

# SPECTRAL THEORY OF DISCRETE PROCESSES

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ABSTRACT. We offer a spectral analysis for a class of transfer operators. These transfer operators arise for a wide range of stochastic processes, ranging from random walks on infinite graphs to the processes that govern signals and recursive wavelet algorithms; even spectral theory for fractal measures. In each case, there is an associated class of harmonic functions which we study. And in addition, we study three questions in depth:

In specific applications, and for a specific stochastic process, how do we realize the transfer operator  $T$  as an operator in a suitable Hilbert space? And how to spectral analyze  $T$  once the right Hilbert space  $\mathcal{H}$  has been selected? Finally we characterize the stochastic processes that are governed by a single transfer operator.

In our applications, the particular stochastic process will live on an infinite path-space which is realized in turn on a state space  $S$ . In the case of random walk on graphs  $G$ ,  $S$  will be the set of vertices of  $G$ . The Hilbert space  $\mathcal{H}$  on which the transfer operator  $T$  acts will then be an  $L^2$  space on  $S$ , or a Hilbert space defined from an energy-quadratic form.

This circle of problems is both interesting and non-trivial as it turns out that  $T$  may often be an unbounded linear operator in  $\mathcal{H}$ ; but even if it is bounded, it is a non-normal operator, so its spectral theory is not amenable to an analysis with the use of von Neumann's spectral theorem. While we offer a number of applications, we believe that our spectral analysis will have intrinsic interest for the theory of operators in Hilbert space.

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*Date:* Mar, 2009.

*1991 Mathematics Subject Classification.* Primary 42C40, 47S50, 62B5, 68U10, 94A08.

*Key words and phrases.* Hilbert space, spectrum, encoding, transfer operator, infinite graphs, wavelets, fractals, stochastic process, path measures, transition probability.

Work supported in part by the U.S. National Science Foundation.

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## 1. INTRODUCTION

In this paper, we consider infinite configurations of vectors  $(f_k)_{k \in \mathbb{Z}}$  in a Hilbert space  $\mathcal{H}$ . Since our Hilbert spaces  $\mathcal{H}$  are typically infinite-dimensional, this can be quite complicated, and it will be difficult to make sense of finite and infinite linear combinations  $\sum_{k \in \mathbb{Z}} c_k f_k$ .

In case the system  $(f_k)$  is orthogonal, the problem is easy, but non-orthogonality serves as an encoding of statistical correlations, which in turn motivates our study. In applications, a particular system of vectors  $f_k$  may often be analyzed with the use of a single unitary operator  $U$  in  $\mathcal{H}$ . This happens if there is a fixed vector  $\varphi \in \mathcal{H}$  such that  $f_k = U^k \varphi$  for all  $k \in \mathbb{Z}$ . When this is possible, the spectral theorem will then apply to this unitary operator. A key idea in our paper is to identify a spectral density function and a transfer operator, both computed directly from the pair  $(\varphi, U)$ .

We show that the study of linear expressions  $\sum_k c_k f_k$  may be done with the aid of the spectral function for a pair  $(\varphi, U)$ . A spectral function for a unitary operator  $U$  is really a system of functions  $(p_\varphi)$ , one for each cyclic subspace  $\mathcal{H}(\varphi)$ . In each cyclic subspace, the function  $p_\varphi$  is a complete unitary invariant for  $U$  restricted to  $\mathcal{H}(\varphi)$ : by this we mean that the function  $p_\varphi$  encodes all the spectral data coming from the vectors  $f_k = U^k \varphi$ ,  $k \in \mathbb{Z}$ . For background literature on spectral function and their applications we refer to [1, 10, 16, 19, 20, 21].

In summary, the spectral representation theorem is the assertion that commuting unitary operators in Hilbert space may be represented as multiplication operators in an  $L^2$ -Hilbert space. The understanding is that this representation is defined as a unitary equivalence, and that the  $L^2$ -Hilbert space to be used allows arbitrary measures, and  $L^2$  will be a Hilbert space of vector valued functions, see e.g., [6]. Because of applications, our systems of vectors will be indexed by an arbitrary discrete set rather than merely integers  $\mathbb{Z}$ .

We will attack this problem via an isometric embedding of  $\mathcal{H}$  into an  $L^2$ -space built on infinite paths in such a way that the vectors  $f_k$  in  $\mathcal{H}$  transform into a system of random variables  $Z_k$ . Specifically, via certain encodings we build a path-space  $\Omega$  for the particular problem at hand as well as a path space measure  $\mathbb{P}$  defined on a  $\sigma$ -algebra of subsets of  $\Omega$ .

If  $\mathcal{H}$  consists of a space of functions  $f$  on a state space  $S$ , we will need the covariance numbers

$$\mathbb{E}((f_1 \circ Z_n) \cdot (f_2 \circ Z_m)) := \int_{\Omega} f_1(Z_n(\gamma)) f_2(Z_m(\gamma)) d\mathbb{P}(\gamma),$$

where  $Z_n : \Omega \rightarrow S$ , i.e., where the stochastic process is  $S$ -valued. The set  $S$  is called the state space.

The paper is organized as follows. In section 2, for later use, we present our path-space approach, and we discuss the path-space measures that we will use in computing transitions for stochastic processes. We prove two theorems making the connection between our path-space measures on the one hand, and the operator theory on the other. Several preliminary results are established proving how the transfer operator governs the process and its applications.

The applications we give in sections 3 and 4 are related. In fact, we unify these applications with the use of an encoding map which is also studied in detail. It is applied to transitions on certain infinite graphs, to dynamics of (non-invertible) endomorphisms (measures on solenoids), to digital filters and their use in wavelets and signals, and to harmonic analysis on fractals.

The remaining sections deal primarily with applications to a sample of concrete cases.

## 2. STOCHASTIC PROCESSES

A key tool in our analysis is the construction of path-space measures on infinite paths, primarily in the case of discrete paths, but the fundamental ideas are the same in the continuous case. Both viewpoints are used in [12]. Readers who wish to review the ideas behind these constructions (stochastic processes and consistent families of measures) are referred to [8, 9, 7] and [18].

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a Borel probability space,  $\Omega$  compact Hausdorff space. (Expectation  $\mathbb{E}(\cdot) = \int_{\Omega} \cdot d\mathbb{P}$ .)

Let  $(Z_k)_{k \geq 0}$  be a stochastic process, and

$$(2.1) \quad \mathcal{F}_n = \sigma\text{-alg.}\{Z_k | k \leq n\}$$

the corresponding filtration. Let  $\mathcal{A}_n :=$  the subspace in  $L^2(\Omega, \mathbb{P})$  generated by  $\mathcal{F}_n$ . Let  $P_n$  be the orthogonal projection of  $L^2(\Omega, \mathbb{P})$  onto  $\mathcal{A}_n$ ; then the conditional expectations  $\mathbb{E}(\cdot | \mathcal{F}_n)$  is simply  $P_n$ .

We say that  $(Z_k)_{k \geq 0}$  has the generalized Markov property if and only if there exists a state space  $S$  (also a compact Borel space):

$$Z_k : \Omega \rightarrow S$$

such that for all bounded functions  $f$  on  $S$ , for all  $n \in \mathbb{N}_{\geq 0}$ ,  $\mathbb{E}(f | \mathcal{F}_n) = \mathbb{E}(f | Z_n)$ .

To make precise the operator theoretic tools going into our construction, we must first introduce the ambient Hilbert spaces. We are restricting here to  $L^2$  processes, so the corresponding stochastic integrals will take values in an ambient  $L^2$ -space of random variables: For our analysis, we must therefore specify a fixed probability space, with  $\sigma$ -algebra and probability measure.

We will have occasion to vary this initial probability space, depending on the particular transition operator that governs the process.

In the most familiar case of Brownian motion, or random walk, the probability space amounts to a somewhat standard construction of Wiener and Kolmogorov, but here with some modification for our problem at hand: The essential axiom in Wiener's case is that all finite samples are jointly Gaussian, but we will drop this restriction and consider general stochastic processes, and so we will not make

restricting assumptions on the sample distributions and on the underlying probability space. For more details, and concrete applications, regarding this stochastic approach and its applications, see sections 2 and 4 below.

We begin here with a particular case of a process taking values in the set of vertices in a fixed infinite graph  $G$ : [13]

### 2.1. Starting Assumptions and Constructions.

- (a)  $G = (G^0, G^1)$  a graph,  $G^0$  = the set of vertices,  $G^1$  = the set of edges.
- (b)  $(S, \mathcal{B}_S, \mu)$  a probability space.
- (c) The *transition matrix* is the function

$$p(x, y) := \mathbb{P}(\{\gamma \in \Omega | Z_n(\gamma) = x, Z_{n+1}(\gamma) = y\})$$

defined for all  $(x, y) \in G^1$ , and we assume that it is independent of  $n$ .

- (d) From (a) and (b), we construct the path space

$$\Omega := \{\gamma = (x_0 x_1 x_2 \cdots) | (x_{i-1} x_i) \in G^1, \forall i \in \mathbb{N}\},$$

and the path-measure  $\mathbb{P} = \mathbb{P}_\mu$ . The cylinder sets given by the following data:  
For  $E_i \in \mathcal{B}_S$ ,  $x_i \in S$ , set

$$\mathbb{P}(C(E_1, \dots, E_n))$$

$$:= \int_{E_0} \int_{E_1} \cdots \int_{E_n} p(x_0, x_1) p(x_1, x_2) \cdots p(x_{n-1}, x_n) d\mu(x_0) d\mu(x_1) \cdots d\mu(x_n)$$

- (e) Starting with  $(\Omega, \mathcal{F}, \mathbb{P})$ , if  $\mathcal{G} \subset \mathcal{F}$  is a subsigma algebra, let  $\mathbb{E}(\cdot | \mathcal{G})$  be the conditional expectation, conditioned by  $\mathcal{G}$ .  
If  $(X_i)$  is a family of random variables, and  $\mathcal{G}$  is the  $\sigma$ -algebra generated by  $(X_i)$  we write  $\mathbb{E}(\cdot | (X_i))$  in place of  $\mathbb{E}(\cdot | \mathcal{G})$ .
- (f) Let  $(\Omega, \mathcal{F}, \mathbb{P}, (Z_n))$  be as above. We say that  $(Z_n)$  is *Markov* if and only if

$$\mathbb{E}(f \circ Z_{n+1} | \{Z_0, \dots, Z_n\}) = \mathbb{E}(f \circ Z_{n+1} | Z_n) \quad \text{for all } n \in \mathbb{N}_0.$$

- (g) From (b) and (d) we define the *transfer operator*  $T$  by

$$(2.2) \quad (Tf)(x) = \int_S p(x, y) f(y) d\mu(y)$$

for measurable functions  $f$  on  $S$ . If  $\mathbb{1}$  denote the constant function 1 on  $S$ , then  $T\mathbb{1} = \mathbb{1}$ .

- (h) Let  $(S, \mathcal{B}_S, \mu)$  and  $T$  be as in (g), see(2.12). A measure  $\mu_0$  on  $S$  is said to be a *Perron-Frobenius measure* if and only if

$$(2.3) \quad \int_S (Tf)(x) d\mu_0(x) = \int_S f(x) d\mu_0(x), \quad \text{abbreviated } \mu_0 \circ T = \mu_0.$$

- (i) Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be as above, and let  $T$  be the transfer operator. If  $\mu_0$  is a Perron-Frobenius measure, let  $\mathbb{P}^{(\mu_0)}$  be the measure on  $\Omega$  determined by using  $\mu_0$  as the first factor, i.e.,

$$\mathbb{P}^{(\mu_0)}(C(E_1, \dots, E_n))$$

$$= \int_{E_0} \int_{E_1} \cdots \int_{E_n} p(x_0, x_1) p(x_1, x_2) \cdots p(x_{n-1}, x_n) d\mu_0(x_0) d\mu(x_1) \cdots d\mu(x_n)$$

$$= \int_{E_0} \mathbb{P}_{x_0}(C(E_1, \dots, E_n)) d\mu(x_0).$$

In many cases, it is possible to choose specific Perron-Frobenius measures  $\mu_0$ , i.e., measures  $\mu_0$  satisfying

$$\mu_0(S) = 1 \text{ and } \int_S (Tf)(x) d\mu_0(x) = \int_S f(x) d\mu_0(x).$$

(Note the normalization!)

**Theorem 2.1.** (*D. Ruelle*) [2] *Suppose there is a norm  $\|\cdot\|$  on bounded measurable functions  $f$  on  $S$  such that the  $\|\cdot\|$ -completion  $L(S)$  is embedded in  $L^\infty(S)$ , and that there are constants  $\alpha \in (0, 1)$ ,  $M \in \mathbb{R}_+$  such that*

$$\|Tf\| \leq \alpha\|f\| + M\|f\|_\infty,$$

where  $\|\cdot\|_\infty$  is the essential supremum-norm. Then  $T$  has a Perron-Frobenius measure.

**Theorem 2.2.** *Let  $(S, \mu)$  be a probability space with  $S$  carrying a separate  $\sigma$ -algebra  $\mathcal{B}_S$  and  $\mu$  defined on  $\mathcal{B}_S$ . Let  $\Omega$  be the path space, and suppose the transfer operator  $T$  has a Perron-Frobenius measure  $\mu_0$ , then*

$$(2.4) \quad \mathbb{E}^{(\mu_0)}((\overline{\varphi} \circ Z_n)(\psi \circ Z_{n+1})) = \langle \varphi, T\psi \rangle_{L^2(\mu_0)}$$

for all  $\varphi, \psi \in L^2(\mu)$ , and all  $n \in \mathbb{N}_0$ . Here  $\mathbb{E}(F) := \int_\Omega F(\omega) d\mathbb{P}(\omega)$  for all integrable random variables  $F : \Omega \rightarrow \mathbb{C}$ ;  $\mathbb{E}$  for expectation.

