_{\{1,3\}}-3\,\zeta_{\{2,3\}}+5\,\zeta_{\{1,2,3\}}$

### The zeon division algorithm

In[•]:= `? zeonDivisionAlgorithm`

zeonDivisionAlgorithm[u,v] returns $\{q,r\}\subset Cl_n^{nil}$ such that u = qv+r, where (i.) qv is homogeneously decomposable and (ii.) r=0 or zeonMinGrade[r] > zeonMinGrade[u].

In[•]:= `{q, r} = zeonDivisionAlgorithm[α, β]`

Out[•]=
$$\left\{\frac{1}{2}+\frac{\zeta_{\{1\}}}{10}+\frac{\zeta_{\{2\}}}{2}+\frac{3\,\zeta_{\{3\}}}{10}+\frac{2\,\zeta_{\{4\}}}{5}-\frac{23}{50}\zeta_{\{1,2\}}+\frac{4}{25}\zeta_{\{1,3\}}+\frac{1}{2}\zeta_{\{1,4\}}+\frac{1}{5}\zeta_{\{2,3\}}+\frac{47}{50}\zeta_{\{2,4\}}+\frac{27}{50}\zeta_{\{3,4\}}-\frac{129}{250}\zeta_{\{1,2,3\}}-\frac{62}{125}\zeta_{\{1,2,4\}}+\frac{43}{50}\zeta_{\{1,3,4\}}+\frac{98}{125}\zeta_{\{2,3,4\}}-\frac{416}{625}\zeta_{\{1,2,3,4\}},\ \frac{5}{2}+\frac{3}{5}\zeta_{\{1,2\}}+\frac{27}{5}\zeta_{\{1,3\}}-\frac{13}{5}\zeta_{\{2,3\}}+\frac{129}{25}\zeta_{\{1,2,3\}}\right\}$$

In[•]:= `clExpand[q ⊙ β + r]`

Out[•]= $5+\zeta_{\{1\}}+\zeta_{\{2\}}-\zeta_{\{3\}}+\zeta_{\{1,2\}}+5\,\zeta_{\{1,3\}}-3\,\zeta_{\{2,3\}}+5\,\zeta_{\{1,2,3\}}$

In[•]:= `α`

Out[•]= $5+\zeta_{\{1\}}+\zeta_{\{2\}}-\zeta_{\{3\}}+\zeta_{\{1,2\}}+5\,\zeta_{\{1,3\}}-3\,\zeta_{\{2,3\}}+5\,\zeta_{\{1,2,3\}}$

## Zeon Norms

### The zeon infinity norm

In[•]:= `? zeonNorm`

zeonNorm[u,m] – returns the m–norm $\|u\|_m$ of $u \in Cl_n^{nil}$.

```
In[•]:= setSignature[4];
α = Table[Random[Integer, {-5, 5}], {j, 2^maxIndex}].zeonBasis;
Print["α = ", α];
Print["‖α‖∞ = ", zeonNorm[α, ∞]];
```

$\alpha = -3 - 4\,\zeta_{\{1\}} - \zeta_{\{2\}} + 3\,\zeta_{\{3\}} - 2\,\zeta_{\{4\}} - 4\,\zeta_{\{1,3\}} + 4\,\zeta_{\{1,4\}} + 3\,\zeta_{\{2,3\}} - \zeta_{\{2,4\}} - 3\,\zeta_{\{3,4\}} + 2\,\zeta_{\{1,2,3\}} + 2\,\zeta_{\{1,2,4\}} - 2\,\zeta_{\{1,3,4\}} + \zeta_{\{2,3,4\}} - 3\,\zeta_{\{1,2,3,4\}}$

$\|\alpha\|_\infty = 4$

## The zeon p-norms

```
In[•]:= ? zeonNorm
```

zeonNorm[u,m] – returns the m–norm $\|u\|_m$ of $u \in Cl_n^{nil}$.

```
In[•]:= α = Table[Random[Integer, {-5, 5}], {j, 2^maxIndex}].zeonBasis;
Print["α = ", α];
Print["‖α‖p = ", zeonNorm[α, p]];
```

$\alpha = -4 + 5\,\zeta_{\{1\}} + 5\,\zeta_{\{2\}} - 4\,\zeta_{\{3\}} - 3\,\zeta_{\{4\}} + 5\,\zeta_{\{1,2\}} + 5\,\zeta_{\{1,4\}} - 3\,\zeta_{\{2,3\}} + 5\,\zeta_{\{2,4\}} - 5\,\zeta_{\{3,4\}} + 3\,\zeta_{\{1,2,3\}} + 4\,\zeta_{\{1,2,4\}} - \zeta_{\{1,3,4\}} + 3\,\zeta_{\{1,2,3,4\}}$

$\|\alpha\|_p = \left(1 + 2 \times 0^p + 4 \times 3^p + 3 \times 4^p + 6 \times 5^p\right)^{\frac{1}{p}}$

The 1-norm is sub-multiplicative; i.e., $\|uv\| \le \|u\|_1\,\|v\|_1$ for all $u,\ v \in Cl_n^{nil}$.

```
In[•]:= α = Table[Random[Integer, {-5, 5}], {j, 2^maxIndex}].zeonBasis;
β = Table[Random[Integer, {-5, 5}], {j, 2^maxIndex}].zeonBasis;
Print["α = ", α];
Print["‖α‖1 = ", a1 = zeonNorm[α, 1]];
Print["β = ", β];
Print["‖β‖1 = ", b1 = zeonNorm[β, 1]];
Print["αβ = ", ab = clExpand[α ⊙ β]];
Print["‖αβ‖1 = ", zeonNorm[ab, 1]];
Print["‖α‖1‖β‖1 = ", a1 b1];
```

$\alpha = -1 - 5\,\zeta_{\{1\}} + 4\,\zeta_{\{2\}} - \zeta_{\{4\}} - \zeta_{\{1,2\}} + 5\,\zeta_{\{1,3\}} - 3\,\zeta_{\{1,4\}} - 3\,\zeta_{\{2,3\}} - 4\,\zeta_{\{2,4\}} - 3\,\zeta_{\{1,2,3\}} + 5\,\zeta_{\{1,2,4\}} - \zeta_{\{1,3,4\}} - \zeta_{\{2,3,4\}} - 5\,\zeta_{\{1,2,3,4\}}$

$\|\alpha\|_1 = 42$

$\beta = 2 - 4\,\zeta_{\{1\}} - 4\,\zeta_{\{2\}} - 3\,\zeta_{\{3\}} + 2\,\zeta_{\{4\}} + 5\,\zeta_{\{1,2\}} - \zeta_{\{1,3\}} - 4\,\zeta_{\{1,4\}} + 3\,\zeta_{\{2,3\}} + 5\,\zeta_{\{2,4\}} + 3\,\zeta_{\{3,4\}} + 2\,\zeta_{\{1,2,3\}} + 2\,\zeta_{\{1,2,4\}} - 3\,\zeta_{\{1,3,4\}} + 2\,\zeta_{\{2,3,4\}} + 4\,\zeta_{\{1,2,3,4\}}$

$\|\beta\|_1 = 49$

$\alpha\beta = -2 - 6\,\zeta_{\{1\}} + 12\,\zeta_{\{2\}} + 3\,\zeta_{\{3\}} - 4\,\zeta_{\{4\}} - 3\,\zeta_{\{1,2\}} + 26\,\zeta_{\{1,3\}} - 8\,\zeta_{\{1,4\}} - 21\,\zeta_{\{2,3\}} - \zeta_{\{2,4\}} - 32\,\zeta_{\{1,2,3\}} - 12\,\zeta_{\{1,2,4\}} + 6\,\zeta_{\{1,3,4\}} + 11\,\zeta_{\{2,3,4\}} - 22\,\zeta_{\{1,2,3,4\}}$

$\|\alpha\beta\|_1 = 169$

$\|\alpha\|_1\|\beta\|_1 = 2058$

### The zeon inner product norm

In[∘]:= 
```
? zeonInnerProduct
```

zeonInnerProduct[u,v] – returns <u,v>= $\sum_I u_I v_I$ inner product <u,v> of zeon elements u,v ∈ $Cl_n^{nil}$.

In[∘]:= 
```
α = Table[Random[Integer, {-5, 5}], {j, 2^maxIndex}].zeonBasis;
Print["α = ", α];
Print["‖α‖2 = ", zeonNorm[α, 2]];
```

$\alpha = 4 - 4\,\zeta_{\{1\}} - 2\,\zeta_{\{4\}} - 3\,\zeta_{\{1,2\}} - 2\,\zeta_{\{1,3\}} - \zeta_{\{1,4\}} - 3\,\zeta_{\{2,3\}} - 4\,\zeta_{\{2,4\}} + 2\,\zeta_{\{3,4\}} - \zeta_{\{1,2,3\}} + \zeta_{\{1,2,4\}} + 2\,\zeta_{\{1,3,4\}} - 3\,\zeta_{\{1,2,3,4\}}$

$\|\alpha\|_2 = \sqrt{94}$

In[∘]:= 
```
Print["Square root of <α,α> : ", Sqrt[zeonInnerProduct[α, α]]];
```

Square root of <α,α> : $\sqrt{158}$

### The zeon spectral seminorm

In[∘]:= 
```
? zeonSpectralSeminorm
```

zeonSpectralSeminorm[u] – returns |ℜu| for u∈$Cl_n^{nil}$.

### Zeon geometric series

In[∘]:= 
```
? zeonGeometricSeries
```

zeonGeometricSeries[a,u,k] – returns partial sum $\sum_{j=0}^{k} a u^j$ of geometric series $\sum_{j=0}^{\infty} a u^j$.

If k is omitted and |ℜu|<1, returns limit of geometric series: $\frac{a}{1-\Re u} \sum_{j=0}^{maxIndex} (1-\Re u)^{-j} (\mathfrak{D}u)^j$.

In[∘]:= 
```
setSignature[4]
```

In[∘]:= 
```
a = 13;
u = 1 / 2 + 3 ζ{2} - 5 ζ{1,3} + 7 ζ{2,4};
```

In[∘]:= 
```
zeonGeometricSeries[a, u]
```

Out[∘]= $26 + 156\,\zeta_{\{2\}} - 260\,\zeta_{\{1,3\}} + 364\,\zeta_{\{2,4\}} - 3120\,\zeta_{\{1,2,3\}} - 7280\,\zeta_{\{1,2,3,4\}}$

In[∘]:= 
```
For[k = 1, k ≤ 10, k++,
 Print["k = ", k, " : ", zeonGeometricSeries[a, u, k] // N];]
```

k = 1 : $19.5 + 39.\ \zeta_{\{2\}} - 65.\ \zeta_{\{1,3\}} + 91.\ \zeta_{\{2,4\}}$

k = 2 : $22.75 + 78.\ \zeta_{\{2\}} - 130.\ \zeta_{\{1,3\}} + 182.\ \zeta_{\{2,4\}} - 390.\ \zeta_{\{1,2,3\}} - 910.\ \zeta_{\{1,2,3,4\}}$

k = 3 : $24.375 + 107.25\ \zeta_{\{2\}} - 178.75\ \zeta_{\{1,3\}} + 250.25\ \zeta_{\{2,4\}} - 975.\ \zeta_{\{1,2,3\}} - 2275.\ \zeta_{\{1,2,3,4\}}$

k = 4 : $25.1875 + 126.75\ \zeta_{\{2\}} - 211.25\ \zeta_{\{1,3\}} + 295.75\ \zeta_{\{2,4\}} - 1560.\ \zeta_{\{1,2,3\}} - 3640.\ \zeta_{\{1,2,3,4\}}$

k = 5 : $25.5938 + 138.938\ \zeta_{\{2\}} - 231.563\ \zeta_{\{1,3\}} + 324.188\ \zeta_{\{2,4\}} - 2047.5\ \zeta_{\{1,2,3\}} - 4777.5\ \zeta_{\{1,2,3,4\}}$

k = 6 : $25.7969 + 146.25\ \zeta_{\{2\}} - 243.75\ \zeta_{\{1,3\}} + 341.25\ \zeta_{\{2,4\}} - 2413.13\ \zeta_{\{1,2,3\}} - 5630.63\ \zeta_{\{1,2,3,4\}}$

k = 7 : $25.8984 + 150.516\ \zeta_{\{2\}} - 250.859\ \zeta_{\{1,3\}} + 351.203\ \zeta_{\{2,4\}} - 2669.06\ \zeta_{\{1,2,3\}} - 6227.81\ \zeta_{\{1,2,3,4\}}$

k = 8 : $25.9492 + 152.953\ \zeta_{\{2\}} - 254.922\ \zeta_{\{1,3\}} + 356.891\ \zeta_{\{2,4\}} - 2839.69\ \zeta_{\{1,2,3\}} - 6625.94\ \zeta_{\{1,2,3,4\}}$

k = 9 : $25.9746 + 154.324\ \zeta_{\{2\}} - 257.207\ \zeta_{\{1,3\}} + 360.09\ \zeta_{\{2,4\}} - 2949.38\ \zeta_{\{1,2,3\}} - 6881.88\ \zeta_{\{1,2,3,4\}}$

k = 10 : $25.9873 + 155.086\ \zeta_{\{2\}} - 258.477\ \zeta_{\{1,3\}} + 361.867\ \zeta_{\{2,4\}} - 3017.93\ \zeta_{\{1,2,3\}} - 7041.84\ \zeta_{\{1,2,3,4\}}$

In[∘]:=
```
Clear[k];
Limit[zeonGeometricSeries[a, u, k], {k → ∞}] // ExpandAll
```

Out[∘]= $26 + 156\ \zeta_{\{2\}} - 260\ \zeta_{\{1,3\}} + 364\ \zeta_{\{2,4\}} - 3120\ \zeta_{\{1,2,3\}} - 7280\ \zeta_{\{1,2,3,4\}}$

# Zeon Functions

## Elementary functions on zeons

The **CliffMath** package includes a number of built-in elementary zeon functions, including exponential, logarithmic, hyperbolic, and trigonometric functions.