*Proof.*

$$\begin{aligned} & \mathbb{E}^{(\mu_0)}((\overline{\varphi} \circ Z_n)(\psi \circ Z_{n+1})) \\ &= \int_S \int_S \cdots \int_S p(x_0, x_1) \cdots p(x_{n-1}, x_n) p(x_n, x_{n+1}) \overline{\varphi}(x_n) \psi(x_{n+1}) d\mu_0(x_0) d\mu(x_1) \cdots d\mu(x_{n+1}) \\ &= \int_S \int_S \cdots \int_S p(x_0, x_1) \cdots p(x_{n-1}, x_n) \overline{\varphi}(x_n) (T\psi)(x_n) d\mu_0(x_0) d\mu(x_1) \cdots d\mu(x_n) \\ &= \int_S T^n(\overline{\varphi} \cdot (T\psi))(x_0) d\mu_0(x_0) \\ &= \int_S \overline{\varphi}(x) (T\psi)(x) d\mu_0(x) \quad \text{by Perron-Frobenius} \\ &= \langle \varphi, T\psi \rangle_{L^2(\mu_0)}. \end{aligned}$$

□

It is not necessary in (2.4) to restrict attention to functions  $\varphi, \psi$  in  $L^2(\mu_0)$ . The important thing is that the integral  $\int_S \overline{\varphi}(x) (T\psi)(x) d\mu_0(x)$  exists, and this quantity may then be used instead on the RHS in (2.4).

Let  $(Z_n)_{n \in \mathbb{N}_0}$  be a stochastic process, and let  $\mathcal{F}_n$  be the  $\sigma$ -algebra generated by  $\{Z_k \mid 0 \leq k \leq n\}$ . Furthermore, let  $\mathbb{E}(\cdot | \mathcal{F}_n)$  be the conditioned expectation conditioned by  $\mathcal{F}_n$ .

**Theorem 2.3.** *Let  $(Z_n)_{n \in \mathbb{N}_0}$  be a stochastic process with stationary transitions and operator  $T$ . Then*

$$(2.5) \quad \mathbb{E}(f \circ Z_{n+1} | \mathcal{F}_n) = (Tf) \circ Z_n$$

for all bounded measurable functions  $f$  on  $S$ , and all  $n \in \mathbb{N}_0$

*Proof.* We may assume that  $f$  is a real valued function on  $S$ . Let  $\mathcal{A}_n :=$  all bounded  $\mathcal{F}_n$ -measurable functions. Then the assertion in (2.5) may be restated as:

$$(2.6) \quad \int_{\Omega} \varphi(f \circ Z_{n+1}) d\mathbb{P} = \int_{\Omega} \varphi((Tf) \circ Z_n) d\mathbb{P}$$

for all  $\varphi \in \mathcal{A}_n$ .

If  $\varphi \in \mathcal{A}_n$ ,  $\varphi(\cdot) = \Phi(x_0, x_1, \dots, x_n)$ ; and then the LHS in (2.6) may be written as

$$\begin{aligned} & \int_S \int_S \cdots \int_S p(x_0, x_1) \cdots p(x_n, x_{n+1}) \Phi(x_0, x_1 \cdots x_n) f(x_{n+1}) d\mu_0(x_0) d\mu(x_1) \cdots d\mu(x_{n+1}) \\ &= \int_S \int_S \cdots \int_S p(x_0, x_1) \cdots p(x_{n-1}, x_n) \Phi(x_0, x_1 \cdots x_n) (Tf)(x_n) d\mu_0(x_0) d\mu(x_1) \cdots d\mu(x_n) \\ &= \int_{\Omega} \varphi \cdot (Tf) \circ Z_n d\mathbb{P}. \end{aligned}$$

Hence (2.5) follows.  $\square$

**Corollary 2.4.** *Let  $(\Omega, \mathcal{F}, \mathbb{P}, (Z_n))$  be as in the theorem. Then the process  $(Z_n)$  is Markov.*

*Proof.* We must show that

$$\mathbb{E}(f \circ Z_{n+1} | \mathcal{F}_n) = \mathbb{E}(f \circ Z_{n+1} | Z_n)$$

By the theorem, we only need to show that

$$\mathbb{E}(f \circ Z_{n+1} | Z_n) = (Tf) \circ Z_n.$$

In checking this we use the transition operator  $T$ . As a result we may now assume that  $\varphi$  has the form  $\varphi = g \circ Z_n$  for  $g$  a measurable function on  $S$ . Hence

$$\begin{aligned} \int_{\Omega} \varphi(f \circ Z_{n+1}) d\mathbb{P} &= \int_{\Omega} (g \circ Z_n)(f \circ Z_{n+1}) d\mathbb{P} = \langle g, Tf \rangle_{L^2(\mu)} \\ &= \int_S g(Tf) d\mu = \int_{\Omega} (g \circ Z_n)((Tf) \circ Z_n) d\mathbb{P} \\ &= \int_{\Omega} \varphi((Tf) \circ Z_n) d\mathbb{P} \end{aligned}$$

which is the desired conclusion.  $\square$

**Definition 2.5.** We say that a measurable function  $f$  on  $S$  is harmonic if  $Tf = f$ .

**Definition 2.6.** A sequence of random variables  $(F_n)$  is said to be a martingale if and only if  $\mathbb{E}(F_{n+1} | \mathcal{F}_n) = F_n$  for all  $n \in \mathbb{N}_0$ .

**Corollary 2.7.** *Let  $(Z_n)_{n \in \mathbb{N}_0}$  be a stochastic process with stationary transitions and operator  $T$ . Let  $f$  be a measurable function on  $S$ .*

*Then  $f$  is harmonic if and only if  $(f \circ Z_n)_{n \in \mathbb{N}_0}$  is a martingale.*

*Proof.* This follows from (2.5) combined with Definition 2.6.  $\square$

**Corollary 2.8.** *Suppose a process  $(Z_n)_{n \in \mathbb{N}_0}$  is stationary with a fixed transition operator  $T : L^2(\mu) \rightarrow L^2(\mu)$ . Then  $\mu = \mathbb{P} \circ Z_n^{-1}$  for all  $n \in \mathbb{N}_0$ .*

*Proof.* Let  $f$  and  $g$  be a pair of functions on  $S$  as specified above. Then we showed that

$$\int_S g f d\mu = \int_{\Omega} (g \circ Z_n)(f \circ Z_n) d\mathbb{P}$$

which is the desired conclusion.  $\square$

**2.2. Martingales and Boundaries.** Let  $G = (G^0, G^1)$  be an infinite graph with a fixed conductance  $c$ , and let the corresponding operators be  $\Delta_c$  and  $T_c$ .

Let  $h : G^0 \rightarrow \mathbb{R}$  is a harmonic function, i.e.,  $\Delta_c h = 0$ , or equivalently  $T_c h = h$ .

As an application of Corollary 2.7, we may then apply a theorem of J. Doob to the associated martingale  $h \circ Z_n$ ,  $n \in \mathbb{N}_0$ . This means that the sequence  $(h \circ Z_n)$  will then have  $\mathbb{P}$ - a. e. limit i.e.,

$$(2.7) \quad \lim_{n \rightarrow \infty} h \circ Z_n = v \quad \text{pointwise } \mathbb{P} \text{ a.e.}$$

The limit function  $v : \Omega \rightarrow \mathbb{R}$  will satisfy  $v(x_0 x_1 x_2 \cdots) = v(x_1 x_2 x_3 \cdots)$ , or equivalently,

$$(2.8) \quad v = v \circ \sigma.$$

The existence of the limit in (2.7) holds if one or the other of the two conditions is satisfied:

- (i)  $h \in L^\infty$ ; or
- (ii)  $\sup_n \int_\Omega |h \circ Z_n|^2 d\mathbb{P} < \infty$ .

**Proposition 2.9.** [11] *If  $h : G^0 \rightarrow \mathbb{R}$  is harmonic and if (i) or (ii) hold, then*

$$(2.9) \quad h(x) = \int_\Omega v d\mathbb{P}_x \quad \text{for all } x \in G^0,$$

where  $\mathbb{P}_x$  = the measure  $\mathbb{P}$  conditioned with  $Z_0(\gamma) = x$ . The converse implication holds as well.

*Proof.* Starting with  $h$  harmonic, if the Doob-limit  $v$  in (2.7) exists, then it is clear that  $v$  satisfies (2.8). By Dominated Convergence, (2.9) will be satisfied.

Conversely, suppose some measurable  $v : \Omega \rightarrow \mathbb{R}$  satisfies (2.8), and the integral in (2.9) exists then

$$\begin{aligned} (T_c h)(x) &= \sum_{y \sim x} p(x, y) h(y) \\ &= \sum_{y \sim x} \mathbb{P}(Z_0 = x, Z_1 = y) \mathbb{E}(v | Z_0(\cdot) = y) \\ &= \sum_{y \sim x} p(x, y) \mathbb{E}_x(v | Z_1(\cdot) = y) \\ &\stackrel{\text{by (2.8)}}{=} \sum_{y \sim x} p(x, y) \mathbb{E}(v | Z_0 = x, Z_1 = y) \\ &= \sum_{y \sim x} p(x, y) \mathbb{E}(v | Z_0 = x, Z_1 = y) \\ &= \mathbb{P}_x(v(\cdots)) \\ &= h(x), \end{aligned}$$

showing that  $h$  is harmonic. □

### 2.3. Solenoids.

**Example 2.10.** Let  $S$  be a compact Hausdorff space, and  $\sigma : S \rightarrow S$  a finite-to-one endomorphism onto  $S$ . Let  $X_\sigma(S)$  be the corresponding solenoid:

$$(2.10) \quad X_\sigma(S) \subset \prod_{n \in \mathbb{N}_0} S, \quad \text{where } \mathbb{N}_0 = \{0\} \cup \mathbb{N} = \{0, 1, 2, 3, \dots\},$$

$$X_\sigma(S) = \{(x_k)_{k \in \mathbb{N}_0} | \sigma(x_{k+1}) = x_k\}.$$

One advantage of a choice of solenoid over the initial endomorphism  $\sigma : S \rightarrow S$  is that  $\sigma$  induces an *automorphism*  $\widehat{\sigma} : X_\sigma(S) \rightarrow X_\sigma(S)$  as follows:

$$\widehat{\sigma}((x_0x_1x_2 \cdots)) = (\sigma(x_0)x_0x_1x_2 \cdots), \quad \text{with inverse} \quad \widehat{\sigma}^{-1}((x_0x_1x_2 \cdots)) = (x_1x_2x_3 \cdots).$$

Let  $W : S \rightarrow [0, 1]$  be a Borel measurable function, and set

$$(2.11) \quad (T_W f)(x) = \sum_{\substack{y \\ \sigma(y)=x}} W(y)f(y), \quad f \in B(S), x \in S.$$

Assume

$$(2.12) \quad \sum_{\sigma(y)=x} W(y) \equiv 1, \forall x \in S.$$

For points  $x \in S$ , set  $D(x) := \#\{y | \sigma(y) = x\}$ . A measure  $\mu$  on  $S$  is said to be *strongly invariant* if

$$\int_S \frac{1}{D(x)} \sum_{\substack{y \\ \sigma(y)=x}} f(y) d\mu(x) = \int_S f(x) d\mu(x).$$

**Lemma 2.11.** *Assume a measure  $\mu$  on  $S$  is strongly invariant, and let  $m$  be a function on  $S$ . Set  $Vf(x) = m(x)f(\sigma(x))$ . Then the adjoint operator*

$$V^* : L^2(\mu) \rightarrow L^2(\mu) \quad \text{is} \quad (V^*f)(x) = \frac{1}{D(x)} \sum_{\substack{y \\ \sigma(y)=x}} \overline{m(y)}f(y)$$

*Proof.* See [11]. □

Set  $\Omega := X_\sigma(S)$  and equip it with the  $\sigma$ -algebra  $\mathcal{F}$  and the topology which is generated by the cylinder sets.

Set  $Z_k : \Omega \rightarrow S$ ,

$$(2.13) \quad Z_k(x_0x_1x_2 \cdots) := x_k, \quad k \in \mathbb{N}_0.$$

Let  $E \subset S$  be a Borel set, and consider

$$(2.14) \quad Z_k^{-1}(E) = \{\omega \in \Omega | Z_k(\omega) \in E\}.$$

Then the  $\sigma$ -algebra  $\mathcal{F}$  on  $\Omega$  is generated by the sets

$$(2.15) \quad Z_k^{-1}(E) \quad \text{as } k \text{ and } E \text{ vary.}$$

Set

$$(2.16) \quad \mathcal{F}_n := \sigma\text{-algebra} \ll Z_k | k \leq n \gg$$

where  $\ll \cdot \gg$  refers to the  $\sigma$ -algebra as specified in (2.14).

In  $\Omega = X_\sigma(S)$ , consider the following random walk: For points  $x, y \in S$ , a transition  $x \rightarrow y$  is possible if and only if  $\sigma(y) = x$ ; and in this case the transition probability is  $p_W(x, y) := W(y)$ .

Let  $\mu$  be a probability measure on  $S$ . In  $\Omega$  we introduce the following Kolmogorov measure  $\mathbb{P} := \mathbb{P}_W$  which is determined on cylinder sets as follows

$$(2.17) \quad \mathbb{P}(C_n) := \mathbb{P}(C(E_0, E_1, E_2 \cdots, E_n))$$

$$(2.18) \quad = \int_{E_0} \int_{E_1} \cdots \int_{E_n} W(x_1)W(x_2) \cdots W(x_n) d\mu(x_0) d\mu(x_1) \cdots d\mu(x_n)$$



More specifically,  $\mathbb{P}$  is a measure on infinite paths, and

$$(2.19) \quad C_n = \{\omega = (\omega_0\omega_1\omega_2\cdots) \mid \sigma(\omega_{i+1}) = \omega_i, Z_k(\omega) \in E_k, \text{ for } 0 \leq k \leq n\}.$$

**Example 2.12.** The following is a solenoid which is used in both number theory (the study of algebraic irrational numbers) and in ergodic systems. [4]. For this family of examples, the solenoids are associated with specific polynomials  $p \in \mathbb{Z}[x]$ .

Let  $S := \mathbb{T}^s$  where  $s \in \mathbb{N}$  is fixed; and let  $p(x) = a_0x^s + a_1x^{s-1} + \cdots + a_s$ ;  $a_0 \neq 0$ , be a polynomial,  $p \in \mathbb{Z}[x]$ . Set

$$F = F_p := \begin{pmatrix} 0 & a_0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_0 \\ -a_s & -a_{s-1} & \cdots & \cdots & -a_2 & -a_1 \end{pmatrix}.$$

Consider the shift  $\sigma$  on the infinite torus  $\prod_{\mathbb{Z}} \mathbb{T}^s = (\mathbb{T}^s)^{\mathbb{Z}}$ , and set

$$X_\sigma := \{(z_n)_{n \in \mathbb{Z}} \in (\mathbb{T}^s)^{\mathbb{Z}} \mid a_0 z_{n+1} = F z_n\}.$$

Then it follows that  $X_\sigma(p)$  is  $\sigma$ -invariant and closed. As a result,  $X_\sigma(p)$  is a compact solenoid.