In[∘]:=
```
setSignature[4];
α = Table[Random[Integer, {-5, 5}], {j, 2^maxIndex}].zeonBasis;
Print["α = ", α];
Print["Exp[α] = ", zeonExp[α]];
Print["----"];
Print["Sin[α] = ", zeonSin[α]];
Print["----"];
Print["Cos[α] = ", zeonCos[α]];
Print["----"];
Print["Tan[α] = ", zeonTan[α]];
Print["----"];
Print["Sinh[α] = ", zeonSinh[α]];
Print["----"];
Print["Cosh[α] = ", zeonCosh[α]];
Print["----"];
Print["Tanh[α] = ", zeonTanh[α]];
Print["----"];
```

$\alpha = -3 + 2\ \zeta_{\{1\}} + 4\ \zeta_{\{2\}} - 4\ \zeta_{\{4\}} + 5\ \zeta_{\{1,2\}} + 5\ \zeta_{\{1,3\}} + 5\ \zeta_{\{1,4\}} - 4\ \zeta_{\{2,3\}} - 5\ \zeta_{\{2,4\}} - 2\ \zeta_{\{3,4\}} - 3\ \zeta_{\{1,2,3\}} + 2\ \zeta_{\{1,2,4\}} + 4\ \zeta_{\{1,3,4\}} - \zeta_{\{2,3,4\}} - 4\ \zeta_{\{1,2,3,4\}}$

$\text{Exp}[\alpha] = \frac{1}{e^3} + \frac{2\ \zeta_{\{1\}}}{e^3} + \frac{4\ \zeta_{\{2\}}}{e^3} - \frac{4\ \zeta_{\{4\}}}{e^3} + \frac{13\ \zeta_{\{1,2\}}}{e^3} + \frac{5\ \zeta_{\{1,3\}}}{e^3} - \frac{3\ \zeta_{\{1,4\}}}{e^3} - \frac{4\ \zeta_{\{2,3\}}}{e^3} - \frac{21\ \zeta_{\{2,4\}}}{e^3} - \frac{2\ \zeta_{\{3,4\}}}{e^3} + \frac{9\ \zeta_{\{1,2,3\}}}{e^3} - \frac{40\ \zeta_{\{1,2,4\}}}{e^3} - \frac{20\ \zeta_{\{1,3,4\}}}{e^3} + \frac{7\ \zeta_{\{2,3,4\}}}{e^3} - \frac{97\ \zeta_{\{1,2,3,4\}}}{e^3}$

----

$$\begin{aligned}\mathrm{Sin}[\alpha] = {} & -\mathrm{Sin}[3] + 2\,\mathrm{Cos}[3]\,\zeta_{\{1\}} + 4\,\mathrm{Cos}[3]\,\zeta_{\{2\}} - 4\,\mathrm{Cos}[3]\,\zeta_{\{4\}} + 5\,\mathrm{Cos}[3]\,\zeta_{\{1,2\}} + \\ & 8\,\mathrm{Sin}[3]\,\zeta_{\{1,2\}} + 5\,\mathrm{Cos}[3]\,\zeta_{\{1,3\}} + 5\,\mathrm{Cos}[3]\,\zeta_{\{1,4\}} - 8\,\mathrm{Sin}[3]\,\zeta_{\{1,4\}} - 4\,\mathrm{Cos}[3]\,\zeta_{\{2,3\}} - \\ & 5\,\mathrm{Cos}[3]\,\zeta_{\{2,4\}} - 16\,\mathrm{Sin}[3]\,\zeta_{\{2,4\}} - 2\,\mathrm{Cos}[3]\,\zeta_{\{3,4\}} - 3\,\mathrm{Cos}[3]\,\zeta_{\{1,2,3\}} + 12\,\mathrm{Sin}[3]\,\zeta_{\{1,2,3\}} + \\ & 34\,\mathrm{Cos}[3]\,\zeta_{\{1,2,4\}} - 10\,\mathrm{Sin}[3]\,\zeta_{\{1,2,4\}} + 4\,\mathrm{Cos}[3]\,\zeta_{\{1,3,4\}} - 24\,\mathrm{Sin}[3]\,\zeta_{\{1,3,4\}} - \\ & \mathrm{Cos}[3]\,\zeta_{\{2,3,4\}} + 8\,\mathrm{Sin}[3]\,\zeta_{\{2,3,4\}} + 60\,\mathrm{Cos}[3]\,\zeta_{\{1,2,3,4\}} - 29\,\mathrm{Sin}[3]\,\zeta_{\{1,2,3,4\}}\end{aligned}$$

----

$$\begin{aligned}\mathrm{Cos}[\alpha] = {} & \mathrm{Cos}[3] + 2\,\mathrm{Sin}[3]\,\zeta_{\{1\}} + 4\,\mathrm{Sin}[3]\,\zeta_{\{2\}} - 4\,\mathrm{Sin}[3]\,\zeta_{\{4\}} - 8\,\mathrm{Cos}[3]\,\zeta_{\{1,2\}} + \\ & 5\,\mathrm{Sin}[3]\,\zeta_{\{1,2\}} + 5\,\mathrm{Sin}[3]\,\zeta_{\{1,3\}} + 8\,\mathrm{Cos}[3]\,\zeta_{\{1,4\}} + 5\,\mathrm{Sin}[3]\,\zeta_{\{1,4\}} - 4\,\mathrm{Sin}[3]\,\zeta_{\{2,3\}} + \\ & 16\,\mathrm{Cos}[3]\,\zeta_{\{2,4\}} - 5\,\mathrm{Sin}[3]\,\zeta_{\{2,4\}} - 2\,\mathrm{Sin}[3]\,\zeta_{\{3,4\}} - 12\,\mathrm{Cos}[3]\,\zeta_{\{1,2,3\}} - 3\,\mathrm{Sin}[3]\,\zeta_{\{1,2,3\}} + \\ & 10\,\mathrm{Cos}[3]\,\zeta_{\{1,2,4\}} + 34\,\mathrm{Sin}[3]\,\zeta_{\{1,2,4\}} + 24\,\mathrm{Cos}[3]\,\zeta_{\{1,3,4\}} + 4\,\mathrm{Sin}[3]\,\zeta_{\{1,3,4\}} - \\ & 8\,\mathrm{Cos}[3]\,\zeta_{\{2,3,4\}} - \mathrm{Sin}[3]\,\zeta_{\{2,3,4\}} + 29\,\mathrm{Cos}[3]\,\zeta_{\{1,2,3,4\}} + 60\,\mathrm{Sin}[3]\,\zeta_{\{1,2,3,4\}}\end{aligned}$$

----

$$\begin{aligned}\mathrm{Tan}[\alpha] = {} & 2\,\mathrm{Sec}[3]^2\,\zeta_{\{1\}} + 4\,\mathrm{Sec}[3]^2\,\zeta_{\{2\}} - 4\,\mathrm{Sec}[3]^2\,\zeta_{\{4\}} + 5\,\mathrm{Sec}[3]^2\,\zeta_{\{1,2\}} + 5\,\mathrm{Sec}[3]^2\,\zeta_{\{1,3\}} + \\ & 5\,\mathrm{Sec}[3]^2\,\zeta_{\{1,4\}} - 4\,\mathrm{Sec}[3]^2\,\zeta_{\{2,3\}} - 5\,\mathrm{Sec}[3]^2\,\zeta_{\{2,4\}} - 2\,\mathrm{Sec}[3]^2\,\zeta_{\{3,4\}} - 3\,\mathrm{Sec}[3]^2\,\zeta_{\{1,2,3\}} + \\ & 2\,\mathrm{Sec}[3]^2\,\zeta_{\{1,2,4\}} - 64\,\mathrm{Sec}[3]^4\,\zeta_{\{1,2,4\}} + 4\,\mathrm{Sec}[3]^2\,\zeta_{\{1,3,4\}} - \mathrm{Sec}[3]^2\,\zeta_{\{2,3,4\}} - \\ & 4\,\mathrm{Sec}[3]^2\,\zeta_{\{1,2,3,4\}} - 128\,\mathrm{Sec}[3]^4\,\zeta_{\{1,2,3,4\}} - \mathrm{Tan}[3] - 16\,\mathrm{Sec}[3]^2\,\zeta_{\{1,2\}}\,\mathrm{Tan}[3] + \\ & 16\,\mathrm{Sec}[3]^2\,\zeta_{\{1,4\}}\,\mathrm{Tan}[3] + 32\,\mathrm{Sec}[3]^2\,\zeta_{\{2,4\}}\,\mathrm{Tan}[3] - 24\,\mathrm{Sec}[3]^2\,\zeta_{\{1,2,3\}}\,\mathrm{Tan}[3] + \\ & 20\,\mathrm{Sec}[3]^2\,\zeta_{\{1,2,4\}}\,\mathrm{Tan}[3] + 48\,\mathrm{Sec}[3]^2\,\zeta_{\{1,3,4\}}\,\mathrm{Tan}[3] - 16\,\mathrm{Sec}[3]^2\,\zeta_{\{2,3,4\}}\,\mathrm{Tan}[3] + \\ & 58\,\mathrm{Sec}[3]^2\,\zeta_{\{1,2,3,4\}}\,\mathrm{Tan}[3] - 128\,\mathrm{Sec}[3]^2\,\zeta_{\{1,2,4\}}\,\mathrm{Tan}[3]^2 - 256\,\mathrm{Sec}[3]^2\,\zeta_{\{1,2,3,4\}}\,\mathrm{Tan}[3]^2\end{aligned}$$

----

$$\begin{aligned}\mathrm{Sinh}[\alpha] = {} & -\mathrm{Sinh}[3] + 2\,\mathrm{Cosh}[3]\,\zeta_{\{1\}} + 4\,\mathrm{Cosh}[3]\,\zeta_{\{2\}} - 4\,\mathrm{Cosh}[3]\,\zeta_{\{4\}} + 5\,\mathrm{Cosh}[3]\,\zeta_{\{1,2\}} - \\ & 8\,\mathrm{Sinh}[3]\,\zeta_{\{1,2\}} + 5\,\mathrm{Cosh}[3]\,\zeta_{\{1,3\}} + 5\,\mathrm{Cosh}[3]\,\zeta_{\{1,4\}} + 8\,\mathrm{Sinh}[3]\,\zeta_{\{1,4\}} - 4\,\mathrm{Cosh}[3]\,\zeta_{\{2,3\}} - \\ & 5\,\mathrm{Cosh}[3]\,\zeta_{\{2,4\}} + 16\,\mathrm{Sinh}[3]\,\zeta_{\{2,4\}} - 2\,\mathrm{Cosh}[3]\,\zeta_{\{3,4\}} - 3\,\mathrm{Cosh}[3]\,\zeta_{\{1,2,3\}} - 12\,\mathrm{Sinh}[3]\,\zeta_{\{1,2,3\}} - \\ & 30\,\mathrm{Cosh}[3]\,\zeta_{\{1,2,4\}} + 10\,\mathrm{Sinh}[3]\,\zeta_{\{1,2,4\}} + 4\,\mathrm{Cosh}[3]\,\zeta_{\{1,3,4\}} + 24\,\mathrm{Sinh}[3]\,\zeta_{\{1,3,4\}} - \\ & \mathrm{Cosh}[3]\,\zeta_{\{2,3,4\}} - 8\,\mathrm{Sinh}[3]\,\zeta_{\{2,3,4\}} - 68\,\mathrm{Cosh}[3]\,\zeta_{\{1,2,3,4\}} + 29\,\mathrm{Sinh}[3]\,\zeta_{\{1,2,3,4\}}\end{aligned}$$

----

$$\begin{aligned}\mathrm{Cosh}[\alpha] = {} & \mathrm{Cosh}[3] - 2\,\mathrm{Sinh}[3]\,\zeta_{\{1\}} - 4\,\mathrm{Sinh}[3]\,\zeta_{\{2\}} + 4\,\mathrm{Sinh}[3]\,\zeta_{\{4\}} + 8\,\mathrm{Cosh}[3]\,\zeta_{\{1,2\}} - \\ & 5\,\mathrm{Sinh}[3]\,\zeta_{\{1,2\}} - 5\,\mathrm{Sinh}[3]\,\zeta_{\{1,3\}} - 8\,\mathrm{Cosh}[3]\,\zeta_{\{1,4\}} - 5\,\mathrm{Sinh}[3]\,\zeta_{\{1,4\}} + 4\,\mathrm{Sinh}[3]\,\zeta_{\{2,3\}} - \\ & 16\,\mathrm{Cosh}[3]\,\zeta_{\{2,4\}} + 5\,\mathrm{Sinh}[3]\,\zeta_{\{2,4\}} + 2\,\mathrm{Sinh}[3]\,\zeta_{\{3,4\}} + 12\,\mathrm{Cosh}[3]\,\zeta_{\{1,2,3\}} + 3\,\mathrm{Sinh}[3]\,\zeta_{\{1,2,3\}} - \\ & 10\,\mathrm{Cosh}[3]\,\zeta_{\{1,2,4\}} + 30\,\mathrm{Sinh}[3]\,\zeta_{\{1,2,4\}} - 24\,\mathrm{Cosh}[3]\,\zeta_{\{1,3,4\}} - 4\,\mathrm{Sinh}[3]\,\zeta_{\{1,3,4\}} + \\ & 8\,\mathrm{Cosh}[3]\,\zeta_{\{2,3,4\}} + \mathrm{Sinh}[3]\,\zeta_{\{2,3,4\}} - 29\,\mathrm{Cosh}[3]\,\zeta_{\{1,2,3,4\}} + 68\,\mathrm{Sinh}[3]\,\zeta_{\{1,2,3,4\}}\end{aligned}$$