### 3. GRAPHS

One additional application of these ideas is to infinite graph systems  $(G, c)$  where  $G$  is a graph and  $c$  is a positive conductance function. A comprehensive study of this class of examples was carried out in the paper [12]. We will adapt the convention from that paper:

- $G^0$  : the set of vertices in  $G$ ;
- $G^1$  : the set of edges in  $G$ ;
- and  $c : G^1 \rightarrow \mathbb{R}_+$  the conductance function.

*Assumptions.*

- (i) *Edge symmetry.* If  $x, y \in G^0$  and  $(x, y) \in G^1$ , then we assume that  $c_{x,y} = c_{y,x}$ . Moreover,  $(x, y) \in G^1 \Leftrightarrow (y, x) \in G^1$ .
- (ii) *Finite neighborhoods.* For all  $x \in G^0$ , the set  $Nbh(x) = \{y \in G^0 \mid (x, y) \in G^1\}$  is finite.
- (iii) *No self-loops.* If  $x \in G^0$ , then  $x \notin Nbh(x)$ .  
Convention: If  $x, y \in G^0$ , we write  $x \sim y$  iff  $(x, y) \in G^1$ .
- (iv) *Connectedness.* For all  $x, y \in G^0$  there exists  $\{x_i\}_{i=0}^n \subset G_0$  such that  $(x_i, x_{i+1}) \in G^1$ ,  $i = 0, 1, \dots, n-1$ ,  $x_0 = x$  and  $x_n = y$ .
- (v) *Choice of origin.* We select an origin  $o \in G^0$ .

**Definition 3.1.**      • The Laplace operator  $\Delta = \Delta_c$ :

$$(\Delta f)(x) := \sum_{y \sim x} c_{x,y} (f(x) - f(y)).$$

- Hilbert spaces:

- (i)  $l^2(G^0)$ : functions  $f : G^0 \rightarrow \mathbb{C}$  such that  $\|f\|_2^2 = \sum_{x \in G^0} |f(x)|^2 < \infty$ .  
Set  $\langle f_1, f_2 \rangle_2 := \sum_{x \in G^0} \overline{f_1(x)} f_2(x)$ . For every  $x \in G^0$ , set  $\delta_x : G^0 \rightarrow \mathbb{R}$ ,

$$\delta_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

Note that  $\{\delta_x\}$  is an orthonormal basis (ONB) in  $l^2(G^0)$ .

- (ii)  $\mathcal{H}_E$ : finite energy functions module constants:

$$(3.1) \quad \|f\|_E^2 = \frac{1}{2} \sum_{\text{all } x \sim y} c_{x,y} |f(x) - f(y)|^2.$$

Set

$$(3.2) \quad \langle f_1, f_2 \rangle_E := \frac{1}{2} \sum_{x \sim y} c_{x,y} (\overline{f_1(x)} - \overline{f_1(y)}) (f_2(x) - f_2(y)).$$

- Dipoles. For all  $x \in G^0$  there is a unique  $v_x \in \mathcal{H}_E$  such that

$$\langle v_x, f \rangle_E = f(x) - f(o), \quad \forall f \in \mathcal{H}_E.$$

In this case,  $v_x$  satisfies  $\Delta v_x = \delta_x - \delta_o$ , and we make the choice  $v_x(o) = 0$ .  
The function  $v_x : G^0 \rightarrow \mathbb{R}$  is called a *dipole*.

**Example 3.2.** *The dyadic tree.*

- $\mathcal{A}$  = the alphabet of two letters, bits  $\{0, 1\} \simeq \mathbb{Z}_2$ .
- $G^0$ : the set of all finite words in  $\mathcal{A}$  :  $o = \emptyset =$  the empty word,  $x = (a_1 a_2 \cdots a_n) \in G^0$ ,  $a_i \in \mathcal{A}$ , a word of length  $n$ ;  $l(x) = n$ .
- $G^1$  := the edges in the dyadic tree. If  $x = \emptyset$ ,  $Nbh(x) = \{0, 1\}$  two one-letter words. If  $l(x) = n > 0$ ,  $x = (a_1 a_2 \cdots a_n)$ ,  $Nbh(x) = \{(a_1 \cdots a_{n-1}), (x0), (x1)\}$ .  
Set  $x^* := (a_1 \cdots a_{n-1})$ .
- *Constant conductance.*

This is the restriction  $c \equiv 1$  on  $G^1$ . Then

$$(\Delta f)(o) = 2f(o) - f(0) - f(1), \quad \text{and}$$

$$(\Delta f)(x) = 3f(x) - f(x^*) - f(x0) - f(x1),$$

if  $x \in G^0$ , and  $l(x) > 0$ .

- *Paths in the tree.* If  $x = (a_1 a_2 \cdots a_n) \in G^0$ , there is a unique path  $\gamma(x)$  from  $\emptyset$  to  $x$ : the path is

$$\gamma(x) = \{(o, a_1), (a_1, (a_1 a_2)), \cdots, ((a_1 \cdots a_{n-1}), x)\}$$

and consists of  $n$  edges.

- *Concatenation of words:* For  $x = (a_1 a_2 \cdots a_n)$ ,  $y = (b_1 b_2 \cdots b_m) \in G^0$ . Set  $z = z(xy) = (a_1 \cdots a_n b_1 \cdots b_m)$ .

The dipoles  $(v_x)$  are indexed by  $x \in G^0 \setminus \{o\}$ , and  $v_x(o) = 0$  where  $o$  is the chosen origin. If  $G =$  the tree, then  $o = \emptyset =$  the empty word.

**Lemma 3.3.** [12] *Let  $x = (a_1 a_2 \cdots a_n)$ ,  $a_i \in A$ ,  $n = l(x)$ ; and  $y = (b_1 b_2 \cdots b_m)$ ,  $b_i \in A$ ,  $m = l(y)$ . Then*

(i)

$$v_x(y) := \begin{cases} 0 & \text{if } y = o \\ 2^{n-m} \cdot (2^m - 1) - \frac{2^n - 1}{2} & \text{if } m \leq n \\ \frac{2^n - 1}{2} & \text{if } m > n \end{cases}$$

$$(ii) \quad v_x \in \mathcal{H}_E, \text{ and } \|v_x\|_E^2 = \frac{2}{3}(2^{2n} - 1).$$

$$(iii) \quad \langle v_x, v_y \rangle_E = \frac{2}{3}(2^{2 \min(l(x), l(y))} - 1) = \#(\gamma(x) \cap \gamma(y)), \text{ for all } x, y \in G^0 \setminus \{o\}.$$

*Proof.* (i) By the uniqueness in Lemma 3.3, it is enough to prove that the function  $v_x$  in (i) satisfies  $\langle v_x, f \rangle_E = f(x) - f(o)$  for all  $f \in \mathcal{H}_E$ , and therefore also

$$(3.3) \quad \Delta v_x = \delta_x - \delta_o;$$

and that (ii)-(iii) hold.

Specifically, we must prove that

$$\begin{aligned} (\Delta v_x)(o) &= -1, \\ (\Delta v_x)(x) &= 1, \text{ and} \\ (\Delta v_x)(y) &= 0, \text{ if } y \notin \{o, x\}. \end{aligned}$$

Each is a computation:

$$\begin{aligned} (\Delta v_x)(o) &= 2v_x(o) - v_x(0) - v_x(1) \\ &= 0 - (2 \cdot 2^{n-1} - (2^n - 1)) \\ &= 1 \\ &= \delta_o(o). \end{aligned}$$

And if  $y \neq o$ , but  $m < n$ , then

$$\begin{aligned} (\Delta v_x)(y) &= 3v_x(y) - v_x(y^*) - v_x(y0) - v_x(y1) \\ &= 3 \cdot 2^{n-m} \cdot (2^m - 1) - 2^{n-m+1} \cdot (2^{m-1} - 1) - 2 \cdot 2^{n-m-1} \cdot (2^{m+1} - 1) \\ &= 0. \end{aligned}$$

Finally, we compute the case  $y = x$  as follows:

$$\begin{aligned} (\Delta v_x)(x) &= 3v_x(x) - v_x(x^*) - v_x(x0) - v_x(x1) \\ &= 3 \cdot (2^n - 1) - 2 \cdot (2^{n-1} - 1) - 2 \cdot (2^n - 1) \\ &= 0 - 3 + 2 + 2 = 1 \\ &= \delta_x(x) - \delta_o(x). \end{aligned}$$

We leave the case  $m = l(y) > n$  to the reader.

(ii) A computation using (3.1) yields

$$\begin{aligned} \|v_x\|_E^2 &= \frac{1}{2} \sum_{m \leq n} (2^{n-m})^2 \\ &= \frac{1}{2} \cdot 2^{2n} \cdot \left( \frac{1 - 2^{-2n}}{1 - 2^{-2}} \right) \\ &= \frac{2}{3}(2^{2n} - 1) \end{aligned}$$

proving (ii).

(iii) Suppose  $m = l(y) < n = l(x)$ ,  $x, y \in G^0 \setminus \{o\}$ . From (3.2), we see that the contribution to  $\langle v_x, v_y \rangle_E$  only includes words  $z$  with  $l(z) \leq m$ .

The desired conclusion

$$\langle v_x, v_y \rangle_E = 2^{-2m} \#(\gamma(x) \cap \gamma(y))$$

follows as in (ii). The possibilities may be illustrated in Figure 1 below.

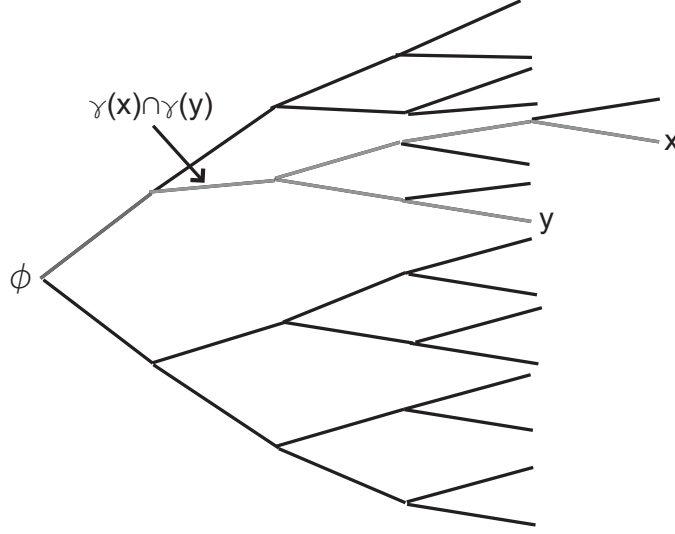


FIGURE 1

□

#### 4. SPECIFIC TRANSITION OPERATORS

**4.1. Transition on Graphs.** Let  $G = (G^0, G^1)$  be a graph with conductance function  $c : G^1 \rightarrow \mathbb{R}^+$ , and transition probabilities

$$p(x, y) := \frac{c(x, y)}{c(x)}, \quad \forall (x, y) \in G^1.$$

Note that  $c(x)p(x, y) = p(x, y)c(y)$ , which makes the corresponding  $p$ -random walk reversible.

**Lemma 4.1.** *Assume that  $\#Nbh(x) < \infty$  for all  $x \in G^0$ . Set*

$$(Tf)(x) := \sum_{y \sim x} p(x, y)f(y),$$

and let  $(Z_n)$  be the random walk on  $G^0$  with transition probabilities  $p(x, y)$  on edges  $(xy)$  in  $G$ , i.e.,

$$\mathbb{P}(\{\gamma | Z_n(\gamma) = x, Z_{n+1}(\gamma) = y\}) = p(x, y) \quad \text{for } (xy) \in G^1$$

Let  $T$  be the transition operator, and for  $\varphi \in l^1(G^0)$ , set

$$\langle \varphi \rangle := \sum_{x \in G^0} \varphi(x)$$

then for pairs of functions  $f_1$  and  $f_2$  on  $G^0$ , we have

$$\mathbb{E}((f_1 \circ Z_n) \cdot (f_2 \circ Z_{n+1})) = \langle T^n(f_1 \cdot T f_2) \rangle$$

with  $f_1$  and  $f_2$  are restricted to make the last sum convergent.

*Proof.* Let  $f_1, f_2$  be a pair of functions (real valued) on  $G^0$  such that the point-wise product  $f_1 \cdot (Tf_2)$  is in  $l^1(G^0)$ . Then for  $n \in \mathbb{N}_0$ , we now compute the  $Z_n$ -expectations: For the  $\mathbb{P}$ -integration on path space  $\Omega$ , we have:

$$\begin{aligned}
& \mathbb{E}((f_1 \circ Z_n) \cdot (f_2 \circ Z_{n+1})) \\
&= \int_{\Omega} (f_1 \circ Z_n) \cdot (f_2 \circ Z_{n+1}) d\mathbb{P} \\
&= \sum_{\substack{x_0 \ x_1 \ \dots \ x_{n+1} \\ \text{such that } x_{i-1} \sim x_i}} p(x_0, x_1) p(x_1, x_2) \cdots p(x_n, x_{n+1}) f_1(x_n) f_2(x_{n+1}) \\
&= \sum_{x_0} \sum_{x_1} \cdots \sum_{x_n} p(x_0, x_1) p(x_1, x_2) \cdots p(x_{n-1}, x_n) f_1(x_n) (Tf_2)(x_n) \\
&= \sum_{x_0 \in G^0} T^n(f_1 \cdot Tf_2)(x_0) \\
&= \langle T^n(f_1 \cdot Tf_2) \rangle
\end{aligned}$$

□

**Theorem 4.2.** Let  $(G, c)$  be a graph with conductance  $c : G^1 \rightarrow \mathbb{R}_+$ . Assume that  $\#Nbh(x) < \infty$  for all  $x \in G^0$ , when  $Nbh(x) := \{y \in G^0 | y \sim x\}$ . Set

$$p(x, y) := \frac{c(x, y)}{c(x)} \quad \text{and} \quad (Tf)(x) := \sum_{y \sim x} p(x, y) f(y).$$

Set

$$l^1(G^0, \mu_c) = \{f : G^0 \rightarrow \mathbb{R} | x \rightarrow c(x)f(x) \in l^1(G^0)\}, \quad \text{and} \quad \langle f \rangle_c := \sum_{x \in G^0} c(x)f(x).$$

Let  $\mathbb{P}^{(c)} = \mathbb{P}^{(\mu_c)}$  be the cylinder path-measure on

$$\Omega := \{(x_0 x_1 x_2 \cdots) | x_i \in G^0, x_{i-1} \sim x_i, i \in \mathbb{N}\}$$

where we use  $\mu_c$  in the first variable  $x_0$ , and counting measure on the remaining variables. Then

$$\mathbb{E}^{(\mu_c)}((f_1 \circ Z_n) \cdot (f_2 \circ Z_{n+1})) = \langle f_1 \cdot Tf_2 \rangle_c$$

*Proof.*

$$\begin{aligned}
& \mathbb{E}^{(\mu_c)}((f_1 \circ Z_n) \cdot (f_2 \circ Z_{n+1})) \\
&= \sum_{\substack{x_0 \ x_1 \ \dots \ x_{n+1} \\ \text{such that } x_{i-1} \sim x_i}} c(x_0) p(x_0, x_1) p(x_1, x_2) \cdots p(x_n, x_{n+1}) f_1(x_n) f_2(x_{n+1}) \\
&= \sum_{x_0} \sum_{x_1} \cdots \sum_{x_n} c(x_0) p(x_0, x_1) p(x_1, x_2) \cdots p(x_{n-1}, x_n) f_1(x_n) (Tf_2)(x_n) \\
&= \sum_{x_0} c(x_0) T^n(f_1 \cdot Tf_2)(x_0) \\
&= \langle T^n(f_1 \cdot Tf_2) \rangle_c \\
&= \langle f_1 \cdot Tf_2 \rangle_c
\end{aligned}$$

In the multiple summations  $\sum_{x_0} \sum_{x_1} \cdots \sum_{x_{n+1}}$ , it is just the first  $\sum_{x_0}$ -summation that is possibly infinite; in case the vertex-set  $G^0$  is infinite. Note that the combined

summations in the beginning of the proof contribute the integration over the set  $\Omega$  of all infinite paths  $\gamma = (x_0 x_1 x_2 \dots)$  specified by  $x_0 \sim x_1, x_1 \sim x_2, x_2 \sim x_3, \dots$ , at each step, moving from  $x_i$  to the next variable, note that  $x_{i+1}$  ranges over the finite set  $Nbh(x_i)$ . For more details on this point, see (4.1), below.

In the last step, we used the following formula which is valid on  $l^1(\mu_c)$ :

$$(4.1) \quad \langle T\varphi \rangle_c = \langle \varphi \rangle_c, \quad \varphi \in l^1(\mu_c).$$

We prove (4.1):

$$\begin{aligned} \langle T\varphi \rangle_c &= \sum_{x \in G^0} c(x) \sum_{y \sim x} p(x, y) \varphi(y) \\ &= \sum_{y \in G^0} \varphi(y) \sum_{x \sim y} c(x) \\ &= \sum_{y \in G^0} \varphi(y) c(y) \\ &= \langle \varphi \rangle_c. \end{aligned}$$

□

**4.2. Transfer Operators.** In section 2, we showed that a stochastic process  $(Z_n)_{n \in \mathbb{N}_0}$  on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  induces a transfer operator  $T$ . The derivation of  $T$  is then essentially canonical.

Here, the strategy will be reversed; but now, starting with  $T$ , there is a variety of choices of associated processes  $(Z_n)_{n \in \mathbb{N}_0}$ .

4.2.1. *Setting.* Let  $S$  be a compact Hausdorff space. Let  $(S, \mathcal{B})_S, \mu$  be a Borel probability measure space, and let  $p : S \times S \rightarrow \mathbb{R}_{\geq 0}$  be a continuous function such that

$$(4.2) \quad \int_S p(x, y) d\mu(y) \equiv 1 \quad \mu \text{ a.e. } x.$$

Set

$$(4.3) \quad (Tf)(x) := \int_S p(x, y) d\mu(y) \quad \text{for all } f \in L^\infty(S).$$

Set

$$(4.4) \quad \Omega := \Omega_p = \{\gamma = (x_0 x_1 x_2 \dots) \mid x_i \in S, \text{ s.t. } p(x_{i-1}, x_i) > 0\},$$

so an infinite path-space with path transitions governed by the function  $p$ .

Let  $\mathbb{P} = (\mathbb{P}_p)$  be the associated cylinder measure on  $\Omega_p$  as defined in section 2. For  $n \in \mathbb{N}_0$  and  $\gamma = (x_0 x_1 x_2 \dots) \in \Omega_p$ , set

$$(4.5) \quad Z_n(\gamma) := x_n; \quad \text{i.e., } Z_n : \Omega_p \rightarrow S$$

is an  $S$  valued random variable for all  $n \in \mathbb{N}_0$ .

**Theorem 4.3.** *Let  $p : S \times S \rightarrow \mathbb{R}_{\geq 0}$  be as stated in (4.2) above. Let  $T$  be the transfer operator (4.3). Then the stochastic process  $(Z_n)_{n \in \mathbb{N}_0}$  in (4.5) satisfies*

$$(4.6) \quad \mathbb{E}^{(p)}((f_1 \circ Z_n) \cdot (f_2 \circ Z_{n+1})) = \int_S (T^n(f_1 \cdot T f_2))(x) d\mu(x) \quad \text{for all } f_1, f_2 \in L^\infty(S).$$

*Proof.* The details in the computation for (4.6) follow those in section 2, but the reasoning is now reversed. Indeed,

$$\begin{aligned}
& \mathbb{E}^{(p)}((f_1 \circ Z_n) \cdot (f_2 \circ Z_{n+1})) \\
&= \int_S \int_S \cdots \int_S p(x_0, x_1) \cdots p(x_n, x_{n+1}) f_1(x_n) f_2(x_{n+1}) d\mu(x_0) d\mu(x_1) \cdots d\mu(x_{n+1}) \\
&= \int_S \int_S \cdots \int_S p(x_0, x_1) \cdots p(x_{n-1}, x_n) f_1(x_n) (Tf_2)(x_n) d\mu_0(x_0) d\mu(x_1) \cdots d\mu(x_n) \\
&= \int_S (T^n(f_1 \cdot Tf_2))(x) d\mu(x).
\end{aligned}$$

□

**Definition 4.4.** Let  $T$  be a transition operator satisfying the conditions (4.2) and (4.3), and suppose there is a Perron-Frobenius measure  $\mu_0$  on  $S$ , i.e.,

$$(4.7) \quad \mu_0 \circ T = \mu_0.$$

We say that  $T$  is *ergodic* if there is only one probability measure  $\mu_0$  on  $(S, \mathcal{B}_S)$  which solves (4.7).

If  $T$  is ergodic, and  $\mu_0$  is the (unique) Perron-Frobenius measure, then it follows from the Pointwise Ergodic Theorem that for all  $f \in L^\infty(S)$ , the limit

$$(4.8) \quad \lim_{n \rightarrow \infty} T^n(f) = \mu_0(f) \mathbb{1}$$

pointwise a.e. exists on  $S$ , where  $\mathbb{1}$  denotes the constant function 1 on  $S$ .

**Corollary 4.5.** Let  $p, T, S, \mathcal{B}_S, \mu$ , and  $(Z_n)$  satisfy the conditions of the theorem. Further assume  $T$  is ergodic with Perron-Frobenius measure  $\mu_0$ . Then

$$(4.9) \quad \lim_{n \rightarrow \infty} \mathbb{E}^{(p)}((f_1 \circ Z_n) \cdot (f_2 \circ Z_{n+1})) = \mu_0(f_1 \cdot Tf_2)$$

is satisfied for all  $f_1, f_2 \in L^\infty(S)$ .

*Proof.* To verify (4.9), note that  $\mathbb{E}^{(p)}((f_1 \circ Z_n) \cdot (f_2 \circ Z_{n+1}))$  is already computed in (4.6) in the theorem.

Since  $\mu$  is a probability measure, the conclusion (4.9) now follows from (4.8), i.e., from an application of the Ergodic Theorem. □

**4.3. Transition on Solenoids.** Let  $(S, \mu)$  be a measure space,  $\sigma : S \rightarrow S$  an endomorphism as specified in section 2. Let  $\Omega := X_\sigma(S)$  be the corresponding solenoid. Let  $W : S \rightarrow [0, 1]$  be a function satisfying

$$(4.10) \quad \sum_{y, \sigma(y)=x} W(y) = 1;$$

and let  $\mathbb{P} = \mathbb{P}_{\mu, \sigma, W}$  be the corresponding path measure.

**Lemma 4.6.** For the solenoid set  $Z_n : \Omega \rightarrow S$ ,  $Z_n(x_0, x_1, x_2, \dots) = x_n$ , and  $(Tf)(x) = \sum_{y, \sigma(y)=x} W(y)f(y)$ , for  $x \in S$ . Suppose  $T$  has a Perron-Frobenius measure  $\mu_0$ . Then  $(Z_n)_{n \in \mathbb{N}_0}$  is stationary with transition operator  $T$ .

*Proof.* Let  $f_1, f_2$  be a pair of functions on  $S$  satisfying the conditions listed above. For the  $\mathbb{P}$ -integration on path space  $\Omega(= X_\sigma(S))$  we then have:

$$\begin{aligned}
& \mathbb{E}^{(\mu_0)}((f_1 \circ Z_n) \cdot (f_2 \circ Z_{n+1})) \\
&= \int_S \sum_{\substack{x_1 \\ \sigma(x_1)=x_0}} \sum_{\substack{x_2 \\ \sigma(x_2)=x_1}} \cdots \sum_{\substack{x_{n+1} \\ \sigma(x_{n+1})=x_n}} W(x_1)W(x_2) \cdots W(x_{n+1})f_1(x_n)f_2(x_{n+1})d\mu_0(x_0) \\
&= \int_S \sum_{\substack{x_1 \\ \sigma(x_1)=x_0}} \sum_{\substack{x_2 \\ \sigma(x_2)=x_1}} \cdots \sum_{\substack{x_n \\ \sigma(x_n)=x_{n-1}}} W(x_1)W(x_2) \cdots W(x_n)f_1(x_n)(Tf_2)(x_n)d\mu_0(x_0) \\
&= \int_S (T^n(f_1 \cdot Tf_2))(x_0)d\mu_0(x_0) \\
&= \mu_0(T^n(f_1 \cdot Tf_2)) = \mu_0(f_1 \cdot Tf_2) \\
&= \langle f_1, Tf_2 \rangle_{L^2(\mu_0)}.
\end{aligned}$$

□

**4.4. Encodings.** Let  $G = (G^0, G^1)$  be a graph where we write  $G^0$  for the vertices and  $G^1$  for the edges. Let  $S$  be a set. We say that  $G$  yields an encoding of the points in  $S$  if there are mappings

$$(4.11) \quad \tau^0 : G^0 \rightarrow S, \quad \text{onto, and}$$

$$(4.12) \quad \tau^1 : G^0 \rightarrow \text{Functions } (S \rightarrow S)$$

such that for every  $e = (x, y) \in G^1$  we have

$$(4.13) \quad \tau^0(y) = \tau^1(e)\tau^0(x).$$

*Examples.*  $G$  = the binary tree,

$$(4.14) \quad S = \mathbb{N}_0 = \{0, 1, 2, \dots\}$$

$$(4.15) \quad = \left\{ \sum_{k=0}^{\text{Finite}} x_k 2^k \mid x_k \in \{0, 1\} \right\}$$

If  $n \in \mathbb{N}_0$  is given the finite word  $(x_0x_1x_2 \cdots)$  in (4.6) is computed from the Euclidean algorithm for division with 2.

Points in  $G^0$  are represented by the empty word  $o$ , and by all finite words  $w = (x_0x_1 \cdots x_p)$ . Set

$$(4.16) \quad \tau^0(w) = \sum_{k=0}^p x_k 2^k = n \in \mathbb{N}_0.$$

Starting with  $w = (x_0x_1 \cdots x_p) \in G^0$ , the three neighbors are  $(w0)$ ,  $(w1)$ , and  $w^* := (x_0x_1 \cdots x_{p-1})$  truncation, see Figure 2.



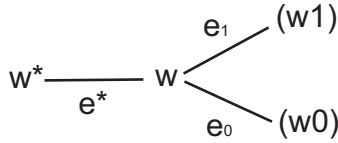


FIGURE 2. Three nearest neighbors and then associated these edges  $e_0, e_1$  and  $e^*$ .

Set

$$(4.17) \quad \begin{cases} \tau^1(e_0) := n \mapsto n; & \text{see (4.16);} \\ \tau^1(e_1) := n \mapsto n + 2^{p+1}; & \text{and} \\ \tau^1(e^*) := n \mapsto \sum_{k=0}^{p-1} x_k 2^k. \end{cases}$$

Note that in this example, there is an additional pair of mappings  $\mathbb{N}_0 \rightarrow \mathbb{N}_0$

$$(4.18) \quad \begin{cases} \sigma^0(n) = 2n \\ \sigma^1(n) = 2n + 1 \end{cases}$$

corresponding to the encoding mappings:

$$(4.19) \quad \begin{cases} \sigma_0 : (x_0 x_1 \cdots x_p) \mapsto \underbrace{(0 x_0 x_1 \cdots x_p)}_{\text{one step longer}} \\ \sigma_1 : (x_0 x_1 \cdots x_p) \mapsto (1 x_0 x_1 \cdots x_p) \end{cases}$$

*Remark 4.7.* The same construction works mutatis mutandis with  $N$ 'adic scaling rather than the dyadic representation of points in  $\mathbb{N}_0$ . Moreover, in the representation

$$(4.20) \quad n = \sum_{k=0}^p x_k N^k,$$

the choices for  $x_k$  may be from any complete set of residues modulo  $N$ , i.e., points in  $\mathbb{N}_0/N \cdot \mathbb{N}_0$ , or  $\mathbb{Z}/N\mathbb{Z}$  = the cyclic group of order  $N$ . The residues  $\{0, 1, \dots, N-1\}$  is only one choice of many.