----

$$\begin{aligned}\mathrm{Tanh}[\alpha] = {} & 2\,\mathrm{Sech}[3]^2\,\zeta_{\{1\}} + 4\,\mathrm{Sech}[3]^2\,\zeta_{\{2\}} - 4\,\mathrm{Sech}[3]^2\,\zeta_{\{4\}} + 5\,\mathrm{Sech}[3]^2\,\zeta_{\{1,2\}} + 5\,\mathrm{Sech}[3]^2\,\zeta_{\{1,3\}} + \\ & 5\,\mathrm{Sech}[3]^2\,\zeta_{\{1,4\}} - 4\,\mathrm{Sech}[3]^2\,\zeta_{\{2,3\}} - 5\,\mathrm{Sech}[3]^2\,\zeta_{\{2,4\}} - 2\,\mathrm{Sech}[3]^2\,\zeta_{\{3,4\}} - 3\,\mathrm{Sech}[3]^2\,\zeta_{\{1,2,3\}} + \\ & 2\,\mathrm{Sech}[3]^2\,\zeta_{\{1,2,4\}} + 64\,\mathrm{Sech}[3]^4\,\zeta_{\{1,2,4\}} + 4\,\mathrm{Sech}[3]^2\,\zeta_{\{1,3,4\}} - \mathrm{Sech}[3]^2\,\zeta_{\{2,3,4\}} - \\ & 4\,\mathrm{Sech}[3]^2\,\zeta_{\{1,2,3,4\}} + 128\,\mathrm{Sech}[3]^4\,\zeta_{\{1,2,3,4\}} - \mathrm{Tanh}[3] + 16\,\mathrm{Sech}[3]^2\,\zeta_{\{1,2\}}\,\mathrm{Tanh}[3] - \\ & 16\,\mathrm{Sech}[3]^2\,\zeta_{\{1,4\}}\,\mathrm{Tanh}[3] - 32\,\mathrm{Sech}[3]^2\,\zeta_{\{2,4\}}\,\mathrm{Tanh}[3] + 24\,\mathrm{Sech}[3]^2\,\zeta_{\{1,2,3\}}\,\mathrm{Tanh}[3] - \\ & 20\,\mathrm{Sech}[3]^2\,\zeta_{\{1,2,4\}}\,\mathrm{Tanh}[3] - 48\,\mathrm{Sech}[3]^2\,\zeta_{\{1,3,4\}}\,\mathrm{Tanh}[3] + 16\,\mathrm{Sech}[3]^2\,\zeta_{\{2,3,4\}}\,\mathrm{Tanh}[3] - \\ & 58\,\mathrm{Sech}[3]^2\,\zeta_{\{1,2,3,4\}}\,\mathrm{Tanh}[3] - 128\,\mathrm{Sech}[3]^2\,\zeta_{\{1,2,4\}}\,\mathrm{Tanh}[3]^2 - 256\,\mathrm{Sech}[3]^2\,\zeta_{\{1,2,3,4\}}\,\mathrm{Tanh}[3]^2\end{aligned}$$

----

### Elementary inverse functions

**CliffMath2018** offers a number of inverse functions, including zeon logarithm, inverse trigonometric functions, and inverse hyperbolic functions. More general inverse functions can be defined by using *zeonExtensionEval*.

In[∘]:= 
```
? zeonLog
```

zeonLog[u] Evaluate logarithm of u ∈ $Cl_n^{nil}$, provided zRe[u]>0.

In[∘]:= 
```
setSignature[4];
α = Table[Random[Integer, {-5, 5}], {j, 2^maxIndex}].zeonBasis;
If[nilpotentZeonQ[α], α = α + 1, If[zRe[α] < 0, α = -α]];
Print["α = ", α];
Print["Log[α] = ", (la = zeonLog[α])];
Print["---"];
Print["Exp[Log[α]] = ", zeonExp[la]];
Print["---"];
Print["Log[Exp[α]] = ", zeonLog[zeonExp[α]]];
```

$\alpha = 1 + \zeta_{\{1\}} - \zeta_{\{2\}} - 4\,\zeta_{\{3\}} - 3\,\zeta_{\{4\}} + \zeta_{\{1,2\}} + 2\,\zeta_{\{1,3\}} - 4\,\zeta_{\{1,4\}} - 2\,\zeta_{\{2,3\}} + 2\,\zeta_{\{2,4\}} + \zeta_{\{3,4\}} + 3\,\zeta_{\{1,2,3\}} - 3\,\zeta_{\{1,2,4\}} - 5\,\zeta_{\{1,3,4\}} + 4\,\zeta_{\{2,3,4\}} - 4\,\zeta_{\{1,2,3,4\}}$

$\text{Log}[\alpha] = \zeta_{\{1\}} - \zeta_{\{2\}} - 4\,\zeta_{\{3\}} - 3\,\zeta_{\{4\}} + 2\,\zeta_{\{1,2\}} + 6\,\zeta_{\{1,3\}} - \zeta_{\{1,4\}} - 6\,\zeta_{\{2,3\}} - \zeta_{\{2,4\}} - 11\,\zeta_{\{3,4\}} + 19\,\zeta_{\{1,2,3\}} + 8\,\zeta_{\{1,3,4\}} - 17\,\zeta_{\{2,3,4\}} + 41\,\zeta_{\{1,2,3,4\}}$

---

$\text{Exp}[\text{Log}[\alpha]] = 1 + \zeta_{\{1\}} - \zeta_{\{2\}} - 4\,\zeta_{\{3\}} - 3\,\zeta_{\{4\}} + \zeta_{\{1,2\}} + 2\,\zeta_{\{1,3\}} - 4\,\zeta_{\{1,4\}} - 2\,\zeta_{\{2,3\}} + 2\,\zeta_{\{2,4\}} + \zeta_{\{3,4\}} + 3\,\zeta_{\{1,2,3\}} - 3\,\zeta_{\{1,2,4\}} - 5\,\zeta_{\{1,3,4\}} + 4\,\zeta_{\{2,3,4\}} - 4\,\zeta_{\{1,2,3,4\}}$

---

$\text{Log}[\text{Exp}[\alpha]] = 1 + \zeta_{\{1\}} - \zeta_{\{2\}} - 4\,\zeta_{\{3\}} - 3\,\zeta_{\{4\}} + \zeta_{\{1,2\}} + 2\,\zeta_{\{1,3\}} - 4\,\zeta_{\{1,4\}} - 2\,\zeta_{\{2,3\}} + 2\,\zeta_{\{2,4\}} + \zeta_{\{3,4\}} + 3\,\zeta_{\{1,2,3\}} - 3\,\zeta_{\{1,2,4\}} - 5\,\zeta_{\{1,3,4\}} + 4\,\zeta_{\{2,3,4\}} - 4\,\zeta_{\{1,2,3,4\}}$

In[∘]:= 
```
setSignature[3];
β = RandomReal[] + Table[RandomInteger[{-5, 5}], {j, 2^maxIndex - 1}].zeonBasis[[2 ;;]];
Print["β = ", β];
Print["---"];
Print["ArcSin[β] = ", zeonArcSin[β]];
Print["---"];
Print["Sin[ArcSin[β]] = ", zeonSin[zeonArcSin[β]]];
Print["---"];
Print["ArcCos[β] = ", zeonArcCos[β]];
Print["---"];
Print["Cos[ArcCos[β]] = ", zeonCos[zeonArcCos[β]]];
Print["---"];
Print["ArcTan[β] = ", zeonArcTan[β]];
Print["---"];
Print["Tan[ArcTan[β]] = ", zeonTan[zeonArcTan[β]]];
Print["---"];
```

$\beta = 0.143471 - \zeta_{\{1\}} + \zeta_{\{3\}} - \zeta_{\{1,2\}} + 5\,\zeta_{\{1,3\}} - \zeta_{\{2,3\}} + 2\,\zeta_{\{1,2,3\}}$

---

$\text{ArcSin}[\beta] = 0.143968 - 1.01045\,\zeta_{\{1\}} + 1.01045\,\zeta_{\{3\}} - 1.01045\,\zeta_{\{1,2\}} + 4.90425\,\zeta_{\{1,3\}} - 1.01045\,\zeta_{\{2,3\}} + 2.02091\,\zeta_{\{1,2,3\}}$

---

$\text{Sin}[\text{ArcSin}[\beta]] = 0.143471 - 1.\,\zeta_{\{1\}} + 1.\,\zeta_{\{3\}} - 1.\,\zeta_{\{1,2\}} + 5.\,\zeta_{\{1,3\}} - 1.\,\zeta_{\{2,3\}} + 2.\,\zeta_{\{1,2,3\}}$

---

$\text{ArcCos}[\beta] = 1.42683 + 1.01045\,\zeta_{\{1\}} - 1.01045\,\zeta_{\{3\}} + 1.01045\,\zeta_{\{1,2\}} - 4.90425\,\zeta_{\{1,3\}} + 1.01045\,\zeta_{\{2,3\}} - 2.02091\,\zeta_{\{1,2,3\}}$

---

$\text{Cos}[\text{ArcCos}[\beta]] = 0.143471 - 1.\,\zeta_{\{1\}} + 1.\,\zeta_{\{3\}} - 1.\,\zeta_{\{1,2\}} + 5.\,\zeta_{\{1,3\}} - 1.\,\zeta_{\{2,3\}} + 2.\,\zeta_{\{1,2,3\}}$

---

$\text{ArcTan}[\beta] = 0.142499 - 0.979831\,\zeta_{\{1\}} + 0.979831\,\zeta_{\{3\}} - 0.979831\,\zeta_{\{1,2\}} + 5.17464\,\zeta_{\{1,3\}} - 0.979831\,\zeta_{\{2,3\}} + 1.95966\,\zeta_{\{1,2,3\}}$

---

$\text{Tan}[\text{ArcTan}[\beta]] = 0.143471 - 1.\,\zeta_{\{1\}} + 1.\,\zeta_{\{3\}} - 1.\,\zeta_{\{1,2\}} + 5.\,\zeta_{\{1,3\}} - 1.\,\zeta_{\{2,3\}} + 2.\,\zeta_{\{1,2,3\}}$

---

### Derivatives and Antiderivatives of Zeon Functions

The theory of zeon differentiation is laid out in *Differential Calculus of Zeon Functions*.

In[•]:= `? generalZeonDerivative`

generalZeonDerivative[f[x],x,k] returns $\phi^{(k)}$(u): the kth order derivative of zeon extension $\phi$ of real function f. If no order is specified, k=1 is assumed.

In[•]:= `generalZeonDerivative[4 x^2 + x Cos[x^2] - 2 x Exp[x^3], x, 1]`

Out[•]=
$$-2\,e^{\mathbb{R}u^3} + 8\,\mathbb{D}u - 8\,e^{\mathbb{R}u^3}\,\mathbb{D}u^3 + 8\,\mathbb{R}u - 24\,e^{\mathbb{R}u^3}\,\mathbb{D}u^2\,\mathbb{R}u - 24\,e^{\mathbb{R}u^3}\,\mathbb{D}u\,\mathbb{R}u^2 - 6\,e^{\mathbb{R}u^3}\,\mathbb{R}u^3 - 132\,e^{\mathbb{R}u^3}\,\mathbb{D}u^3\,\mathbb{R}u^3 - 81\,e^{\mathbb{R}u^3}\,\mathbb{D}u^2\,\mathbb{R}u^4 - 18\,e^{\mathbb{R}u^3}\,\mathbb{D}u\,\mathbb{R}u^5 - 144\,e^{\mathbb{R}u^3}\,\mathbb{D}u^3\,\mathbb{R}u^6 - 27\,e^{\mathbb{R}u^3}\,\mathbb{D}u^2\,\mathbb{R}u^7 - 27\,e^{\mathbb{R}u^3}\,\mathbb{D}u^3\,\mathbb{R}u^9 + \text{Cos}[\mathbb{R}u^2] - 10\,\mathbb{D}u^3\,\mathbb{R}u\,\text{Cos}[\mathbb{R}u^2] - 12\,\mathbb{D}u^2\,\mathbb{R}u^2\,\text{Cos}[\mathbb{R}u^2] - 4\,\mathbb{D}u\,\mathbb{R}u^3\,\text{Cos}[\mathbb{R}u^2] + \frac{8}{3}\,\mathbb{D}u^3\,\mathbb{R}u^5\,\text{Cos}[\mathbb{R}u^2] - 3\,\mathbb{D}u^2\,\text{Sin}[\mathbb{R}u^2] - 6\,\mathbb{D}u\,\mathbb{R}u\,\text{Sin}[\mathbb{R}u^2] - 2\,\mathbb{R}u^2\,\text{Sin}[\mathbb{R}u^2] + \frac{40}{3}\,\mathbb{D}u^3\,\mathbb{R}u^3\,\text{Sin}[\mathbb{R}u^2] + 4\,\mathbb{D}u^2\,\mathbb{R}u^4\,\text{Sin}[\mathbb{R}u^2]$$

In[•]:= `? zeonDerivative`

zeonDerivative[f[x],x,u,k] returns $\phi^{(k)}$(u): the kth order derivative of zeon extension $\phi$ of real function f, evaluated at u ∈ $Cl_n^{nil}$. If no order is specified, k=1 is assumed.

In[•]:= $u = 3 - 4\,\zeta_{\{1,2\}}$;
`zeonDerivative[4 x^2 + x Cos[x^2] - 2 x Exp[x^3], x, u]`

Out[•]= $24 - 164\,e^{27} + \text{Cos}[9] - 18\,\text{Sin}[9] - 32\,\zeta_{\{1,2\}} + 18\,360\,e^{27}\,\zeta_{\{1,2\}} + 432\,\text{Cos}[9]\,\zeta_{\{1,2\}} + 72\,\text{Sin}[9]\,\zeta_{\{1,2\}}$

In[∘]:= ? generalZeonAntiDerivative

generalZeonAntiDerivative[f[x],x] returns Φ[u]+k, where
Φ[u] is the zeon extension of real function F[x], the antiderivative of f[x].