*Encoding of  $\mathbb{Z}$ .* The representation used in (4.16) above works for  $\mathbb{Z}$  as well, but with the following modification:

$$(4.21) \quad \tau^0(x_0 x_1 x_2 \cdots x_p) := -2^p + \sum_{k=0}^p x_k 2^k.$$

Explanation:

$$\begin{aligned}\tau^0(\underbrace{111 \cdots 1}_{p+1 \text{ times}}) &= -2^p + \sum_{k=0}^p x_k 2^k \quad \text{with } x_k = 1, 0 \leq k \leq p \\ &= -2^p + 2^{p+1} - 1 \\ &= 2^p - 1.\end{aligned}$$

Hence, with this convention we arrive at an encoding of  $\mathbb{Z}$ .

*Graphs vs Compactification:* In the examples, we represent points in the vertex sets  $G^0$  on a graph  $G$  by finite words in a specific finite alphabets. A choice of compactification  $\Omega$  of  $G^0$  is the set of infinite paths  $\gamma$ , i.e.,  $\gamma = (x_0 x_1 x_2 \cdots)$  where  $x_i \in G^0$ , and  $(x_{i-1}, x_i) \in G^1$  for all  $i \in \mathbb{N}$ .

In each of the examples we present, we build measure  $\mathbb{P}$  on the compactifications  $\Omega$  with use of Kolmogorov's extension principle. This is a projective limit construction which proceeds in three steps [11]:

- (i) First specify  $\mathbb{P}$  only on finite words, i.e., on cylinder sets over  $G^0$
- (ii) Check that the prescription of  $\mathbb{P}$  on cylinders is consistent.
- (iii) With Kolmogorov's theorem then extend  $\mathbb{P}$  to the Borel  $\sigma$ -algebra of subsets in  $\Omega$  generated by the cylinder-sets [15, 11].

**Definition 4.8.** In later applications, the following two cases for  $\mathbb{P}$  will play a role: Consider the subset  $\Omega_{\text{Fin}}$  in  $\Omega$  consisting of paths  $\gamma = (x_0 x_1 x_2 \cdots)$  which terminate in infinite repetitions, i.e.,  $\gamma \in \Omega_{\text{Fin}} \Leftrightarrow \exists n$  such that  $x_i = x_n \forall i > n$ . The measure  $\mathbb{P}$  is said to be *tight* if and only if  $\mathbb{P}(\Omega_{\text{Fin}}) = 1$ . Alternatively,  $\mathbb{P}(\Omega_{\text{Fin}}) < 1$ .

*Examples Resumed:* Wavelets. We adopt the standard terminology for dyadic wavelets in  $L^2(\mathbb{R})$ , specifically  $\varphi$  for a choice of scaling function; see [11]. Let  $(a_k)_{k \in \mathbb{Z}}$  represent a wavelet filter, i.e., satisfying the following three conditions:

$$(4.22) \quad \sum_{k \in \mathbb{Z}} \bar{a}_k a_{k+2l} = \frac{1}{2} \delta_{0,l},$$

$$(4.23) \quad \sum_{k \in \mathbb{Z}} a_k = 1, \quad \text{and}$$

$$(4.24) \quad \varphi(x) = 2 \sum_{k \in \mathbb{Z}} a_k \varphi(2x - k).$$

The function  $\varphi$  is in  $L^2(\mathbb{R})$  and

$$(4.25) \quad \int_{\mathbb{R}} \varphi(x) dx = 1$$

is a chosen normalization.

Let  $\hat{\varphi}$  be the  $\mathbb{R}$ -Fourier transform.

The following result is from [11]. Let  $\Omega :=$  the set of all infinite words, and view  $\Omega$  as a compactification of the vertex set  $G^0$  of all finite dyadic words.

**Lemma 4.9.** *For every  $t \in \mathbb{R}$ , there is a measure  $\mathbb{P}_t$  on  $\Omega$  such that*

$$(4.26) \quad \mathbb{P}_t(x_0 x_1 \cdots x_p) = \left| \hat{\varphi}(t + \tau^0(x_0 x_1 \cdots x_p)) \right|^2$$

where  $\tau^0 : G^0 \rightarrow \mathbb{Z}$  is the encoding of (4.21).

**Lemma 4.10.** (See [11].)

(a) Consider the process  $(Z_n)$  in  $(\Omega, \mathbb{P}_t)$  from (4.26) with

$$Z_n(\underbrace{x_0 x_1 x_2 \cdots}_{\text{infinite word}}) := x_n \in \{0, 1\}.$$

Then there is a transfer operator  $T$  such that the process is  $T$ -stationary.

(b) Let

$$(4.27) \quad W(e^{it}) := \widetilde{W}(t) = \left| \sum_{k \in \mathbb{Z}} a_k e^{ikt} \right|^2,$$

where functions  $W$  on  $\mathbb{T}$  are identified with  $2\pi$ -periodic functions  $\widetilde{W}$  on  $\mathbb{R}$ , and where  $(a_k)$  is some wavelet filter as in (4.22)-(4.24). The transfer operator  $T$  is then given by

$$(T_W f)(t) = W\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right) + W\left(\frac{t}{2} + \pi\right)f\left(\frac{t}{2} + \pi\right).$$

We say that  $W$  has scaling-degree 2.

Following (4.18), let a transition from  $n$  to  $n + 1$  be given by a choice of  $x \in \{0, 1\}$ .

Then

$$(4.28) \quad \mathbb{E}_t(Z_n Z_{n+1}) = \widetilde{W}(t + x\pi).$$

**Proposition 4.11.** Let  $\varphi \in L^2(\mathbb{R})$  satisfying (4.24), and suppose  $\|\varphi\|_2 \leq 1$ . Let  $\Omega$  be the compactification derived from the encoding  $\tau^0$  of  $\mathbb{Z}$  in (4.21) and let  $t \in (-\pi, \pi]$ . Let  $\mathbb{P}_t$  be the measure on  $\Omega$  from (4.26).

Part I. Then the following affirmations are equivalent:

- (a) The translates  $\{\varphi(\cdot - k) | k \in \mathbb{Z}\}$  form an orthonormal family in  $L^2(\mathbb{R})$ .
- (b) The measures  $\mathbb{P}_t$  are tight measures on  $\Omega$  for all  $t$ .
- (c)  $\sum_{n \in \mathbb{Z}} |\widehat{\varphi}(t + n)|^2 = 1$  for all  $t \in \mathbb{R}$ .

Part II. If the measures  $\mathbb{P}_t$  are not tight, then the translates  $\{\varphi(\cdot - k)\}_{k \in \mathbb{Z}}$  still form a Parseval frame for the closed subspace  $V(\varphi)$  they span, i.e., we have the identity

$$\sum_{k \in \mathbb{Z}} \left| \int_{\mathbb{R}} \overline{\varphi}(x - k) f(x) dx \right|^2 = \int_{\mathbb{R}} |f(x)|^2 dx \quad \text{for all } f \in V(\varphi).$$

*Proof.* See [11]. □

**Definition 4.12.** Functions  $W$  on  $(-\pi, \pi]$  arising as in (4.27) for a system of wavelet coefficients  $(a_k)_{k \in \mathbb{Z}}$  (4.24), are called *wavelet filters*. A wavelet filter  $W$  is said to be *low-pass* if  $\mu_0 := \delta_0$ , i.e., the Dirac measure at  $\theta = 0$ , is a Perron-Frobenius measure for  $T_W$ .

In general, if  $W$  is a Lipschitz function, it is known that  $T_W$  has a Perron-Frobenius measure [3].

**Example 4.13.** [5] Set

$$(4.29) \quad W_F(z) := \frac{1}{6} |1 + z^2| \quad \text{for } z = e^{i\theta}.$$

Then  $W_F$  is a wavelet-filter under scaling by 3, but it is *not* a low-pass filter.

Indeed, the following scaling law holds for  $W_F$ :

$$\sum_{w^3=z} W_F(w) = 1, \quad \forall z = e^{i\theta} \in \mathbb{T}^1.$$

We say that  $W_F$  has scaling degree 3.

It is proved in [5] that  $W_F$  induces a wavelet representation on an  $L^2$ -space built from the middle-third-Cantor construction, ‘‘Cantor-dust’’  $CD_3$  in  $\mathbb{R}$  with Hausdorff measure  $\mathcal{H}^\alpha$ ,  $\alpha = \frac{\ln 2}{\ln 3}$ , i.e., on  $L^2(\text{Cantor dust}, \mathcal{H}^\alpha)$ .

*Cantor Dust  $CD_3$ .* The points  $x \in CD_3 \subset \mathbb{R}$  are encoded by

$$x = a_{-k}3^k + a_{-k+1}3^{k-1} + \cdots + a_0 + \sum_{i=0}^{\infty} \frac{a_i}{3^i}$$

where  $k$  varies in  $\mathbb{N}_0$ , and where  $a_j \in \{0, 1, 2\}$  for  $j \in \mathbb{Z}$  such that  $-k \leq j$ ; but where  $a_j$  attains the value 1 only for at most a finite number of places.

The Perron-Frobenius measure  $\mu_0$  for  $T_{W_F}$  is singular with support  $(\mu_0) = \mathbb{T}$ .

## 5. REPROCITY RULE FOR THE SPECTRUM

In the previous section we saw that a wide class of processes are governed by a transfer operator  $T$ . If the process in question takes places on a graph  $G = (G^0, G^1)$  with conductance  $c$ , then harmonic analysis on  $G$  is phrased in terms of a Laplace operator  $\Delta_c$  as follows:

$$(\Delta_c f)(x) = \sum_{y \sim x} c(x, y)(f(x) - f(y)), \quad \text{for } x \in G^0.$$

**Lemma 5.1.** *Let  $(G, c)$  and  $\Delta_c$  be as above. Set  $p(x, y) = \frac{c(x, y)}{c(x)}$  for  $(x, y) \in G^1$  and let*

$$(T_c f)(x) = \sum_{y \sim x} p(x, y)f(y),$$

then

$$(\Delta_c f)(x) = c(x)\{f(x) - (T_c f)(x)\}.$$

And conversely,

$$(T_c f)(x) = f(x) - \frac{1}{c(x)}(\Delta_c f)(x).$$

*Proof.* Left to the reader. □

Because of reference to harmonic analysis, we present the results in this section in terms of  $\Delta_c$ , but the lemma makes a translation between  $\Delta_c$  and  $T_c$  immediate: For example, a function  $f$  on  $G^0$  satisfies  $\Delta_c f = 0$  if and only if  $T_c f = f$ . Solution  $f$  to either one of these equations are called *harmonic*.

**Definition 5.2.** Let  $\mathcal{H}$  be a Hilbert space, and  $\mathcal{D}$  a dense linear subspace. An operator  $\Delta$  defined on  $\mathcal{D}$  is said to be *formally selfadjoint* if and only if

$$\langle \Delta u, v \rangle = \langle u, \Delta v \rangle$$

holds for all  $u, v \in \mathcal{D}$ .

A further advantage of  $\Delta_c$  over  $T_c$  is that  $\Delta_c$  is formally selfadjoint, (while  $T_c$  is not!).

When we say that  $\Delta_c$  is formally selfadjoint, this applies to either one of the two Hilbert spaces  $l^2(G^0)$ , and  $\mathcal{H}_E :=$  the energy Hilbert space.

In the case of  $\mathcal{H}_E$ , we take for  $\mathcal{D}$  the linear span of the family  $\{v_x | x \in G^0\} \subset \mathcal{H}_E$ ; see Lemma 5.3 and 5.4.

We continue the setup from the previous section:  $G = (G^0, G^1)$  a fixed graph with vertices  $G^0$  and edges  $G^1$ . Let  $c : G^1 \rightarrow \mathbb{R}_+$  be a fixed conductance function. Let  $\Delta = \Delta_c$  be the Laplace operator. Fix an origin  $o$  in  $G^0$ , and let  $\{v_x\}_{x \in G^0 \setminus \{o\}}$  be the system of dipoles.

**Lemma 5.3.** [12] (*Reproducing Kernel*) *The system  $\{v_x\}_{x \in G^0 \setminus \{o\}}$  forms a reproducing kernel in the sense:*

$$(5.1) \quad \langle v_x, f \rangle_E = f(x) - f(o) \quad \text{for all } f \in \mathcal{H}_E,$$

where  $\mathcal{H}_E$  is the energy Hilbert space.

*Proof.* The existence of  $\{v_x\}$  is established with an application of Riesz's lemma: If  $x \in G^0$ , there is a path  $\gamma(x) = x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_n$ ,  $e_i = (x_i, x_{i+1}) \in G^1$ , (generally not unique) such that  $x_0 = 0$  and  $x_n = x$ .

By Cauchy-Schwarz, we get

$$(5.2) \quad |f(x) - f(o)|^2 \leq \sum_i \frac{1}{c(e_i)} \|f\|_E^2.$$

Riesz's lemma applied to  $\mathcal{H}_E$ , then yields  $\exists v_x \in \mathcal{H}_E$  such that (5.1) is satisfied.

We claim that  $v_x$  satisfies the dipole equation

$$(5.3) \quad \Delta v_x = \delta_x - \delta_o, \quad x \in G^0 \setminus \{o\}.$$

This implies (5.3), and if  $\Delta h = 0$ , then  $w_x := v_x + h$  solves (5.3) as well; and vice versa.  $\square$

**Lemma 5.4.** [12] *Let  $\mathcal{D}_0 := \text{span}_{\mathbb{C}}\{\delta_x\}_{x \in G^0}$ , and  $\mathcal{D}_E := \text{span}_{\mathbb{C}}\{v_x\}_{x \in G^0 \setminus \{o\}}$ .*

By "span" we mean finite complex linear combinations, so we consider all finite summations

$$(5.4) \quad \mathcal{D}_0 = \left\{ \sum_x a_x \delta_x \right\}, \quad \text{and} \quad \mathcal{D}_E = \left\{ \sum_x b_x v_x \right\},$$

where  $\{a_x\}$  and  $\{b_x\}$  denote finite systems of scalars,  $a_x, b_x \in \mathbb{C}$ .

Then  $\Delta$  yields a density defined hermitian (i.e., formally selfadjoint) operator in each of the Hilbert spaces  $l^2(G^0)$  and  $\mathcal{H}_E$ .

Specifically,  $\mathcal{D}_0$  is dense in  $l^2(G^0)$  and

$$(5.5) \quad \langle u, \Delta v \rangle_{l^2} = \langle \Delta u, v \rangle_{l^2}, \quad \forall u, v \in \mathcal{D}_0.$$

Moreover,  $\mathcal{V}$  is dense in  $\mathcal{H}_E$ , and

$$(5.6) \quad \langle u, \Delta v \rangle_E = \langle \Delta u, v \rangle_E, \quad \forall u, v \in \mathcal{D}_E$$

*Proof.* The symmetry property (5.5) is immediate from the definition of  $\Delta$ .

We now prove (5.6): Since both sides in (5.5) are sesquilinear, it is enough, by (5.4), to prove

$$(5.7) \quad \langle v_x, \Delta v_y \rangle_E = \langle \Delta v_x, v_y \rangle_E, \quad \forall x, y \in G^0 \setminus \{o\}.$$

We have

$$\begin{aligned}
\langle v_x, \Delta v_y \rangle_E &\stackrel{\text{by (5.3)}}{=} \langle v_x, \delta_y - \delta_0 \rangle_E \\
&\stackrel{\text{by (5.1)}}{=} (\delta_y - \delta_0)(x) - (\delta_y - \delta_0)(o) \\
&= \delta_x(y) + 1 \\
&\stackrel{\text{by symmetry}}{=} \langle \delta_x - \delta_0, v_y \rangle_E \\
&\stackrel{\text{by (5.3)}}{=} \langle \Delta v_x, v_y \rangle_E
\end{aligned}$$

which is the desired eq. (5.7).  $\square$

**5.1. Two Hilbert Spaces.** Let  $G = (G^0, G)$  be as above; and let  $c : G^1 \rightarrow \mathbb{R}_+$  be a fixed conductance function. Let  $\Delta$  and  $T$  be the corresponding operators,  $\Delta = \Delta_c$  the Laplace operator, and

$$(5.8) \quad (Tf)(x) = f(x) - \frac{1}{c(x)}(\Delta f)(x), \quad x \in G^0.$$

Pick a fixed  $o \in G^0$ , and let  $(v_x)_{x \in G^0 \setminus \{o\}}$  be the corresponding reproducing kernel.

It is important to understand the two operators in the two Hilbert spaces  $l^2(G^0)$  and  $\mathcal{H}_E$ . By (5.8), it is enough to consider just  $\Delta$ .

As an operator in  $l^2(G^0)$ , the operator  $\Delta$  has as its domain

$$\begin{aligned}
\mathcal{D}_0 &:= \text{all finite linear combinations of } \{\delta_x\}_{x \in G^0} \\
&= \text{span } \{\delta_x\}_{x \in G^0};
\end{aligned}$$

while the domain in  $\mathcal{H}_E$  is

$$\mathcal{D}_E := \text{span } \{v_x | x \in G^0 \setminus \{o\}\}.$$

**Theorem 5.5.** (a) *The domains in  $l^2$  and in  $\mathcal{H}_E$ :*

- (i)  $\mathcal{D}_0$  is a dense subspace in  $l^2(G^0)$ ; and
- (ii)  $\mathcal{D}_E$  is a dense subspace in  $\mathcal{H}_E$ .
- (iii) If  $\#Nbh(x) < \infty$  for all  $x \in G^0$ , then  $\Delta$  maps  $\mathcal{D}_0$  into itself; and  $\Delta_E$  maps  $\mathcal{D}_E$  into itself.

(b) *For all vectors  $\varphi, \psi \in \mathcal{D}_0$ , we have:*

(i)

$$\langle \varphi, \Delta \varphi \rangle_{l^2} = \sum_{x \in G^0} c(x) |\varphi(x)|^2 - \sum_{\substack{x \quad y \\ x \sim y}} c(x, y) \overline{\varphi(x)} \varphi(y);$$

(ii)  $\langle \varphi, \Delta \varphi \rangle_{l^2} \geq 0$ ; and

(iii)  $\langle \varphi, \Delta \psi \rangle_{l^2} = \langle \Delta \varphi, \psi \rangle_{l^2}$ .

(c) *For all vectors  $\varphi, \psi \in \mathcal{D}_E$ , we have:*

(i)

$$\langle \varphi, \Delta \varphi \rangle_{\mathcal{H}_E} = \sum_{x \in G^0 \setminus \{o\}} |(\Delta \varphi)(x)|^2 + \left| \sum_{x \in G^0 \setminus \{o\}} (\Delta \varphi)(x) \right|^2;$$

(ii)  $\langle \varphi, \Delta \varphi \rangle_{\mathcal{H}_E} \geq 0$ ; and

(iii)  $\langle \varphi, \Delta \psi \rangle_{\mathcal{H}_E} = \langle \Delta \varphi, \psi \rangle_{\mathcal{H}_E}$ .

*Proof.* The proof of (b)(ii) is a sequence of steps with repeated application of Cauchy-Schwarz's inequality. The proof of (a)(i) is an application of the last equation in the proof of Lemma 5.4.  $\square$

*Remark 5.6.* The operator  $\Delta_{l^2}$  in  $l^2$ , or  $\Delta_E$  in  $\mathcal{H}_E$ , may be bounded or unbounded. In all cases  $\Delta_{l^2}$  is essentially selfadjoint in  $l^2$  [12]; but  $\Delta_E$  may have defect-subspaces.

## 5.2. Dichotomy.

*Remark 5.7.* [12] For the graph system  $(G, c) = (tree, \mathbb{1})$  the Laplace operator  $(\Delta, \mathcal{D}_0)$  is bounded and selfadjoint in  $l^2(G^0)$ . For the energy Hilbert space  $\mathcal{H}_E(tree)$ ,  $(\Delta, \mathcal{D}_E)$  is an *unbounded* Hermitian operator. In fact,  $\Delta$  is *not* essentially selfadjoint on  $\Delta$ ; i.e.,  $(\Delta, \mathcal{D}_E)$  has a infinite family of distinct selfadjoint extensions in the Hilbert space  $\mathcal{H}_E$ .

**Lemma 5.8.** *Let  $\mathcal{H}(\langle \cdot, \cdot \rangle)$  be a complex Hilbert space, and let  $\mathcal{D}$  be a dense linear subspace in  $\mathcal{H}$ .*

*Let  $L$  be a closed Hermitian operator defined on  $\mathcal{D}$ , i.e.,  $L$  is linear and satisfies*

$$(5.9) \quad \langle u, Lv \rangle = \langle Lu, v \rangle \quad \forall u, v \in \mathcal{D}.$$

*Then the spectrum of  $\Delta$  is the closure of the set*

$$(5.10) \quad NS(L) := \left\{ \frac{\langle u, Lu \rangle}{\|u\|^2} \mid u \in \mathcal{D} \setminus \{o\} \right\}.$$

*Proof.* The Hermitian property (5.9) implies that the spectrum of  $L$  is contained in  $\mathbb{R}$ .

Now suppose  $\lambda_0 \in \mathbb{R}$ , and that

$$(5.11) \quad \text{dist}(\lambda_0, NS(L)) = \epsilon_1 > 0.$$

We will show that  $\lambda$  must then be in

$$\begin{aligned} \mathbb{R} \setminus \text{spec}(L) &= \text{the complement of the spectrum} \\ &= \text{the resolvent set.} \end{aligned}$$

Let  $u \in \mathcal{D} \setminus \{o\}$ . Then

$$\|\lambda_0 u - Lu\|^2 = \lambda_0^2 \|u\|^2 - 2\lambda_0 \langle u, Lu \rangle + \|Lu\|^2$$

Setting  $x_1 := \frac{\langle u, Lu \rangle}{\|u\|^2} \in NS(L)$ , we get

$$(5.12) \quad \|\lambda_0 u - Lu\|^2 = \|u\|^2 \cdot (\lambda_0 - x_1)^2 - \|u\|^2 x_1^2 + \|Lu\|^2$$

$$(5.13) \quad \begin{aligned} &\geq \|u\|^2 \cdot \epsilon_1^2 + \|Lu\|^2 - \frac{\langle u, Lu \rangle^2}{\|u\|^2} \\ &\text{by (5.11)} \end{aligned}$$

$$(5.14) \quad \geq \|u\|^2 \cdot \epsilon_1^2$$

where we used Schwarz' inequality in the last step; viz.,

$$\langle u, Lu \rangle^2 \leq \|u\|^2 \cdot \|Lu\|^2;$$

or

$$\|Lu\|^2 - \frac{\langle u, Lu \rangle^2}{\|u\|^2} \geq 0.$$

By virtue of the inequality (2.11), we may define an operator

$$R_0 = R(\lambda_0) : \text{range}(\lambda_0 I - L) \longrightarrow \mathcal{H}$$

by

$$(5.15) \quad R_0(\lambda u - Lu) = u.$$

Extend  $R_0$  by setting it = 0 on the ortho-complement

$$(5.16) \quad (\text{range}(\lambda_0 I - L))^\perp = N(\lambda_0 - L^*).$$

Here  $L^*$  denotes the adjoint operator.

From (5.15), we calculate that  $R_0 : \mathcal{H} \rightarrow \mathcal{H}$  defines a bounded inverse to  $\lambda_0 I - L$ , and so  $\lambda_0 \in \text{resolvent}(L)$ ; and conversely.  $\square$

Let  $\{v_x\}_{x \in G^0 \setminus \{0\}}$  be the system of dipoles, and set

$$(5.17) \quad M := (\langle v_x, v_y \rangle_E)$$

viewed as a Hermitian matrix,  $x = \text{row index}$ ,  $y = \text{column index}$ .

If  $\xi = (\xi_x) \in \mathcal{F} \subset l^2(G^0)$ , set

$$(5.18) \quad (M\xi)_x = \sum_y M_{x,y} \xi_y,$$

matrix multiplication, where

$$M_{x,y} := \langle v_x, v_y \rangle_E.$$

Then  $M$  is a density defined Hermitian operator in  $l^2(G^0)$ .

**Theorem 5.9.** *Let  $(G, \mu)$  be given and let  $\Delta$  be the corresponding density defined Hermitian operator in  $\mathcal{H}_E$ . Then*

$$(5.19) \quad \text{spec}_{\mathcal{H}_E}(\Delta) \subset [0, \infty)$$

and

$$(5.20) \quad \text{spec}_{\mathcal{H}_E}(\Delta) = (\text{spec}_{l^2}(M))^{-1}$$

where we use the characters  $\frac{1}{0} = \infty$ , and  $\frac{1}{\infty} = 0$ .

Moreover,

$$(5.21) \quad (\text{spec}_{l^2}(M))^{-1} = \{1/\lambda \mid \lambda \in \text{spec}_{l^2}(M)\}.$$

*Proof.* For  $(\xi_x) \in \mathcal{F}$ , set

$$(5.22) \quad u := \sum_{x \in G^0 \setminus \{0\}} \xi_x v_x.$$

Then  $u \in \mathcal{V}$ , and

$$(5.23) \quad \langle u, \Delta u \rangle_{\mathcal{H}_E} = \sum_x \sum_y \bar{\xi}_x \xi_y \langle v_x, \Delta v_y \rangle_E$$

$$(5.24) \quad = \sum_x \sum_y \bar{\xi}_x \xi_y (\delta_x(y) + 1)$$

$$(5.25) \quad = \sum_x |\xi_x|^2 + \left| \sum_x \xi_x \right|^2 \geq 0$$

Since vectors in  $\mathcal{H}_E$  are equivalence classes modulo the constant function on  $G^0$ , we may add the restriction  $\sum_x \xi_x = 0$  in (5.22), and the operator  $\Delta$  will be unchanged.

The modified equation (2.22) then needs

$$(5.26) \quad \langle u, \Delta u \rangle_E = \|\xi\|_2^2$$



**Claim 5.10.**

$$(5.27) \quad \|u\|_{\mathcal{H}_E}^2 = \langle \xi, M\xi \rangle_{l^2}.$$

*Proof.* (of Claim 2.6). We compute:

$$\begin{aligned} \|u\|_E^2 &= \langle u, u \rangle_E \\ &= \sum_x \sum_y \bar{\xi}_x \xi_y \langle v_x, v_y \rangle \\ &\stackrel{\text{by 5.22}}{=} \sum_x \bar{\xi}_x (M\xi)_x \\ &= \langle \xi, M\xi \rangle_{l^2}, \end{aligned}$$

as claimed.  $\square$

The desired conclusion (5.20) now follows: If  $u \in \mathcal{V} \setminus (o)$  is given by (5.22), then

$$(5.28) \quad \frac{\langle u, \Delta u \rangle_E}{\|u\|_E^2} = \frac{\|\xi\|_2^2}{\langle \xi, M\xi \rangle}.$$

By taking closure, we obtain the sets on the two sides in (5.20)  $\square$

**Corollary 5.11.** *If  $\xi = (\xi_x) \in \mathcal{F}(G^0 \setminus (o))$ , then the representation*

$$(5.29) \quad u = \sum_x \xi_x v_x$$

*is unique; in particular, the system  $(v_x)_{x \in G^0 \setminus (o)}$  is linearly independent.*

*Proof.* Let  $u \in \mathcal{V}$  have a representation (5.29) as a finite summation with  $\xi_x \in \mathbb{C}$ .

Let  $y \in G^0 \setminus (o)$ . Then

$$\begin{aligned} \langle \delta_y, u \rangle_E &= \sum_x \xi_x \langle \delta_y, v_x \rangle_E \\ &\stackrel{\text{by (5.1)}}{=} \sum_x \xi_x (\delta_y(x) - \delta_y(o)) \\ &= \xi_y. \end{aligned}$$

In particular, if  $u = 0$ , then  $\xi_y = 0$ ,  $\forall y \in G^0 \setminus (o)$ .  $\square$

**Corollary 5.12.** *If  $F \subset G^0 \setminus (o)$  is a finite subset, then 0 is not in the spectrum of the matrix*

$$(5.30) \quad M_F := (\langle v_x, v_y \rangle_E)_{x, y \in F}.$$

Suppose  $o \in \text{spec}(M_F)$  where  $F$  is a fixed as in the statement of the Corollary 5.12. Then

$$\exists \xi \in l^2(F \setminus (o))$$

such that

$$(5.31) \quad (M\xi)_x = \sum_{y \in F} \langle v_x, v_y \rangle_E \xi_y = 0.$$

Setting  $u := \sum_{y \in F} \xi_y v_y$  we note that

$$(5.32) \quad u \in (\{v_x\}_{x \in F})^\perp$$

**Claim 5.13.**

$$(5.33) \quad u \in (\{v_x\}_{x \in G^0 \setminus (o)})^\perp.$$

We need to prove this only if  $x \in G^0 \setminus F$ .