In[∘]:= generalZeonAntiDerivative[4 x^2 + x Cos[x^2] - 2 x Exp[x^3], x]

Out[∘]=
$$-e^{\Re u^3}\,\mathfrak{D}u^2+\frac{4\,\mathfrak{D}u^3}{3}-2\,e^{\Re u^3}\,\mathfrak{D}u\,\Re u+4\,\mathfrak{D}u^2\,\Re u+4\,\mathfrak{D}u\,\Re u^2-4\,e^{\Re u^3}\,\mathfrak{D}u^3\,\Re u^2+\frac{4\,\Re u^3}{3}-3\,e^{\Re u^3}\,\mathfrak{D}u^2\,\Re u^3-3\,e^{\Re u^3}\,\mathfrak{D}u^3\,\Re u^5+\kappa+\frac{1}{2}\,\mathfrak{D}u^2\,\mathrm{Cos}[\Re u^2]+\mathfrak{D}u\,\Re u\,\mathrm{Cos}[\Re u^2]-\frac{2}{3}\,\mathfrak{D}u^3\,\Re u^3\,\mathrm{Cos}[\Re u^2]-\frac{2\,(-\Re u^3)^{1/3}\,\mathrm{Gamma}[\frac{2}{3},-\Re u^3]}{3\,\Re u}+\frac{\mathrm{Sin}[\Re u^2]}{2}-\mathfrak{D}u^3\,\Re u\,\mathrm{Sin}[\Re u^2]-\mathfrak{D}u^2\,\Re u^2\,\mathrm{Sin}[\Re u^2]$$

In[∘]:= ? zeonAntiDerivative

zeonAntiDerivative[f[x],x,u] returns Φ[u], where
Φ[x] is the zeon extension of real function F[x], the antiderivative of f[x].

In[∘]:= zeonAntiDerivative[4 x^2 + x Cos[x^2] - 2 x Exp[x^3], x, u]

Out[∘]=
$$36-\frac{2}{3}\,(-1)^{1/3}\,\mathrm{Gamma}\left[\frac{2}{3},-27\right]+\frac{\mathrm{Sin}[9]}{2}-144\,\zeta_{\{1,2\}}+24\,e^{27}\,\zeta_{\{1,2\}}-12\,\mathrm{Cos}[9]\,\zeta_{\{1,2\}}$$

# Zeon Polynomials

## Zeon Quadratic Equations

**CliffMath2018** offers a number of tools for working with quadratic equations with zeon coefficients. Of particular interest are equations of the form $\alpha u^2+\beta u+\gamma=0$, where $\alpha,\beta,\gamma\in \mathrm{Cl}_n^{\mathrm{nil}}$, and $\alpha$ is invertible (i.e. $\Re\alpha\neq 0$).

In[∘]:= ? zeonDiscriminant

zeonDiscriminant[f[x],x] computes the zeon discriminant of $\phi[x]=\alpha x^2+\beta x+\gamma$, where $\alpha,\beta,\gamma\ \varepsilon\ \mathrm{Cl}_n^{\mathrm{nil}}$.

In[∘]:= ? zeonQuadraticFormula

zeonQuadraticFormula[ϕ[x],x] applies the zeon quadratic formula to $\phi[x]=\alpha x^2+\beta x+\gamma$,
where $\alpha,\beta,\gamma\ \varepsilon\ \mathrm{Cl}_n^{\mathrm{nil}}$. When Δ=0, includes arbitrary κ assumed to be nilpotent of index 2.

In[∘]:= setSignature[2]

In[∘]:=
```
Clear[u];
α1 = Table[Random[Integer, {-5, 5}], {j, 2^maxIndex}].zeonBasis;
α = If[nilpotentZeonQ[α1], α1 + Random[Integer, {1, 5}], zeonAbs[α1]];
β = Table[Random[Integer, {-5, 5}], {j, 2^maxIndex}].zeonBasis;
γ = Table[Random[Integer, {-5, 5}], {j, 2^maxIndex}].zeonBasis;
g = Collect[α u^2 + β u + γ, u];
Print["ϕ[u] = ", g];
```

$$\phi[u]\ =\ -3\,\zeta_{\{1\}}-\zeta_{\{2\}}+u\,(3-3\,\zeta_{\{1\}}+4\,\zeta_{\{2\}}-5\,\zeta_{\{1,2\}})+u^2\,(1-5\,\zeta_{\{1\}}-\zeta_{\{2\}}+2\,\zeta_{\{1,2\}})$$

In[•]:= `Δ = zeonDiscriminant[g, u]`

Out[•]= $9 - 6\,\zeta_{\{1\}} + 28\,\zeta_{\{2\}} - 86\,\zeta_{\{1,2\}}$

If ℜΔ>0, then $\phi$[u]=0 has exactly two zeon solutions. If ℜΔ<0, then $\phi$[u]=0 has no solutions. When Δ is nilpotent, the equation has either infinitely many solutions or no solutions. When Δ=0, an uncountable family of solutions is obtained.

In[•]:= `sols = zeonQuadraticFormula[g, u]`

Out[•]= $\left\{\zeta_{\{1\}} + \frac{\zeta_{\{2\}}}{3} - \frac{14}{3}\,\zeta_{\{1,2\}},\ -3 + 2\,\zeta_{\{1\}} - \frac{13\,\zeta_{\{2\}}}{3} + \frac{29}{3}\,\zeta_{\{1,2\}}\right\}$

Solutions can be verified by evaluating the polynomial $\phi$[u] at the specified values.

In[•]:= `zeonQuadraticEval[α u^2 + β u + γ, u, sols[[1]]]`

Out[•]= $-\frac{31}{3}\,\zeta_{\{1,2\}}$

In[•]:= `zeonQuadraticEval[α x^2 + β x + γ, x, sols[[2]]]`

Out[•]= $-45\,\zeta_{\{1\}} - 9\,\zeta_{\{2\}} - \frac{331}{3}\,\zeta_{\{1,2\}}$

## Matrix Representations of Zeon Algebras

The basic building blocks of matrix representations of zeon generators are the "$\eta$ matrices", which are 2×2 null-square matrices obtained from Pauli matrices.

In[•]:= `? etaMatrix`

etaMatrix – Returns 2×2 matrix $\eta = \sigma_z + i\sigma_y$ used to construct representations of zeon generators.

In[•]:= `MatrixForm[etaMatrix]`

Out[•]//MatrixForm=

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

Matrix representations of zeon generators are obtained as Kronecker products of eta matrices with identity matrices.

In[•]:= `? zeonGeneratorRep`

zeonGeneratorRep[j] – $2^n \times 2^n$ matrix representation of zeon generator $\zeta_{\{j\}} \to \sigma_0{}^{\otimes(j-1)} \otimes \eta \otimes \sigma_0{}^{\otimes(n-j)}$, where $\eta=\sigma_z+i\sigma_y$, and $\sigma_0$, $\sigma_y$, and $\sigma_z$ are Pauli matrices.

In[•]:=
```
setSignature[3];
MatrixForm[zeonGeneratorRep[3]]
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$

To see the general matrix representation of a zeon element, we can first generate the general canonical expansion of an element using *generalZeon*, as follows. In this example, the 4-dimensional algebra $Cl_2^{nil}$ is used to simplify the notation.

In[•]:=
```
? generalZeonElement
```

generalZeonElement[α] – Canonical expansion $\sum_{I\in 2^{[n]}} \alpha_I \zeta_I$, where $\alpha$ is a symbol.

In[•]:=
```
setSignature[2];
Clear[a];
generalZeonElement[a]
```

Out[•]= $a_0 + a_{\{1\}}\,\zeta_{\{1\}} + a_{\{2\}}\,\zeta_{\{2\}} + a_{\{1,2\}}\,\zeta_{\{1,2\}}$

The matrix representation of an arbitrary zeon element is then obtained using *zeonRep*, as in the following example.

In[•]:=
```
? zeonRep
```

zeonRep[u] – Returns $2^n \times 2^n$ real matrix representation of $u \in Cl_n{}^{nil}$.

In[•]:=
```
zeonRep[generalZeonElement[a]] // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} a_0 + a_{\{1\}} + a_{\{2\}} + a_{\{1,2\}} & a_{\{2\}} + a_{\{1,2\}} & a_{\{1\}} + a_{\{1,2\}} & a_{\{1,2\}} \\ -a_{\{2\}} - a_{\{1,2\}} & a_0 + a_{\{1\}} - a_{\{2\}} - a_{\{1,2\}} & -a_{\{1,2\}} & a_{\{1\}} - a_{\{1,2\}} \\ -a_{\{1\}} - a_{\{1,2\}} & -a_{\{1,2\}} & a_0 - a_{\{1\}} + a_{\{2\}} - a_{\{1,2\}} & a_{\{2\}} - a_{\{1,2\}} \\ a_{\{1,2\}} & -a_{\{1\}} + a_{\{1,2\}} & -a_{\{2\}} + a_{\{1,2\}} & a_0 - a_{\{1\}} - a_{\{2\}} + a_{\{1,2\}} \end{pmatrix}$$

## Zeon Matrices

Computations with matrices having entries in $Cl_n{}^{nil}$ is supported.

In[•]:=
```
? zeonMatrixInverse
```

zeonMatrixInverse[A] – Given square zeon
matrix A, returns zeon matrix U such that AU=UA=I, provided |ℜA|≠0.

In[•]:=
```
(A = {{1 + ζ{1,2} - 2 ζ{3}, 4 + ζ{1} - ζ{2,3}}, {ζ{1,2,3}, 5 + 3 ζ{1}}}) // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 1 - 2\,\zeta_{\{3\}} + \zeta_{\{1,2\}} & 4 + \zeta_{\{1\}} - \zeta_{\{2,3\}} \\ \zeta_{\{1,2,3\}} & 5 + 3\,\zeta_{\{1\}} \end{pmatrix}$$

A zeon matrix $A = \Re A + \mathfrak{D} A$ is invertible if and only if the real matrix $\Re A$ is invertible.

In[•]:=
```
zRe[A] // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 1 & 4 \\ 0 & 5 \end{pmatrix}$$

When $A$ is invertible, the inverse is given by $A^{-1} = (\Re A)^{-1} \sum_{j=0}^{k-1} \left(-\mathfrak{D}A\ (\Re A)^{-1}\right)^j$, where $k=\kappa\left(\mathfrak{D}A\ (\Re A)^{-1}\right)$ is the index of nilpotency of the matrix $\mathfrak{D}A\ (\Re A)^{-1}$. In $Cl_n{}^{nil}$, the index of nilpotency is never greater than $n$.

In[•]:=
```
(U = zeonMatrixInverse[A]) // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 1 + 2\,\zeta_{\{3\}} - \zeta_{\{1,2\}} - \frac{16}{5}\,\zeta_{\{1,2,3\}} & -\frac{4}{5} + \frac{7\,\zeta_{\{1\}}}{25} - \frac{8\,\zeta_{\{3\}}}{5} + \frac{4}{5}\,\zeta_{\{1,2\}} + \frac{14}{25}\,\zeta_{\{1,3\}} + \frac{1}{5}\,\zeta_{\{2,3\}} + \frac{61}{25}\,\zeta_{\{1,2,3\}} \\ -\frac{1}{5}\,\zeta_{\{1,2,3\}} & \frac{1}{5} - \frac{3\,\zeta_{\{1\}}}{25} + \frac{4}{25}\,\zeta_{\{1,2,3\}} \end{pmatrix}$$

In[•]:=
```
clExpand[cliffordMatrixProduct[A, U]] // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

### The zeon-Frobenius norm of a zeon matrix

```
In[∘]:= ? zeonFrobeniusNorm
```

zeonFrobeniusNorm[A] – returns $\sqrt{\mathfrak{R}(\mathrm{tr}[A^\dagger A]^*)} = \sqrt{\sum \| a_{ij} \|_2{}^2}$.

Here are some zeon-Frobenius norm examples.

```
In[∘]:= setSignature[3];
For[lp = 1, lp ≤ 5, lp++,
  rows = Random[Integer, {1, 3}];
  cols = Random[Integer, {1, 3}];
  Print[Subscript["C", lp], " = ",
   MatrixForm[C_lp = Table[Table[Random[Integer] Random[Integer, {-1, 2}], {j, 2^maxIndex}].
       zeonBasis, {i, rows}, {j, cols}]]];
  Print[Superscript[Subscript["C", lp], "†"], " = ",
   MatrixForm[ad = Transpose[zeonDual[C_lp]]]];
  Print[Superscript[Subscript["C", lp], "†"], Subscript["C", lp], " = ",
   MatrixForm[adA = clExpand[cliffordMatrixProduct[C_lp, Transpose[zeonDual[C_lp]]]]]];
  Print["tr[", Superscript[Subscript["C", lp], "†"], Subscript["C", lp], "] = ",
   x = Tr[zeonDual[clExpand[cliffordMatrixProduct[C_lp, Transpose[zeonDual[C_lp]]]]]]];
  Print["‖", Subscript["C", lp], "‖zF = ",
   DisplayForm[RowBox[{SqrtBox[RowBox[{"ℛ", RowBox[{"(", SuperscriptBox[
            RowBox[{"tr", "[", RowBox[{SuperscriptBox[Subscript["C", lp], "†"], Subscript[
                  "C", lp]}], "]"}], "*"], ")"}]}]], " = ", zf = Sqrt[zRe[x]]}]]];
  Print["----"];
  Print["zeonFrobeniusNorm[", Subscript["C", lp], "] = ", zeonFrobeniusNorm[C_lp]];
  Print["----"];
 ];
```

$C_1 = \begin{pmatrix} 2\zeta_{\{1,2\}} + \zeta_{\{1,2,3\}} & 2\zeta_{\{2\}} + 2\zeta_{\{1,2,3\}} \\ 2\zeta_{\{1\}} - \zeta_{\{1,2\}} & 2\zeta_{\{2\}} + \zeta_{\{1,3\}} + \zeta_{\{1,2,3\}} \\ 1 + 2\zeta_{\{3\}} + 2\zeta_{\{1,2\}} & 2\zeta_{\{1\}} + 2\zeta_{\{1,3\}} \end{pmatrix}$