Combining (5.27) and (5.31), we get

$$\begin{aligned} \|u\|_E^2 &= \langle \xi, M\xi \rangle_{l^2} \\ &= \langle \xi, 0 \rangle_{l^2} \\ &= 0, \end{aligned}$$

so  $u = 0$  is a constant function on  $G^0$ , and (5.33) is satisfied.

### 6. THE ENERGY-INNER PRODUCT

$(G, c) = (\text{tree}T, 1)$ ,  $o = \emptyset$ ,  $c \equiv 1$ . Explicitly form for  $v_x$ ,  $x \in G^0 \setminus (o)$ . Set

$$(6.1) \quad x = (a_1 a_2 a_3 \cdots a_n) \in G^0 \setminus (o) \quad a_i \in A = \{0, 1\}.$$

$$(6.2) \quad \gamma(x) = \{(oa_1), (a_1 a_2), (a_2 a_3), \dots, (a_{n-2} a_{n-1}), (a_{n-1} a_n)\}$$

where  $\gamma(x)$  is a path. Note  $\gamma(x) \subset G^1 = \text{edges in } T$ .

**Example 6.1.**  $x = 101$  vertex,  $\{(\varphi, 1), (1, 10), (10, 101)\} = \gamma(x)$   $\#\gamma(x) = 3$ .

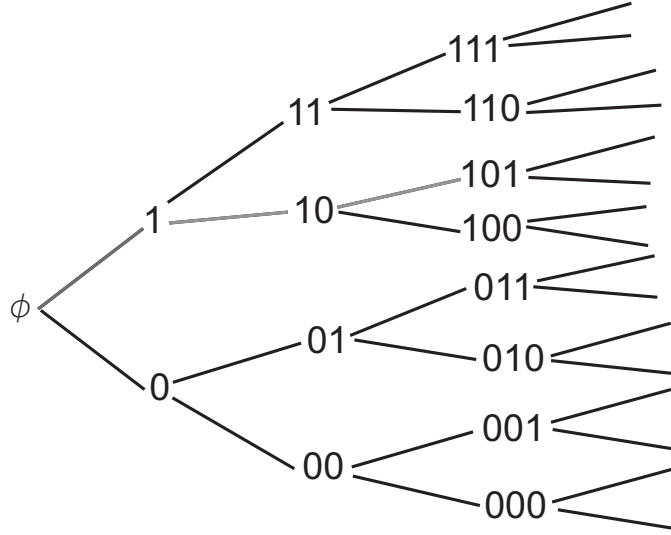


FIGURE 3

**Theorem 6.2.** Let  $(T, 1)$  be as usual,  $o = \emptyset$ , and let  $\mathcal{H}_E =$  the 0 energy span

$$(6.3) \quad \|f\|_E^2 = \frac{1}{2} \sum_x \sum_{\substack{y \\ x \sim y}} (f(x) - f(y))^2$$

but with edges  $(\underline{x}, \underline{xb}) = e$ ,  $x \in G^0$ ,  $b \in A = \{0, 1\}$ ,  $c(e) \equiv 1$ . Then the function

$$(6.4) \quad v_x(y) := \#\gamma(x) \cap \gamma(y)$$

solves

$$(6.5) \quad \langle v_x, f \rangle_E = f(x) - f(o), \quad \forall f \in \mathcal{H}_E$$

$$(6.6) \quad \Delta v_x = \delta_x - \delta_o, \quad x \in G^0 \setminus (o)$$

and

$$(6.7) \quad \langle v_x, v_y \rangle_E = \#(\gamma(x) \cap \gamma(y)) \quad \forall x, y \in G^0 \setminus (o)$$

*Proof. Proof of (6.6).* By (6.4)  $x = (a_1 a_2 \cdots a_n) \in G^0 \setminus (o)$ . Let  $x$  be as in (6.2). Set  $\gamma(x) = \text{RHS in (6.2)} \subseteq G^1$ . Neighbors of

$$\begin{aligned} x &\longrightarrow a_1 \cdots a_{n-1} \\ &\longrightarrow x0 \\ &\longrightarrow x1 \end{aligned}$$

If  $x = a$ ,  $n = 1$ ,  $\text{Nbh}(x) = \{o, a0, a1\}$

*Cases.*

n=1 See Figure 4

$$\begin{aligned} (\Delta v_x)(o) &= 2v_x(o) - v_x(0) - v_x(1) \\ &= 0 - 1 \\ &= \delta_x(o) - \delta_o(o) \end{aligned}$$

$$\begin{aligned} (\Delta v_x)(x) &= 3v_x(x) - v_x(o) - v_x(x0) - v_x(x1) \\ &\stackrel{\text{use(6.4)}}{=} 3 - 0 - 1 - 1 = 1 \\ &= (\delta_x - \delta_o)(x) \end{aligned}$$

Now, let  $y \in G^0 \setminus \{o, x\}$ .  $y = (b_1 b_2 \cdots b_k)$ ,  $b_i \in A = \{0, 1\}$ . Suppose  $x \subseteq y$

$$\Delta v_x(y) \stackrel{\text{by(6.4)}}{=} 3 - 1 - 1 - 1 = 0$$

More cases are  $\equiv 0$ .

n>1  $x = (a_1 a_2 \cdots a_n)$ . A computation yields

$$\Delta_x(o) = 0 - v_x(0) - v_x(1) = 0 - 1 = -1$$

$$\begin{aligned} \Delta_x(x) &= 3n - (n-1) - 2n = 1 \\ &= (\delta_x - \delta_o)(x) \end{aligned}$$

$$\Delta_x(y) = 0 \quad y \in G^0 \setminus \{o, x\}$$

Several cases e.g.  $y \leq x$ , etc.

$$\begin{aligned} \Delta_x(y) &= 3v_x(y) - v_x(b_1 \cdots b_{k-1}) - v_x(y0) - v_x(y1) \\ &= 3k - (k-1) - (k+1) - k = 0, \text{ etc.} \end{aligned}$$

Computation of

$$\begin{aligned}
 \|v_x\|_E^2 &= E(v_x) \\
 &= \frac{1}{2} \sum_s \sum_{t \sim s} c_{s,t} (v_x(s) - v_x(t))^2 \\
 &= \langle v_x, \Delta v_x \rangle_{l^2} \stackrel{\text{by (6.6)}}{=} \langle v_x, \delta_x - \delta_o \rangle_{l^2} \\
 &= v_x(x) - v_x(o) = n - 0 \\
 &= \#(\gamma(x))
 \end{aligned}$$

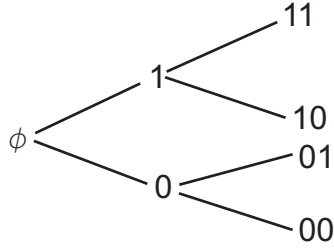


FIGURE 4. Case 1

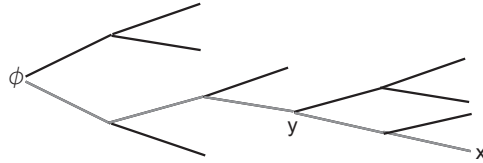


FIGURE 5. Case 2

$$\begin{aligned}
 \langle v_x, v_y \rangle_E &\stackrel{\text{all finite functions, see (6.4)}}{=} \langle v_x, \Delta v_y \rangle_{l^2} \\
 &\stackrel{\text{by (6.6)}}{=} \langle v_x, \delta_y - \delta_o \rangle_{l^2} \\
 &\stackrel{\text{by (6.4)}}{=} v_x(y) - v_x(o) = 0 \\
 &\stackrel{\text{by (6.4)}}{=} \#(\gamma(x) \cap \gamma(y))
 \end{aligned}$$

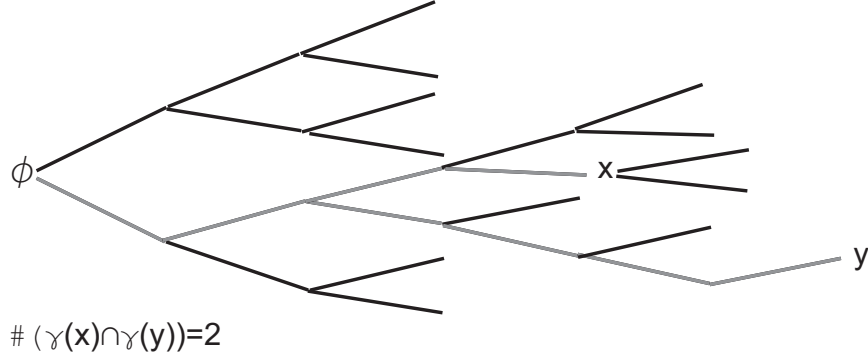


FIGURE 6

□

Set  $M = \langle (v_x, v_y)_E \rangle = (\#(\gamma(x) \cap \gamma(y)))$ ,  $x, y \in G^0 \setminus (o)$ . Given

$$\text{spec}_{l_2}(M) = (\text{spec}_{\mathcal{H}_E}(\Delta))_{\text{spec}_{\mathcal{H}_E}(\Delta) \rightarrow \infty}^{-1}$$

From our theorem above  $\Delta$  (unbounded spectrum),  $\text{closure} \Delta = \mathcal{V}$ ,  $\mathcal{V} \subset \mathcal{H}_E$ .

**Corollary 6.3.**  $\forall \epsilon \exists F \subset G^0 \setminus (o)$  finite,  $\exists \lambda \in \text{spec}_{l_2}(M_F)$  such that  $\lambda < \epsilon$ .

Note

$$\begin{aligned} M_F &= \langle (v_x, v_y)_E \rangle_{x,y \in F} \\ &= (\#(\gamma(x) \cap \gamma(y)))_{x,y \in F} \end{aligned}$$

and  $0 \notin \text{spec}_{l_2}(M_F)$ .

Problem: Find a systematic way of selecting  $F$ . See Figure 8.

It is much easier to find  $M_F$  with  $\text{spec}_{l_2} M_F \rightarrow \infty$ .

**Example 6.4.**

$$(6.8) \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 1 \\ 1 & n \end{pmatrix},$$

$$\lambda_n^\pm = \frac{n+1 \pm \sqrt{(n+1)^2 - 4(n-1)}}{2}, \quad \text{and}$$

$$\begin{aligned} \lambda_n^- &= \frac{n+1 - \sqrt{(n+1)^2 - 4(n-1)}}{2} \\ &= \frac{2(n-1)}{n+1 - \sqrt{n^2 - 2n + 2}} \\ &\xrightarrow{n \rightarrow \infty} 1 \end{aligned}$$

Actually both expand part of  $\text{spec}_{l_2} M$  as intervals.

		Column Index															
		0	1	00	01	10	11	000	001	010	011	100	101	110	111		
Index	0	1	0	1	1	0	0	1	1	1	1	0	0	0	0		
	1	0	1	0	0	1	1	0	0	0	0	1	1	1	1		
	00	1	0	2	1	0	0	2	2	1	1	0	0	0	0		
	01	1	0	1	2	0	0	1	1	2	2	0	0	0	0		
	10	0	1	0	0	2	1	0	0	0	0	2	2	1	1		
	11	0	1	0	0	1	2	0	0	0	0	1	1	2	2		
	000	1	0	2	1	0	0	3	2	1	1	0	0	0	0		
	001	1	0	2	1	0	0	2	3	1	1	0	0	0	0		
	010	1	0	1	2	0	0	1	1	3	2	0	0	0	0		
	011	1	0	1	2	0	0	1	1	2	3	0	0	0	0		
	100	0	1	0	0	2	1	0	0	0	0	3	2	1	1		
	101	0	1	0	0	2	1	0	0	0	0	2	3	1	1		
	110	0	1	0	0	1	2	0	0	0	0	1	1	3	2		
	111	0	1	0	0	1	2	0	0	0	0	1	1	2	3		

FIGURE 7. Google matrix.

## 7. KARHUNEN-LOÈVE

**Definition 7.1.**  $F \subset G^0 \setminus (o)$   $F$  finite,  $G^0$  infinite,  $(G, c)$  fixed. Fix  $o \in G^0 \rightsquigarrow \Delta = \Delta_c$ ,  $\mathcal{H}_E$  energy Hilbert space  $\langle v_x, f \rangle_E = f(x) - f(o)$ ,  $\forall f \in \mathcal{H}_E$ .

**Definition 7.2.**  $\mathcal{H}_E(F) := \overline{\text{span}}_{x \in F} \{v_x\}$

General  $(G, c) \rightarrow$  Fix point  $o \in G^0 \rightarrow \Delta_c$ ,  $\mathcal{H}_E$ ,  $G^0$  infinite. Fix  $v_x$ , for  $x \in G^0 \setminus (o)$ , determined from Riesz applied to  $\mathcal{H}_E$ .

$$(7.1) \quad \langle v_x, f \rangle_E = f(x) - f(o), \quad x \in G^0 \setminus (o),$$

and consider the infinite matrix

$$(7.2) \quad M = (\langle v_x, v_y \rangle_E), \quad x, y \in G^0 \setminus (o)$$

and its finite  $F \times F$  submatrices

$$(7.3) \quad M_F = (\langle v_x, v_y \rangle_E), \quad x, y \in F$$

so the matrices are  $\infty \times \infty$ , or  $|F| \times |F|$ .

**Important Formula.** Observe

$$(7.4) \quad \langle v_x, v_y \rangle_E = v_y(x) - v_y(o)$$

$$(7.5) \quad = v_y(x) = v_x(y);$$

in other words  $k_E(x, y) := v_x(y)$  is a reproducing kernel.

Since  $x, y \in G^0 \setminus (o)$ ; and  $v_x : G^0 \rightarrow \mathbb{R}$  (i.e., real valued) convention:  $v_x(o) = 0$ . Diagonalization motivated by the classical Karhunen-Loève theorem, see [14] and [17].