$C_1{}^\dagger = \begin{pmatrix} 1 + 2\zeta_{\{3\}} & -\zeta_{\{3\}} + 2\zeta_{\{2,3\}} & 2\zeta_{\{3\}} + 2\zeta_{\{1,2\}} + \zeta_{\{1,2,3\}} \\ 2 + 2\zeta_{\{1,3\}} & 1 + \zeta_{\{2\}} + 2\zeta_{\{1,3\}} & 2\zeta_{\{2\}} + 2\zeta_{\{2,3\}} \end{pmatrix}$

$C_1{}^\dagger C_1 = \begin{pmatrix} 4\zeta_{\{2\}} + 2\zeta_{\{1,2\}} + 13\zeta_{\{1,2,3\}} & 2\zeta_{\{2\}} + 4\zeta_{\{1,2,3\}} \\ 2\zeta_{\{1\}} + 4\zeta_{\{2\}} - \zeta_{\{1,2\}} + 6\zeta_{\{1,3\}} + 4\zeta_{\{1,2,3\}} & 2\zeta_{\{2\}} - \zeta_{\{1,3\}} + 11\zeta_{\{1,2,3\}} \\ 1 + 4\zeta_{\{1\}} + 4\zeta_{\{3\}} + 2\zeta_{\{1,2\}} + 4\zeta_{\{1,3\}} + 4\zeta_{\{1,2,3\}} & 2\zeta_{\{1\}} - \zeta_{\{3\}} + 2\zeta_{\{1,2\}} + 2\zeta_{\{1,3\}} + 2\zeta_{\{2,3\}} & 2\zeta_{\{3\}} + 6 \end{pmatrix}$

$\mathrm{tr}[C_1{}^\dagger C_1] = 41 - \zeta_{\{2\}} + 8\zeta_{\{3\}} + 2\zeta_{\{1,2\}} + 6\zeta_{\{1,3\}}$

$\| C_1 \|_{zF} = \sqrt{\mathfrak{R}\left(\mathrm{tr}\left[C_1{}^\dagger C_1\right]^*\right)} = \sqrt{41}$

----

zeonFrobeniusNorm[$C_1$] = $\sqrt{41}$

----

$C_2 = \begin{pmatrix} -\zeta_{\{1\}} + \zeta_{\{3\}} - \zeta_{\{2,3\}} & 2 \end{pmatrix}$

$$C_2{}^{\dagger} = \begin{pmatrix} -\zeta_{\{1\}} + \zeta_{\{1,2\}} - \zeta_{\{2,3\}} \\ 2\,\zeta_{\{1,2,3\}} \end{pmatrix}$$

$$C_2{}^{\dagger}C_2 = \begin{pmatrix} -\zeta_{\{1,3\}} + 7\,\zeta_{\{1,2,3\}} \end{pmatrix}$$

$$\mathrm{tr}[C_2{}^{\dagger}C_2] = 7 - \zeta_{\{2\}}$$

$$\|C_2\|_{zF} = \sqrt{\mathbb{R}\left(\mathrm{tr}\left[C_2{}^{\dagger}\,C_2\right]^{*}\right)} = \sqrt{7}$$

----

zeonFrobeniusNorm[$C_2$] = $\sqrt{7}$

----

$$C_3 = \begin{pmatrix} 2 - \zeta_{\{2\}} - \zeta_{\{1,3\}} + \zeta_{\{1,2,3\}} & -1 - \zeta_{\{3\}} + 2\,\zeta_{\{2,3\}} + \zeta_{\{1,2,3\}} \\ 2 - \zeta_{\{2\}} & 1 + \zeta_{\{1,2\}} + \zeta_{\{1,3\}} + 2\,\zeta_{\{2,3\}} \end{pmatrix}$$

$$C_3{}^{\dagger} = \begin{pmatrix} 1 - \zeta_{\{2\}} - \zeta_{\{1,3\}} + 2\,\zeta_{\{1,2,3\}} & -\zeta_{\{1,3\}} + 2\,\zeta_{\{1,2,3\}} \\ 1 + 2\,\zeta_{\{1\}} - \zeta_{\{1,2\}} - \zeta_{\{1,2,3\}} & 2\,\zeta_{\{1\}} + \zeta_{\{2\}} + \zeta_{\{3\}} + \zeta_{\{1,2,3\}} \end{pmatrix}$$

$$C_3{}^{\dagger}C_3 = \begin{pmatrix} 1 - 2\,\zeta_{\{1\}} - 3\,\zeta_{\{2\}} - \zeta_{\{3\}} + \zeta_{\{1,2\}} - 5\,\zeta_{\{1,3\}} + 2\,\zeta_{\{2,3\}} + 14\,\zeta_{\{1,2,3\}} & -2\,\zeta_{\{1\}} - \zeta_{\{2\}} - \zeta_{\{3\}} - 4\,\zeta_{\{1,3\}} - \zeta_{\{2,3 \\ 3 + 2\,\zeta_{\{1\}} - 3\,\zeta_{\{2\}} - \zeta_{\{1,3\}} + 2\,\zeta_{\{2,3\}} + 8\,\zeta_{\{1,2,3\}} & 2\,\zeta_{\{1\}} + \zeta_{\{2\}} + \zeta_{\{3\}} - 2\,\zeta_{\{1,3\}} + 12 \end{pmatrix}$$

$$\mathrm{tr}[C_3{}^{\dagger}C_3] = 26 + 2\,\zeta_{\{1\}} - 7\,\zeta_{\{2\}} + \zeta_{\{3\}} - 2\,\zeta_{\{1,3\}} + \zeta_{\{1,2,3\}}$$

$$\|C_3\|_{zF} = \sqrt{\mathbb{R}\left(\mathrm{tr}\left[C_3{}^{\dagger}\,C_3\right]^{*}\right)} = \sqrt{26}$$

----

zeonFrobeniusNorm[$C_3$] = $\sqrt{26}$

----

$$C_4 = \begin{pmatrix} 2\,\zeta_{\{1\}} - \zeta_{\{3\}} + \zeta_{\{1,3\}} - \zeta_{\{1,2,3\}} & -1 + 2\,\zeta_{\{2,3\}} + \zeta_{\{1,2,3\}} & 2 + 2\,\zeta_{\{2\}} + 2\,\zeta_{\{1,2\}} + 2\,\zeta_{\{2,3\}} \\ 1 + \zeta_{\{2\}} & 2\,\zeta_{\{2\}} + \zeta_{\{1,2\}} & -1 + 2\,\zeta_{\{3\}} + 2\,\zeta_{\{1,2,3\}} \end{pmatrix}$$

$$C_4{}^{\dagger} = \begin{pmatrix} -1 + \zeta_{\{2\}} - \zeta_{\{1,2\}} + 2\,\zeta_{\{2,3\}} & \zeta_{\{1,3\}} + \zeta_{\{1,2,3\}} \\ 1 + 2\,\zeta_{\{1\}} - \zeta_{\{1,2,3\}} & \zeta_{\{3\}} + 2\,\zeta_{\{1,3\}} \\ 2\,\zeta_{\{1\}} + 2\,\zeta_{\{3\}} + 2\,\zeta_{\{1,3\}} + 2\,\zeta_{\{1,2,3\}} & 2 + 2\,\zeta_{\{1,2\}} - \zeta_{\{1,2,3\}} \end{pmatrix}$$

$$C_4{}^{\dagger}C_4 = \begin{pmatrix} -1 + 5\,\zeta_{\{3\}} + 6\,\zeta_{\{1,2\}} + 3\,\zeta_{\{1,3\}} + 5\,\zeta_{\{2,3\}} + 29\,\zeta_{\{1,2,3\}} & 4 + 4\,\zeta_{\{2\}} - \zeta_{\{3\}} + 8\,\zeta_{\{1,2\}} - 2\,\zeta_{\{1 \\ -1 - 2\,\zeta_{\{1\}} + 2\,\zeta_{\{2\}} - 2\,\zeta_{\{3\}} + 4\,\zeta_{\{1,2\}} + 2\,\zeta_{\{1,3\}} + 2\,\zeta_{\{2,3\}} - 2\,\zeta_{\{1,2,3\}} & -2 + 4\,\zeta_{\{3\}} - 2\,\zeta_{\{1,2\}} + \zeta_{\{1,3\}} + \end{pmatrix}$$

$$\mathrm{tr}[C_4{}^{\dagger}C_4] = 45 + 7\,\zeta_{\{1\}} + 4\,\zeta_{\{2\}} + 4\,\zeta_{\{3\}} + 9\,\zeta_{\{1,2\}} - 3\,\zeta_{\{1,2,3\}}$$

$$\|C_4\|_{zF} = \sqrt{\mathbb{R}\left(\mathrm{tr}\left[C_4{}^{\dagger}\,C_4\right]^{*}\right)} = 3\sqrt{5}$$

----

zeonFrobeniusNorm[$C_4$] = $3\sqrt{5}$

----

$$C_5 = \begin{pmatrix} 2\,\zeta_{\{1,2,3\}} \end{pmatrix}$$

$$C_5{}^{\dagger} = \begin{pmatrix} 2 \end{pmatrix}$$

$$C_5{}^{\dagger}C_5 = \begin{pmatrix} 4\,\zeta_{\{1,2,3\}} \end{pmatrix}$$

$$\mathrm{tr}[C_5{}^{\dagger}C_5] = 4$$

$$\|C_5\|_{zF} = \sqrt{\mathbb{R}\left(\mathrm{tr}\left[C_5{}^{\dagger}\,C_5\right]^{*}\right)} = 2$$

----

zeonFrobeniusNorm[$C_5$] = 2

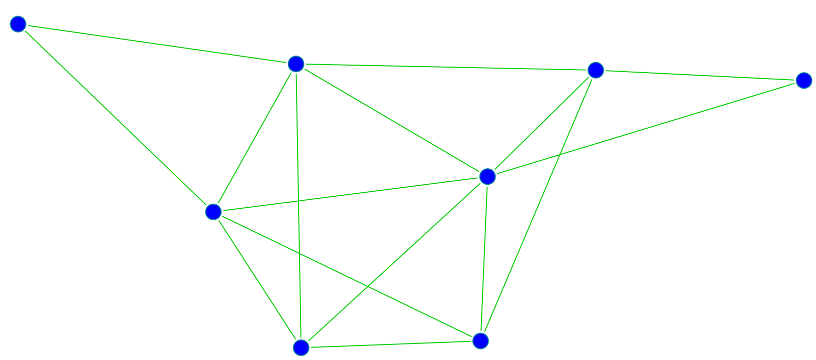
----

## Zeons in Graph Theory

In[∘]:=
```
(* Generate adjacency matrix for random graph on n vertices *)
GenRandGraph[n_, p_] := Module[{i, j, A, rw, co},
   A = Table[0, {i, 1, n}, {j, 1, n}]; For[rw = 1, rw ≤ n, rw++,
    For[co = 1, co < rw, co++,
     If[RandomReal[] ≤ p, A⟦rw, co⟧ = 1;
       A⟦co, rw⟧ = 1;, Null];]];
   Return[A];];
```

In[∘]:=
```
vertices = 8;
setSignature[vertices, 0];
m = GenRandGraph[vertices, .5];
g = AdjacencyGraph[m, VertexStyle → {Blue}, EdgeStyle → {RGBColor[0, .8, 0]}]
```

Out[∘]=



In[∘]:=
```
(A = m.DiagonalMatrix[Table[ζ{i}, {i, 1, vertices}]]) // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & \zeta_{\{2\}} & \zeta_{\{3\}} & 0 & 0 & \zeta_{\{6\}} & \zeta_{\{7\}} & \zeta_{\{8\}} \\ \zeta_{\{1\}} & 0 & \zeta_{\{3\}} & \zeta_{\{4\}} & 0 & \zeta_{\{6\}} & 0 & 0 \\ \zeta_{\{1\}} & \zeta_{\{2\}} & 0 & \zeta_{\{4\}} & 0 & \zeta_{\{6\}} & 0 & \zeta_{\{8\}} \\ 0 & \zeta_{\{2\}} & \zeta_{\{3\}} & 0 & 0 & \zeta_{\{6\}} & \zeta_{\{7\}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_{\{6\}} & \zeta_{\{7\}} & 0 \\ \zeta_{\{1\}} & \zeta_{\{2\}} & \zeta_{\{3\}} & \zeta_{\{4\}} & \zeta_{\{5\}} & 0 & \zeta_{\{7\}} & 0 \\ \zeta_{\{1\}} & 0 & 0 & \zeta_{\{4\}} & \zeta_{\{5\}} & \zeta_{\{6\}} & 0 & 0 \\ \zeta_{\{1\}} & 0 & \zeta_{\{3\}} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Powers of nilpotent adjacency matrices can be used to count cycles. In this example, the graph's 5-cycles are enumerated along the main diagonal of $A^5$.

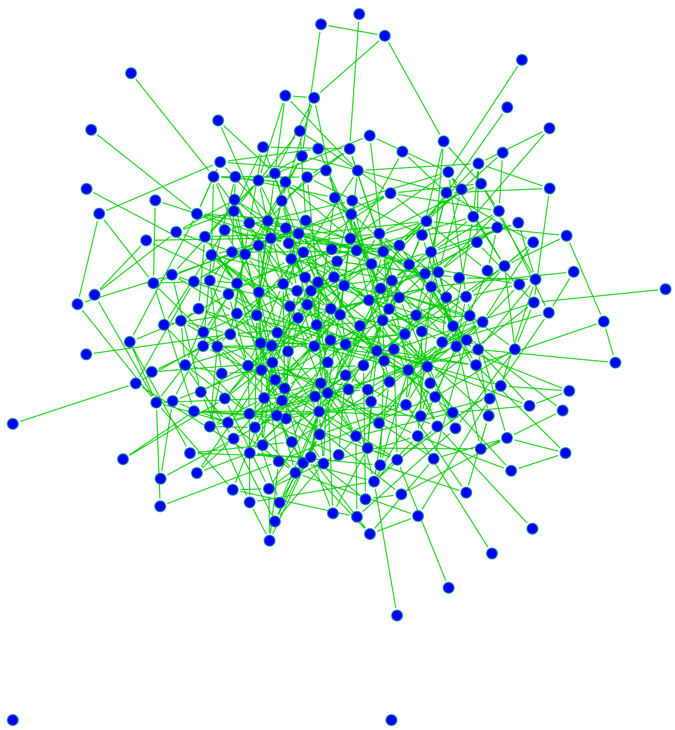
In[∘]:=
```
Print[(t = Tr[cliffordMatrixPower[A, 5]])];
Print["Number of 5-cycles: ", scalarSum[t] / 10];
```

$$60\,\zeta_{\{1,2,3,4,6\}} + 20\,\zeta_{\{1,2,3,4,7\}} + 10\,\zeta_{\{1,2,3,4,8\}} + 20\,\zeta_{\{1,2,3,6,7\}} + 20\,\zeta_{\{1,2,3,6,8\}} + 40\,\zeta_{\{1,2,4,6,7\}} + 10\,\zeta_{\{1,2,5,6,7\}} + 40\,\zeta_{\{1,3,4,6,7\}} + 10\,\zeta_{\{1,3,4,6,8\}} + 10\,\zeta_{\{1,3,4,7,8\}} + 10\,\zeta_{\{1,3,5,6,7\}} + 10\,\zeta_{\{1,3,6,7,8\}} + 20\,\zeta_{\{2,3,4,6,7\}} + 10\,\zeta_{\{2,4,5,6,7\}} + 10\,\zeta_{\{3,4,5,6,7\}}$$

```
Number of 5-cycles: 30
```

In[◦]:=
```
vertices = 256;
setSignature[vertices, 0];
m = GenRandGraph[vertices, .02];
g = AdjacencyGraph[m, VertexStyle → {Blue}, EdgeStyle → {RGBColor[0, .8, 0]}]
```

Out[◦]=



In[◦]:=
```
(A = m.DiagonalMatrix[Table[ξ{i}, {i, 1, vertices}]]);
```

In[◦]:=
```
v0 = Random[Integer, {1, vertices}];
Print["Min hop cycles based at vertex ", v0, ":"];
u0 = UnitVector[vertices, v0];
u1 = u0.A;
u1 = clExpand[cliffordMatrixProduct[u1, A]];
loop = 2;
While[(loop ≤ vertices ∧ clExpand[u1[[v0]]] === 0) ∨ loop == 2,
  u1 = clExpand[cliffordMatrixProduct[u1, A]];
  loop++;
  Print[loop, "-cycles based at vertex ", v0, ": ", clExpand[ u1[[v0]]];];
```

Min hop cycles based at vertex 30:

3-cycles based at vertex 30: 0

4-cycles based at vertex 30: 0

5-cycles based at vertex 30: $2\,\zeta_{\{30,38,90,229,238\}} + 2\,\zeta_{\{30,90,194,214,227\}} + 2\,\zeta_{\{30,91,214,238,242\}}$

In[•]:=
```
v0 = Random[Integer, {1, vertices}];
v1 = Random[Integer, {1, vertices}];
Print["Hop-minimal paths from vertex ", v0, " to vertex ", v1, ":"];
u0 = UnitVector[vertices, v0];
u1 = u0.A;
loop = 1;
While[(loop ≤ vertices) ∧ clExpand[ζ{v0} ⊙ u1[[v1]]] === 0,
  u1 = clExpand[cliffordMatrixProduct[u1, A]];
  loop++;];
Print[loop, "-paths ", v0, "→", v1, ": ", clExpand[ζ{v0} ⊙ u1[[v1]]]];
```

Hop-minimal paths from vertex 172 to vertex 83:

3-paths 172→83: $\zeta_{\{39,62,83,172\}}$

# 4

# The “Sym-Clifford” and “Idem-Clifford” algebras

## 4.1. The sym-Clifford algebra

Canonical generators for $\mathrm{Cl}_{p,q,r}{}^{\mathrm{sym}}$ are denoted $\varsigma_{\{i\}}$ for $i$=1,..., $n$=$p$+$q$+$r$. The generators *commute* and their squares satisfy the following:

$e_{\{i\}}{}^2 = 1 \ (1 \le i \le p)$

$e_{\{i\}}{}^2 = -1 \ (p+1 \le i \le p+q)$

$e_{\{i\}}{}^2 = 0 \ (p+q+1 \le i \le p+q+r)$.

In[∘]:=
```
setSignature[2, 1];
Print["Multiplication table for basis blades of Cl2,1sym"];
Print[MatrixForm[
  M = Table[clExpand[symBasis[[i]] ⊙ symBasis[[j]]], {j, 1, 2^maxIndex}, {i, 1, 2^maxIndex}]]];
```

$$\begin{pmatrix}
1 & \varsigma_{\{1\}} & \varsigma_{\{2\}} & \varsigma_{\{1,2\}} & \varsigma_{\{3\}} & \varsigma_{\{1,3\}} & \varsigma_{\{2,3\}} & \varsigma_{\{1,2,3\}} \\
\varsigma_{\{1\}} & 1 & \varsigma_{\{1,2\}} & \varsigma_{\{2\}} & \varsigma_{\{1,3\}} & \varsigma_{\{3\}} & \varsigma_{\{1,2,3\}} & \varsigma_{\{2,3\}} \\
\varsigma_{\{2\}} & \varsigma_{\{1,2\}} & 1 & \varsigma_{\{1\}} & \varsigma_{\{2,3\}} & \varsigma_{\{1,2,3\}} & \varsigma_{\{3\}} & \varsigma_{\{1,3\}} \\
\varsigma_{\{1,2\}} & \varsigma_{\{2\}} & \varsigma_{\{1\}} & 1 & \varsigma_{\{1,2,3\}} & \varsigma_{\{2,3\}} & \varsigma_{\{1,3\}} & \varsigma_{\{3\}} \\
\varsigma_{\{3\}} & \varsigma_{\{1,3\}} & \varsigma_{\{2,3\}} & \varsigma_{\{1,2,3\}} & -1 & -\varsigma_{\{1\}} & -\varsigma_{\{2\}} & -\varsigma_{\{1,2\}} \\
\varsigma_{\{1,3\}} & \varsigma_{\{3\}} & \varsigma_{\{1,2,3\}} & \varsigma_{\{2,3\}} & -\varsigma_{\{1\}} & -1 & -\varsigma_{\{1,2\}} & -\varsigma_{\{2\}} \\
\varsigma_{\{2,3\}} & \varsigma_{\{1,2,3\}} & \varsigma_{\{3\}} & \varsigma_{\{1,3\}} & -\varsigma_{\{2\}} & -\varsigma_{\{1,2\}} & -1 & -\varsigma_{\{1\}} \\
\varsigma_{\{1,2,3\}} & \varsigma_{\{2,3\}} & \varsigma_{\{1,3\}} & \varsigma_{\{3\}} & -\varsigma_{\{1,2\}} & -\varsigma_{\{2\}} & -\varsigma_{\{1\}} & -1
\end{pmatrix}$$

Multiplication table for basis blades of $\mathrm{Cl}_{2,1}{}^{\mathrm{sym}}$

$$\begin{pmatrix}
1 & \varsigma_{\{1\}} & \varsigma_{\{2\}} & \varsigma_{\{1,2\}} & \varsigma_{\{3\}} & \varsigma_{\{1,3\}} & \varsigma_{\{2,3\}} & \varsigma_{\{1,2,3\}} \\
\varsigma_{\{1\}} & 1 & \varsigma_{\{1,2\}} & \varsigma_{\{2\}} & \varsigma_{\{1,3\}} & \varsigma_{\{3\}} & \varsigma_{\{1,2,3\}} & \varsigma_{\{2,3\}} \\
\varsigma_{\{2\}} & \varsigma_{\{1,2\}} & 1 & \varsigma_{\{1\}} & \varsigma_{\{2,3\}} & \varsigma_{\{1,2,3\}} & \varsigma_{\{3\}} & \varsigma_{\{1,3\}} \\
\varsigma_{\{1,2\}} & \varsigma_{\{2\}} & \varsigma_{\{1\}} & 1 & \varsigma_{\{1,2,3\}} & \varsigma_{\{2,3\}} & \varsigma_{\{1,3\}} & \varsigma_{\{3\}} \\
\varsigma_{\{3\}} & \varsigma_{\{1,3\}} & \varsigma_{\{2,3\}} & \varsigma_{\{1,2,3\}} & -1 & -\varsigma_{\{1\}} & -\varsigma_{\{2\}} & -\varsigma_{\{1,2\}} \\
\varsigma_{\{1,3\}} & \varsigma_{\{3\}} & \varsigma_{\{1,2,3\}} & \varsigma_{\{2,3\}} & -\varsigma_{\{1\}} & -1 & -\varsigma_{\{1,2\}} & -\varsigma_{\{2\}} \\
\varsigma_{\{2,3\}} & \varsigma_{\{1,2,3\}} & \varsigma_{\{3\}} & \varsigma_{\{1,3\}} & -\varsigma_{\{2\}} & -\varsigma_{\{1,2\}} & -1 & -\varsigma_{\{1\}} \\
\varsigma_{\{1,2,3\}} & \varsigma_{\{2,3\}} & \varsigma_{\{1,3\}} & \varsigma_{\{3\}} & -\varsigma_{\{1,2\}} & -\varsigma_{\{2\}} & -\varsigma_{\{1\}} & -1
\end{pmatrix}$$

Multiplication table for basis blades of $\mathrm{Cl}_{2,1}{}^{\mathrm{sym}}$

In[∘]:=
```
setSignature[1, 2];
Print["Multiplication table for basis blades of Cl2,1sym"];
Print[MatrixForm[
  M = Table[clExpand[symBasis[[i]] ⊙ symBasis[[j]]], {j, 1, 2^maxIndex}, {i, 1, 2^maxIndex}]]];
```

Multiplication table for basis blades of $Cl_{2,1}{}^{sym}$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | ς{1} | ς{2} | ς{1,2} | ς{3} | ς{1,3} | ς{2,3} | ς{1,2,3} |
| ς{1} | 1 | ς{1,2} | ς{2} | ς{1,3} | ς{3} | ς{1,2,3} | ς{2,3} |
| ς{2} | ς{1,2} | -1 | -ς{1} | ς{2,3} | ς{1,2,3} | -ς{3} | -ς{1,3} |
| ς{1,2} | ς{2} | -ς{1} | -1 | ς{1,2,3} | ς{2,3} | -ς{1,3} | -ς{3} |
| ς{3} | ς{1,3} | ς{2,3} | ς{1,2,3} | -1 | -ς{1} | -ς{2} | -ς{1,2} |
| ς{1,3} | ς{3} | ς{1,2,3} | ς{2,3} | -ς{1} | -1 | -ς{1,2} | -ς{2} |
| ς{2,3} | ς{1,2,3} | -ς{3} | -ς{1,3} | -ς{2} | -ς{1,2} | 1 | ς{1} |
| ς{1,2,3} | ς{2,3} | -ς{1,3} | -ς{3} | -ς{1,2} | -ς{2} | ς{1} | 1 |

In[∘]:=
```
setSignature[3, 0];
Print["Multiplication table for basis blades of Cl2,1^sym"];
Print[MatrixForm[
  M = Table[clExpand[symBasis[[i]] ⊙ symBasis[[j]]], {j, 1, 2^maxIndex}, {i, 1, 2^maxIndex}]]];
```

Multiplication table for basis blades of $Cl_{2,1}{}^{sym}$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | ς{1} | ς{2} | ς{1,2} | ς{3} | ς{1,3} | ς{2,3} | ς{1,2,3} |
| ς{1} | 1 | ς{1,2} | ς{2} | ς{1,3} | ς{3} | ς{1,2,3} | ς{2,3} |
| ς{2} | ς{1,2} | 1 | ς{1} | ς{2,3} | ς{1,2,3} | ς{3} | ς{1,3} |
| ς{1,2} | ς{2} | ς{1} | 1 | ς{1,2,3} | ς{2,3} | ς{1,3} | ς{3} |
| ς{3} | ς{1,3} | ς{2,3} | ς{1,2,3} | 1 | ς{1} | ς{2} | ς{1,2} |
| ς{1,3} | ς{3} | ς{1,2,3} | ς{2,3} | ς{1} | 1 | ς{1,2} | ς{2} |
| ς{2,3} | ς{1,2,3} | ς{3} | ς{1,3} | ς{2} | ς{1,2} | 1 | ς{1} |
| ς{1,2,3} | ς{2,3} | ς{1,3} | ς{3} | ς{1,2} | ς{2} | ς{1} | 1 |

In[∘]:=
```
setSignature[1, 1, 1];
Print["Multiplication table for basis blades of Cl2,1^sym"];
Print[MatrixForm[
  M = Table[clExpand[symBasis[[i]] ⊙ symBasis[[j]]], {j, 1, 2^maxIndex}, {i, 1, 2^maxIndex}]]];
```

Multiplication table for basis blades of $Cl_{2,1}{}^{sym}$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | ς{1} | ς{2} | ς{1,2} | ς{3} | ς{1,3} | ς{2,3} | ς{1,2,3} |
| ς{1} | 1 | ς{1,2} | ς{2} | ς{1,3} | ς{3} | ς{1,2,3} | ς{2,3} |
| ς{2} | ς{1,2} | -1 | -ς{1} | ς{2,3} | ς{1,2,3} | -ς{3} | -ς{1,3} |
| ς{1,2} | ς{2} | -ς{1} | -1 | ς{1,2,3} | ς{2,3} | -ς{1,3} | -ς{3} |
| ς{3} | ς{1,3} | ς{2,3} | ς{1,2,3} | 0 | 0 | 0 | 0 |
| ς{1,3} | ς{3} | ς{1,2,3} | ς{2,3} | 0 | 0 | 0 | 0 |
| ς{2,3} | ς{1,2,3} | -ς{3} | -ς{1,3} | 0 | 0 | 0 | 0 |
| ς{1,2,3} | ς{2,3} | -ς{1,3} | -ς{3} | 0 | 0 | 0 | 0 |

In[∘]:=
```
setSignature[4];
(A = (Table[1, {i, 1, 4}, {j, 1, 4}] -
     IdentityMatrix[4]).DiagonalMatrix[Table[ς{i}, {i, 1, 4}]]) // MatrixForm
```

Out[∘]//MatrixForm=

| | | | |
|---|---|---|---|
| 0 | ς{2} | ς{3} | ς{4} |
| ς{1} | 0 | ς{3} | ς{4} |
| ς{1} | ς{2} | 0 | ς{4} |
| ς{1} | ς{2} | ς{3} | 0 |

In[∘]:=
```
Tr[cliffordMatrixPower[A, 3]]
```

Out[∘]= $6\,ς_{\{1,2,3\}} + 6\,ς_{\{1,2,4\}} + 6\,ς_{\{1,3,4\}} + 6\,ς_{\{2,3,4\}}$

In[∘]:=
```
cliffordMatrixPower[A, 2] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \varsigma_{\{1,2\}} + \varsigma_{\{1,3\}} + \varsigma_{\{1,4\}} & \varsigma_{\{2,3\}} + \varsigma_{\{2,4\}} & \varsigma_{\{2,3\}} + \varsigma_{\{3,4\}} & \varsigma_{\{2,4\}} + \varsigma_{\{3,4\}} \\ \varsigma_{\{1,3\}} + \varsigma_{\{1,4\}} & \varsigma_{\{1,2\}} + \varsigma_{\{2,3\}} + \varsigma_{\{2,4\}} & \varsigma_{\{1,3\}} + \varsigma_{\{3,4\}} & \varsigma_{\{1,4\}} + \varsigma_{\{3,4\}} \\ \varsigma_{\{1,2\}} + \varsigma_{\{1,4\}} & \varsigma_{\{1,2\}} + \varsigma_{\{2,4\}} & \varsigma_{\{1,3\}} + \varsigma_{\{2,3\}} + \varsigma_{\{3,4\}} & \varsigma_{\{1,4\}} + \varsigma_{\{2,4\}} \\ \varsigma_{\{1,2\}} + \varsigma_{\{1,3\}} & \varsigma_{\{1,2\}} + \varsigma_{\{2,3\}} & \varsigma_{\{1,3\}} + \varsigma_{\{2,3\}} & \varsigma_{\{1,4\}} + \varsigma_{\{2,4\}} + \varsigma_{\{3,4\}} \end{pmatrix}$$

## 4.2. The idem-Clifford algebra

Canonical generators for $\mathrm{Cl}_n^{\mathrm{idem}}$ are denoted $\varepsilon_{\{i\}}$ for $i$=1,..., $n=p+q+r$. The generators commute and are idempotent; i.e., their squares satisfy

$$\varepsilon_{\{i\}}{}^2 = \varepsilon_{\{i\}} \ (1 \le i \le n)$$

In[∘]:=
```
setSignature[3];
Print["Multiplication table for Cl_n^idem"];
Print[MatrixForm[
   M = Table[clExpand[idemBasis[[i]] ⊙ idemBasis[[j]]], {j, 1, 2^maxIndex}, {i, 1, 2^maxIndex}]]];
```

Multiplication table for $\mathrm{Cl}_n{}^{\mathrm{idem}}$

$$\begin{pmatrix} 1 & \varepsilon_{\{1\}} & \varepsilon_{\{2\}} & \varepsilon_{\{1,2\}} & \varepsilon_{\{3\}} & \varepsilon_{\{1,3\}} & \varepsilon_{\{2,3\}} & \varepsilon_{\{1,2,3\}} \\ \varepsilon_{\{1\}} & \varepsilon_{\{1\}} & \varepsilon_{\{1,2\}} & \varepsilon_{\{1,2\}} & \varepsilon_{\{1,3\}} & \varepsilon_{\{1,3\}} & \varepsilon_{\{1,2,3\}} & \varepsilon_{\{1,2,3\}} \\ \varepsilon_{\{2\}} & \varepsilon_{\{1,2\}} & \varepsilon_{\{2\}} & \varepsilon_{\{1,2\}} & \varepsilon_{\{2,3\}} & \varepsilon_{\{1,2,3\}} & \varepsilon_{\{2,3\}} & \varepsilon_{\{1,2,3\}} \\ \varepsilon_{\{1,2\}} & \varepsilon_{\{1,2\}} & \varepsilon_{\{1,2\}} & \varepsilon_{\{1,2\}} & \varepsilon_{\{1,2,3\}} & \varepsilon_{\{1,2,3\}} & \varepsilon_{\{1,2,3\}} & \varepsilon_{\{1,2,3\}} \\ \varepsilon_{\{3\}} & \varepsilon_{\{1,3\}} & \varepsilon_{\{2,3\}} & \varepsilon_{\{1,2,3\}} & \varepsilon_{\{3\}} & \varepsilon_{\{1,3\}} & \varepsilon_{\{2,3\}} & \varepsilon_{\{1,2,3\}} \\ \varepsilon_{\{1,3\}} & \varepsilon_{\{1,3\}} & \varepsilon_{\{1,2,3\}} & \varepsilon_{\{1,2,3\}} & \varepsilon_{\{1,3\}} & \varepsilon_{\{1,3\}} & \varepsilon_{\{1,2,3\}} & \varepsilon_{\{1,2,3\}} \\ \varepsilon_{\{2,3\}} & \varepsilon_{\{1,2,3\}} & \varepsilon_{\{2,3\}} & \varepsilon_{\{1,2,3\}} & \varepsilon_{\{2,3\}} & \varepsilon_{\{1,2,3\}} & \varepsilon_{\{2,3\}} & \varepsilon_{\{1,2,3\}} \\ \varepsilon_{\{1,2,3\}} & \varepsilon_{\{1,2,3\}} & \varepsilon_{\{1,2,3\}} & \varepsilon_{\{1,2,3\}} & \varepsilon_{\{1,2,3\}} & \varepsilon_{\{1,2,3\}} & \varepsilon_{\{1,2,3\}} & \varepsilon_{\{1,2,3\}} \end{pmatrix}$$

# A
# Glossary of commands

Help is available for most **CliffMath** commands by placing “?” before the command.

In[∘]:= `? CircleDot`

CircleDot[u,v] – returns product u ⊙ v in Clifford
algebra or subalgebra determined by canonical basis elements. >>

In[∘]:= `? clGrade`

clGrade[u] – Maximal clGrade of nonzero terms in canonical expansion of u.

In[∘]:= `? clGramSchmidt`

clGramSchmidt[L] – Given set L of linearly
independent vectors, returns set B of vectors pairwise–orthogonal
with respect to the Clifford inner product <u,v> = $\frac{1}{2}$(uv+vu).

In[∘]:= `? cliffordConjugate`

cliffordConjugate[u] – Clifford conjugate of u ∈ $Cl_{p,q,r}$.   u CTRL& _

In[∘]:= `? clIdemNorm`

clIdemNorm[u,m] – returns $\|u\|_m$ m–norm of u ∈ $Cl_n{}^{idem}$.

In[∘]:= `? cliffordDecomp`

cliffordDecomp[u] – Returns scalar vectors {$v_i$ : 1 ≤ i ≤clGrade[u]} such that u= $\prod_{i=1}^{clGrade[u]} v_i$.

Note: Assumes u is decomposable *a priori*. If in doubt, apply cliffordDecomposableQ first.

In[∘]:= `? cliffordDecomposableQ`

cliffordDecomposableQ[u] – Decomposability
test for Clifford algebra element. Necessary in indefinite signatures

In[∘]:= ? cliffordGeneratorRep

cliffordGeneratorRep[j] – $2^n \times 2^n$ real matrix representation of generator $e_{\{j\}} \in Cl_{p,q,r}$:
$e_{\{j\}} \to \sigma_x{\otimes}^{(j-1)} \otimes \sigma_z \otimes \sigma_0{\otimes}^{(n-j)}$ $(1 \le j \le p)$
$e_{\{j\}} \to \sigma_x{\otimes}^{(j-1)} \otimes i\sigma_y \otimes \sigma_0{\otimes}^{(n-j)}$ $(p < j \le p+q)$
$e_{\{j\}} \to \sigma_x{\otimes}^{(j-1)} \otimes \eta \otimes \sigma_0{\otimes}^{(n-j)}$ $(p+q < j \le n)$
where $\eta = \sigma_z + i\sigma_y$, and $\sigma_0$, $\sigma_y$, and $\sigma_z$ are Pauli matrices.

In[∘]:= ? cliffordInnerProduct

cliffordInnerProduct[u,v] – returns $<u,v>=\sum_I u_I v_I$ inner product of Clifford elements $u,v \in Cl_{p,q,r}$.

In[∘]:= ? cliffordInverse

cliffordInverse[u] – returns $\frac{\tilde{u}}{u\,\tilde{u}}$, the multiplicative inverse of $u \in Cl_{p,q,r}$ when u is invertible.

In[∘]:= ? cliffordInvertibleQ

cliffordInvertibleQ[u] – Invertibility test for Clifford algebra element. Necessary in indefinite signatures

In[∘]:= ? cliffordListProduct

cliffordListProduct[{$u_1$, $u_2$, ..., $u_k$}] returns the ordered product $u_1 \odot u_2 \odot \ldots \odot u_k$.

In[∘]:= ? cliffordMatrixPower

cliffordMatrixPower[A,m] – Computes $A^{\odot m}$ for square matrix $A$ having entries from $Cl_{p,q,r}$ or $Cl_*{}^*$.

In[∘]:= ? cliffordMatrixProduct

cliffordMatrixProduct[A,B] Compute matrix product AB
using multiplication in appropriate algebra $Cl_{p,q,r}$, $Cl_n{}^{nil}$, $Cl_n{}^{idem}$, or $Cl_{p,q,r}{}^{sym}$.

In[∘]:= ? cliffordNorm

cliffordNorm[u,m] – returns $\|u\|_m$ m–norm of $u \in Cl_{p,q,r}$.

In[∘]:= ? cliffordPower

cliffordPower[u,m] – Compute $u^{\odot m}$ for nonnegative integer m and $u \in Cl_{p,q,r}$ or other $Cl_*{}^*$.

In[∘]:= ? conjugatePseudoInverse

conjugatePseudoInverse[u] – returns grade involution of $u^{-1}$, i.e., $\frac{\overline{u}}{u\,\tilde{u}}$ $(u \in Cl_{p,q,r})$.

In[∘]:= ? cliffordRep

cliffordRep[u] – Returns $2^n \times 2^n$ (real) matrix representation of $u \in Cl_{p,q,r}$.

In[∘]:= ? cliffordReversion

cliffordReversion[u] – Clifford reversion of $u \in Cl_{p,q,r}$.  u CTRL & ~

In[∘]:= ? clMu

clMu[i,J] – Counts number of elements in multi–index J that are greater than $i \in \mathbb{N}_0$.

In[∘]:= ? clRecursionLimit

clRecursionLimit[m_] – sets recursion limit for particular CliffMath procedures.
Mathematica's default value is 1024.  In some
cases, higher limits are desirable, but 1024 is minimum allowed.

In[∘]:= ? clSymNorm

clSymNorm[u,m] – returns $\|u\|_m$ m–norm of $u \in {Cl_{p,q,r}}^{sym}$.

In[∘]:= ? etaMatrix

etaMatrix – Returns 2×2 matrix $\eta = \sigma_z + i\sigma_y$ used to construct representations of zeon generators.

In[∘]:= ? fastBladeFactor

fastBladeFactor[B] – Returns scalar $\alpha$ and vectors {b_i:
1≤i≤clGrade(B)} such that $B=\alpha \prod_{i=1}^{clGrade(B)} b_i$, provided $B \in Cl_{p,q,r}$ is a blade.

In[∘]:= ? generalCliffordElement

generalCliffordElement[$\alpha$] – Canonical expansion $\sum_{I\in 2^{[n]}} \alpha_I e_I$, where $\alpha$ is a symbol.

In[∘]:= ? generalIdemElement

generalIdemElement[$\alpha$] – Canonical expansion $\sum_{I\in 2^{[n]}} \alpha_I \varepsilon_I$, where $\alpha$ is a symbol.

In[∘]:= ? generalSymElement

generalSymElement[$\alpha$] – Canonical expansion $\sum_{I\in 2^{[n]}} \alpha_I \varsigma_I$, where $\alpha$ is a symbol.

In[∘]:= ? generalZeonElement

generalZeonElement[$\alpha$] – Canonical expansion $\sum_{I\in 2^{[n]}} \alpha_I \zeta_I$, where $\alpha$ is a symbol.

In[∘]:= ? generalZeonAntiDerivative

generalZeonAntiDerivative[f[x],x] returns Φ[u]+k, where
Φ[u] is the zeon extension of real function F[x], the antiderivative of f[x].

In[∘]:= ? generalZeonDerivative

generalZeonDerivative[f[x],x,k] returns $\phi^{(k)}(u)$: the kth order derivative
of zeon extension $\phi$ of real function f. If no order is specified, k=1 is assumed.

In[∘]:= ? getIndex

getIndex[u] – Extract all indices appearing among multi–indices of u.

In[∘]:= ? getSignature

getSignature – Returns $Cl_{p,q,r}$ , where p,q,r represent the currently set signature.

In[∘]:= ? gradeBasis

gradeBasis[k,symb] – Returns basis k–blades for positive integer k.
Parameter symb specifies the algebra: e (default): $Cl_{p,q,r}$; $\zeta$: $Cl_n^{nil}$; $\varepsilon$: $Cl_n^{idem}$; $\varsigma$: $Cl_{p,q,r}^{sym}$.

In[∘]:= ? gradeInvolution

gradeInvolution[u] – Clifford grade involution of $u \in Cl_{p,q,r}$. u CTRL& ^

In[∘]:= ? gradeKPart

gradeKPart[u,k] – Returns grade–k part $\langle u\rangle_k$ of $u \in Cl_{p,q,r}$ or $Cl_*^*$.

In[∘]:= ? idemBasis

idemBasis – Returns ordered basis blades $\{\varepsilon_I : I\in 2^{[n]}\} \subseteq Cl_n^{nil}$.

In[∘]:= ? idemInnerProduct

idemInnerProduct[u,v] – returns $\langle u,v\rangle = \sum_I u_I v_I$ inner product $\langle u,v\rangle$ of elements $u,v \in Cl_n^{idem}$.

In[∘]:= ? leftContract

leftContract[u,v] – Clifford left contraction $u \rfloor v$.

In[∘]:= `? matrixKroneckerPower`

matrixKroneckerPower[M,k] – returns *k*th Kronecker power of Matrix M, i.e., $M^{\otimes k}$.

In[∘]:= `? nilpotentZeonQ`

nilpotentZeonQ[u] returns True if u is nilpotent; False if u is invertible.

In[∘]:= `? OverBar`

OverBar[u] – Clifford conjugate of u ∈ $Cl_{p,q,r}$ ≫

In[∘]:= `? OverHat`

OverHat[u] – grade involution of u ∈ $Cl_{p,q,r}$ ≫

In[∘]:= `? OverTilde`

OverTilde[u] – Reversion of u ∈ $Cl_{p,q,r}$ ≫

In[∘]:= `? parityCheck`

parityCheck[u] – True if all terms of u are of even grade or all terms are of odd grade.

In[∘]:= `? repMatrixProduct`

repMatrixProduct[M,q] – Compute ordered product of matrices in list M containing q matrices.

In[∘]:= `? rightContract`

rightContract[u,v] – Clifford right contraction $u \llcorner v$.

In[∘]:= `? scalarSum`

scalarSum[u] returns sum of scalar coefficients in canonical expansion of u ∈ $Cl_{p,q,r}$ or u ∈ $Cl_{\star}{}^{*}$.

In[∘]:= `? setSignature`

setSignature[p,q,r] – Initializes Clifford multiplication in $Cl_{p,q,r}$, as well as `commutative Clifford' multiplication in $Cl_{p,q,r}{}^{sym}$. For n=p+q+r, initializes exterior (Grassmann) multiplication in $\Lambda R^n$, `idempotent–Clifford' multiplication in $Cl_n{}^{idem}$ and `zeon multiplication' in $Cl_n{}^{nil}$

In[∘]:= `? setSymmetricDifference`

setSymmetricDifference[A,B] – returns (A∪B))∩(A∩B)', the symmetric difference of sets A and B.

In[∘]:= `? signedSeminormSquare`

signedSeminormSquare[u] – returns $\langle u\tilde{u}\rangle_0$.

In[•]:= ? symBasis

symBasis – Returns ordered basis blades $\{\varsigma_I : I\in 2^{[n]}\} \subseteq Cl_n{}^{nil}$.

In[•]:= ? symInnerProduct

symInnerProduct[u,v] – returns $<u,v>=\sum_I u_I v_I$ inner product <u,v> of elements $u,v \in Cl_{p,q,r}{}^{sym}$.

In[•]:= ? versorFactor

versorFactor[u] – Factor versor $u \in Cl_n{}^{nil}$ (Euclidean Clifford algebra) via geometric considerations.

Returns vectors $\{v_i : 1\le i\le \text{clGrade}[u]\}$ and scalar $\alpha$ such that $u= \alpha \prod_{i=1}^{\text{clGrade}[u]} v_i$.

In[•]:= ? wedge

wedge[u,v] – returns exterior (wedge) product $u\wedge v$.

In[•]:= ? wedgeListProduct

wedgeListProduct[{$u_1$, $u_2$, ..., $u_k$}] computes wedge product $u_1 \wedge u_2 \wedge ... \wedge u_k$

In[•]:= ? wedgeMatrixPower

wedgeMatrixPower[A,m] – Computes $A^{\wedge m}$ for square matrix $A$ having entries from $Cl_{p,q,r}$.

In[•]:= ? wedgeMatrixProduct

wedgeMatrixProduct[A,B] Compute wedge(exterior) matrix product AB.

In[•]:= ? wedgePower

wedgePower[u,m] – Compute $u^{\wedge m}$ for nonegative integer m.

In[•]:= ? zDu

zDu[u] – Returns dual part, $\mathfrak{D}u$ (`soul'), of $u \in Cl_n{}^{nil}$

In[•]:= ? zRe

zRe[u] – Returns real part, $\mathfrak{R}u$ (`body'), of $u \in Cl_n{}^{nil}$

In[•]:= ? zeonAntiDerivative

zeonAntiDerivative[f[x],x,u] returns Φ[u], where

Φ[x] is the zeon extension of real function F[x], the antiderivative of f[x].

In[∘]:= ? **zeonAbs**

zeonAbs[u] – returns `zeon absolute value' of u: = $\sqrt{u^2}$ .

In[∘]:= ? **zeonArcCos**

zeonArcCos[u] Evaluates arccosine at $u \in Cl_n^{nil}$, provided $-1 \le zRe[u] \le 1$.

In[∘]:= ? **zeonArcSin**

zeonArcSin[u] Evaluates arcsine at $u \in Cl_n^{nil}$, provided $-1 \le zRe[u] \le 1$.

In[∘]:= ? **zeonArcTan**

zeonArcTan[u] Evaluates arctangent at $u \in Cl_n^{nil}$.

In[∘]:= ? **zeonBasis**

zeonBasis – Returns ordered basis blades $\{\zeta_I : I \in 2^{[n]}\} \subseteq Cl_n^{nil}$.

In[∘]:= ? **zeonCos**

zeonCos[u] evaluates the cosine function at $u \in Cl_n^{nil}$.

In[∘]:= ? **zeonCosh**

zeonCosh[u] – Hyperbolic cosine evaluated at $u \in Cl_n^{nil}$.

In[∘]:= ? **zeonDerivative**

zeonDerivative[f[x],x,u,k] returns $\phi^{(k)}(u)$: the kth order derivative of zeon extension $\phi$ of real function f, evaluated at $u \in Cl_n^{nil}$. If no order is specified, k=1 is assumed.

In[∘]:= ? **zeonDiscriminant**

zeonDiscriminant[f[x],x] computes the zeon discriminant of $\phi[x]=\alpha x^2+\beta x+\gamma$, where $\alpha,\beta,\gamma \in Cl_n^{nil}$.

In[∘]:= ? **zeonDivisionAlgorithm**

zeonDivisionAlgorithm[u,v] returns $\{q,r\} \subset Cl_n^{nil}$ such that u = qv+r, where (i.) qv is homogeneously decomposable and (ii.) r=0 or zeonMinGrade[r] > zeonMinGrade[u].

In[∘]:= ? **zeonDual**

zeonDual[u] – Returns Zeon–Hodge dual u* given by linear extension of $\zeta_I{}^*=\zeta_{[n]\setminus I}$ in $Cl_n^{nil}$.
Satisfies $\langle\alpha \odot \beta^*\rangle_n = \langle\alpha, \beta\rangle\, \zeta_{[n]}$.

In[∘]:= `? zeonElementaryFactorization`

zeonElementaryFactorization[u] returns factors a, $\{(1+a_I\zeta_I): I\in 2^{[n]}, I\neq\emptyset\}$ such that $u = a\prod(1+a_I\zeta_I)$

In[∘]:= `? zeonExp`

zeonExp[u] returns the exponential of $u \in Cl_n{}^{nil}$.

In[∘]:= `? zeonExtension`

zeonExtension[f[x],x] returns $\phi$[u], the series
representation of zeon extension $\phi$ of (suitably differentiable) real function f.

In[∘]:= `? zeonExtensionEval`

zeonExtensionEval[f[x],x,u] evaluates the zeon
extension $\phi$ (of suitably differentiable) real function f at $u \in Cl_n{}^{nil}$.

In[∘]:= `? zeonFrobeniusNorm`

zeonFrobeniusNorm[A] – returns $\sqrt{\Re\left(\mathrm{tr}[A^\dagger A]^*\right)} = \sqrt{\sum \| a_{ij} \|_2{}^2}$ .

In[∘]:= `? zeonGeneratorRep`

zeonGeneratorRep[j] – $2^n \times 2^n$ matrix representation of zeon generator $\zeta_{\{j\}} \to \sigma_0{}^{\otimes(j-1)} \otimes \eta \otimes \sigma_0{}^{\otimes(n-j)}$,
where $\eta=\sigma_z+i\sigma_y$, and $\sigma_0$, $\sigma_y$, and $\sigma_z$ are Pauli matrices.

In[∘]:= `? zeonGeometricSeries`

zeonGeometricSeries[a,u,k] – returns partial sum $\sum_{j=0}^{k} a u^j$ of geometric series $\sum_{j=0}^{\infty} a u^j$.

If k is omitted and $|\Re u|<1$, returns limit of geometric series: $\frac{a}{1-\Re u} \sum_{j=0}^{\mathrm{maxIndex}} (1-\Re u)^{-j} (\mathfrak{D} u)^j$.

In[∘]:= `? zeonInnerProduct`

zeonInnerProduct[u,v] – returns $<u,v>=\sum_I u_I v_I$ inner product $<u,v>$ of zeon elements $u,v \in Cl_n{}^{nil}$.

In[∘]:=

`? zeonInverse`

zeonInverse[u] – Multiplicative inverse of $u \in Cl_n{}^{nil}$, provided Re[u]≠0.

In[∘]:= `? zeonKthRoot`

zeonKthRoot[u] Recursive computation of principal kth root of $u \in Cl_n{}^{nil}$, provided Re[u] ≥ 0.

In[∘]:= ? zeonLeftContraction

zeonLeftContraction[u,v] – Zeon contraction
u ⌉ v defined by linear extension of $\zeta_I \rceil \zeta_J = \zeta_{J\setminus I}$ if $I \subseteq J$, 0 otherwise.

In[∘]:= ? zeonLog

zeonLog[u] Evaluate logarithm of $u \in Cl_n{}^{nil}$, provided zRe[u]>0.

In[∘]:= ? zeonMatrixInverse

zeonMatrixInverse[A] – Given square zeon
matrix A, returns zeon matrix U such that AU=UA=I, provided |ℜA|≠0.

In[∘]:= ? zeonMaxGenDecomp

zeonMaxGenDecomp[u] – Given zeon u with
maximum generator index m, return $\{\omega,\psi\}$ such that $u = \omega + \psi\zeta_{\{m\}}$.

In[∘]:= ? zeonMaxGenerator

zeonMaxGenerator[u] – Maximum generator index appearing in $u \in Cl_n{}^{nil}$.

In[∘]:= ? zeonMinGrade

zeonMinGrade[u] Returns 0 if u is trivial (i.e, real), otherwise returns minimal grade of the dual part of u.

In[∘]:= ? zeonNilpotency

zeonNilpotency[u] returns least positive integer $\kappa$ such that
$(\mathfrak{D}u)^{\kappa}=0$.

In[∘]:= ? zeonNorm

zeonNorm[u,m] – returns the m–norm $\|u\|_m$ of $u \in Cl_n{}^{nil}$.

In[∘]:= ? zeonPower

zeonPower[u,r] returns $u^r$ (rth power of zeon u) for $r\in\mathbb{R}$ and $u \in Cl_n{}^{nil}$. zRe[u]≠0 required if r<0.

In[∘]:= ? zeonPowerZeon

zeonPowerZeon[u,v] returns zeon to zeon power $u^v$, where $u,v \in Cl_n{}^{nil}$ and Re u > 0.

In[∘]:= ? zeonQuadraticEval

zeonQuadraticEval[$\phi$[x],x,u] evaluates the quadratic function $\phi[x]=\alpha x^2+\beta x+\gamma$ at u, where $\alpha,\beta,\gamma,u \in Cl_n{}^{nil}$.

In[∘]:= ? zeonQuadraticFormula

zeonQuadraticFormula[$\phi$[x],x] applies the zeon quadratic formula to $\phi[x]=\alpha x^2+\beta x+\gamma$, where $\alpha,\beta,\gamma \in Cl_n^{nil}$. When $\Delta=0$, includes arbitrary $\kappa$ assumed to be nilpotent of index 2.

In[∘]:= ? zeonRep

zeonRep[u] – Returns $2^n \times 2^n$ real matrix representation of $u \in Cl_n^{nil}$.

In[∘]:= ? zeonRightContraction

zeonRightContraction[u,v] – Zeon contraction
$u \llcorner v$ defined by linear extension of $\zeta_I \llcorner \zeta_J = \zeta_{I \setminus J}$ if $J \subseteq I$, 0 otherwise.

In[∘]:= ? zeonSin

zeonSin[u] evaluates the sine function at $u \in Cl_n^{nil}$.

In[∘]:= ? zeonSinh

zeonSinh[u] – Hyperbolic sine evaluated at $u \in Cl_n^{nil}$.

In[∘]:= ? zeonSpectralSeminorm

zeonSpectralSeminorm[u] – returns $|\mathfrak{R}u|$ for $u \in Cl_n^{nil}$.

In[∘]:= ? zeonSqRoot

zeonSqRoot[u] Recursive computation of principal square root of $u \in Cl_n^{nil}$, provided $Re[u] \geq 0$.

In[∘]:= ? zeonTan

zeonTan[u] evaluates the tangent function at $u \in Cl_n^{nil}$, provided cos(u) is invertible; i.e., $Re[\cos(u)] \neq 0$.

In[∘]:= ? zeonTanh

zeonTanh[u] – Hyperbolic tangent evaluated at $u \in Cl_n^{nil}$.

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