**7.1. Finite-dimensional Approximation.** Apply the Spectral Theorem to  $M$  and  $M_F$ . The Hilbert space is  $l^2(G^0)$  or  $l^2(F) \simeq \mathbb{C}^{|F|}$  with  $\langle \xi, \eta \rangle_2 = \sum_x \overline{\xi_x} \eta_x$  as inner product.

For  $(M_F, l^2(F))$  the spectrum is always discrete, and for some cases i.e.,  $(M, l^2(G^0))$  it may not be discrete.

In the discrete case, there exists  $M = Mf$  ONB  $\xi_1, \xi_2, \dots \in l^2(G^0)$  or  $l^2(F)$  eigenvectors

$$(7.6) \quad \langle \xi_j, \xi_k \rangle_2 = \sum_x \overline{\xi_j(x)} \xi_k(x) = \delta_{j,k} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$

$\xi = \xi^F \in l^2(F)$  such that

$$(7.7) \quad M^F \xi_j = \lambda_j \xi_j, \quad \lambda_1 \geq \lambda_2 \geq \dots > 0 \quad \xi_j \in l^2(F), \|\xi_j\|_2 = 1 \quad (\text{in the } F\text{-case})$$

In the infinite case  $\text{spec}(M)$  for  $l^2(G^0)$  may accumulate both at 0 and at  $\infty$ .

Since  $M_{xy}^F = \langle v_x, v_y \rangle_{E\Lambda} \in \mathbb{R}$ , we may take all  $\xi_k : G^0 \rightarrow \mathbb{R}$  real valued. Fix  $F \subset G^0 \setminus (0)$ :  $\xi_k^F \in l^2(F)$ . Set

$$w_k^F(\cdot) = \frac{1}{\lambda_k} \sum_{x \in F} \xi_k^F(x) v_x(\cdot)$$

i.e.,

$$(7.8) \quad w_k^F(z) = \frac{1}{\lambda_k} \sum_{x \in F} \xi_k^F(x) v_x(z), \quad \forall z \in G^0$$

**Lemma 7.3.** *If  $F$  is fixed then  $\xi_k^F \in l^2(F)$  is an ONB. Set*

$$(7.9) \quad M_F \xi_k^F = \lambda_k^F \xi_k^F,$$

then

$$w_k^F : G^0 \rightarrow \mathbb{R}, \quad w_k^F \in \mathcal{H}_E$$

is an extension of  $\xi_k^F : F \rightarrow \mathbb{R}$  from  $F$  to  $G^0$ .

*Proof.* By (7.8) if  $z \in F$ :

$$\begin{aligned} w_k^F(z) &= \frac{1}{\lambda_k} \sum_{x \in F} v_x(z) \xi_k^F(x) \\ &\stackrel{\text{by(4.4)}}{=} \frac{1}{\lambda_k} \sum_{x \in F} M_{z,x} \xi_k^F(x) \\ &\stackrel{\text{by(7.9)}}{=} \frac{1}{\lambda_k} (M_F \xi_k^F)_z \\ &= \frac{\lambda_k}{\lambda_k} \xi_k^F(z) \\ &= \xi_k^F(z) \end{aligned}$$

□

**Lemma 7.4.** *Fix  $F \subset G^0 \setminus (0)$  finite, and let*

$$(7.10) \quad w_k^F(\cdot) = \frac{1}{\lambda_k} \sum_{x \in F} \xi_k^F(x) v_x(\cdot), \quad k \in (1, 2, \dots, |F|) \quad \text{as in Lemma 7.3.}$$

Then  $\{w_k^F\}_k$  is an orthonormal system in  $H_E$  (thus in each of the Hilbert spaces) i.e., with the inner product

$$(7.11) \quad \langle u, v \rangle_E := \frac{1}{2} \sum_{\text{all } xy} \sum_{x \sim y} c_{xy} (\overline{u(x)} - \overline{u(y)}) (\overline{v(x)} - \overline{v(y)}).$$

We have

$$(7.12) \quad \langle w_j^F, w_k^F \rangle_E = \frac{1}{\lambda_k} \delta_{j,k} = \begin{cases} \frac{1}{\lambda_k} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

*Proof.* We have:

$$\begin{aligned} \langle w_j^F, w_k^F \rangle_E &\stackrel{\text{by (7.10)}}{=} \frac{1}{\lambda_j \lambda_k} \sum_{xy \in F} \xi_j^F(x) \xi_k^F(y) \langle v_x, v_y \rangle_E \\ &\stackrel{\text{by (7.3)}}{=} \frac{1}{\lambda_j \lambda_k} \sum_{x \in F} \xi_j^F(x) \sum_{y \in F} M_{xy}^F \xi_k^F(y) \\ &= \frac{1}{\lambda_j \lambda_k} \langle \xi_j^F, M^F \xi_k^F \rangle_2 \\ &= \frac{1}{\lambda_j} \langle \xi_j^F, \xi_k^F \rangle_2 \\ &\stackrel{\text{by (7.6)}}{=} \frac{1}{\lambda_j} \delta_{j,k} = \begin{cases} \frac{1}{\lambda_j} & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}. \end{aligned}$$

□

Set  $u_j^F = \sqrt{\lambda_j} w_j^F$ ; then

$$\langle u_j^F, u_k^F \rangle_E = \delta_{j,k}, \quad j, k \in \{1, 2, \dots, |F|\}.$$

**7.2. Normalization.** The following different normalization  $u_j^F = \sqrt{\lambda_j} w_j^F$  satisfies

$$(7.13) \quad \|u_j^F\|_{H_E} = 1,$$

so

$$(7.14) \quad u_j^F(\cdot) = \frac{1}{\lambda_j} \sum_{x \in F} \xi_j(x) v_x(\cdot).$$

Note that the

$$(7.15) \quad u_j^F|_F = \sqrt{\lambda_j} \xi_j(\cdot) \quad \text{on } F.$$

**7.3. Projection Valued Measures.** Set

$$(7.16) \quad P^F(\lambda_j) := |u_j^F\rangle\langle u_j^F|; \quad \text{Dirac notation for rank-one projection,}$$

so a projection in  $H_E$  on the one-dimensional subspace  $\mathbb{C}u_j^F$ . Then  $P^F(\cdot)$  is an orthonormal projection system, and it has a limit as  $F \rightarrow \infty$  which is a global spectral measure.

We claim that

$$(7.17) \quad s_\Delta(u_j^F) = \langle u_j^F, \Delta u_j^F \rangle \in \text{spec}_{H_E}(\Delta v)$$



**Lemma 7.5.** (*Spectral Reciprocity*)

$$(7.18) \quad s_{\Delta}(u_j^F) = \frac{1}{\sqrt{\lambda_j}} \left( 1 + \left| \sum_{x \in F} \xi_j(x) \right|^2 \right)$$

*Proof.*

$$\begin{aligned} s_{\Delta}(u_j^F) &\stackrel{\text{by(7.14)}}{=} \frac{1}{\sqrt{\lambda_j}} \left\langle \left( \sum_{x \in F} \xi_j(x) v_x \right), \Delta \left( \sum_{y \in F} \xi_j(y) v_y \right) \right\rangle_E \\ &= \frac{1}{\sqrt{\lambda_j}} \sum_{x \in F} \sum_{y \in F} \xi_j(x) \xi_j(y) \langle v_x, \Delta v_y \rangle_E \\ &= \frac{1}{\sqrt{\lambda_j}} \sum_{x \in F} \sum_{y \in F} \xi_j(x) \xi_j(y) (\delta_x(y) + 1) \\ &= \frac{1}{\sqrt{\lambda_j}} \left( \|\xi_j\|_2^2 + \left| \sum_{x \in F} \xi_j(x) \right|^2 \right) \\ &\stackrel{\text{by(7.13)}}{=} \frac{1}{\sqrt{\lambda_j}} \left( 1 + \left| \sum_{x \in F} \xi_j(x) \right|^2 \right) \end{aligned}$$

□

**Example 7.6.**

$$M^F = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad sp = \{1, 3\},$$

same spectrum, but different  $M^F$ .

$$\xi_{\lambda=1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_{\lambda=3} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \langle \xi_1 \rangle = \langle \xi_3 \rangle = 1.$$

Set  $R_F(\lambda) := \frac{1}{\lambda}(1 + |\langle \xi_{\lambda}^F \rangle|^2)$ . Then  $\langle u_{\lambda}, \Delta u_{\lambda} \rangle = R_F(\lambda)$ ; see Lemma 7.5.  
In the examples:

$$M^F = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \begin{cases} R_F(1) = 1 \\ R_F(3) = \frac{1}{3}(1+2) = 1 \end{cases} \quad \text{smaller for } M^F \text{ off-diagonal.}$$

$$M^F = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad \begin{cases} R_F(1) = \frac{1}{1}(1+1^2) = 2 \\ R_F(3) = \frac{1}{3}(1+1^2) = \frac{2}{3} \end{cases}$$

$$M^F = \begin{pmatrix} 3 & 3 \\ 3 & 7 \end{pmatrix}, \quad \lambda_{\pm} = 5 \pm \sqrt{13}$$

$$M^F = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}, \quad \lambda_{\pm} = \frac{7 \pm \sqrt{5}}{2}$$

$$\lambda = 5 \pm \sqrt{13} \Rightarrow R(\lambda) = \frac{1}{\lambda} + \frac{\lambda}{1 + \left(\frac{2 - \sqrt{13}}{3}\right)^2} < \frac{1}{\lambda} + \lambda$$

$$M^F = \begin{pmatrix} 1 & 1 \\ 1 & m \end{pmatrix}, \quad m \rightarrow \infty \quad \lambda_{\pm} = \frac{m+1 \pm \sqrt{(m+1)^2 - 4(m-1)}}{2}$$

In both cases, we have:

$$R_F(\lambda) = \frac{1}{\lambda} + \frac{\lambda}{1 + (\lambda - 1)^2}$$

$$R_F(\lambda_-) = \frac{\lambda_+}{m - 1} + \frac{\lambda_-}{1 + (\lambda_- - 1)^2} \sim 4 \text{ as } m \rightarrow \infty.$$

	0	1	10	11	100	101	110	111
0	1	0	0	0	0	0	0	0
1	0	1	1	1	1	1	1	1
10	0	1	2	1	2	2	1	1
11	0	1	1	2	1	1	2	2
100	0	1	2	1	3	2	1	1
101	0	1	2	1	2	3	1	1
110	0	1	1	2	1	1	3	2
111	0	1	1	2	1	1	2	3

FIGURE 8

We now illustrate by an example that points in the spectrum can go into  $\infty$ :

$$s_\Delta(u) = \frac{\langle u, \Delta u \rangle}{\|u\|_E^2} \rightarrow \infty$$

If  $\lambda \in \text{spec}_{l^2}(M^F)$  set  $u_\lambda = \frac{1}{\sqrt{\lambda}} \sum_{x \in F} \xi_\lambda(x) v_x(\cdot)$   $M\xi_\lambda = \lambda\xi_\lambda$ ,  $\|\xi_\lambda\|_2 = 1 \Rightarrow \|u_\lambda\|_E = 1$  so  $s_\Delta(u) = \langle u, \Delta u \rangle = \frac{1}{\lambda}(1 + \|P_\lambda e\|_2^2)$ ,  $e = e_F = \chi_F(\cdot)$ ,  $P_\lambda e = \langle \xi_\lambda, e \rangle_2 \xi_\lambda$

**Theorem 7.7.** *The truncated operators  $P_{\mathcal{H}_E(F)} \Delta_{\mathcal{D}_E} P_{\mathcal{H}_E(F)}$  has spectral growth  $\simeq \mathcal{O}(\#F)$ ; so  $\Delta_E$  is unbounded in  $\mathcal{H}_E$ .*

*Proof.* The idea is to perform a diagonalization of an infinite matrix  $(M_{x,y})$   $x, y \in G^0 \setminus (o)$ ; a method inspired by Karhunen-Loève [14, 17]. Here  $F \subset G^0 \setminus (o)$  is fixed and finite. The following computations refer to  $F$ :  $(\xi_k)$  is an ONB in  $l^2(F)$  satisfying (7.19) below; set  $w_k = \frac{1}{\lambda_k} \sum_{x \in F} \xi_k(x) v_x$ , and  $v_k = \sqrt{\lambda_k} w_k = \frac{1}{\sqrt{\lambda_k}} \sum_{x \in F} \xi_k(x) v_x$ . Then

$$(7.19) \quad M^F \xi_k = \lambda_k \xi_k, \quad \text{and} \quad \langle \xi_j, \xi_k \rangle_{l^2(F)} = \delta_{j,k}.$$

We may now compute the matrices:

$$\begin{aligned} \langle u_j, \Delta u_k \rangle_E &= \frac{1}{\sqrt{\lambda_j \lambda_k}} \sum_{F \times F} \xi_j(x) \xi_k(x) \langle v_x, \Delta v_y \rangle_E \\ &= \frac{1}{\sqrt{\lambda_j \lambda_k}} \sum_{F \times F} \xi_j(x) \xi_k(x) (\delta_x(y) + 1). \end{aligned}$$

Set  $\sum_{x \in F} \xi_j(x) = \langle \xi_j, e \rangle_2 = \langle \xi_j \rangle$  where  $e = e^F = \chi_F$ .

Then the matrix entries are: Off-Diagonal:

$$\langle u_j, \Delta u_k \rangle_E = \frac{1}{\sqrt{\lambda_j \lambda_k}} (\delta_{j,k} + \langle \xi_j \rangle \langle \xi_k \rangle);$$

and Diagonal:

$$\langle u_j, \Delta u_j \rangle_E = \frac{1}{\lambda_j} (1 + \langle \xi_j \rangle^2).$$

We further used the following identity:

$$\begin{aligned} \sum_{F \times F} \delta_x(y) \xi_j(x) \xi_k(y) &= \langle \xi_j, \xi_k \rangle_{l^2(F)} \\ &= \delta_{j,k} \quad \text{by (7.19)}. \end{aligned}$$

This may be summarized in the following matrix form:

$$\begin{pmatrix} \frac{1}{\lambda_1}(1 + \langle \xi_1 \rangle^2) & \frac{\langle \xi_1 \rangle \langle \xi_2 \rangle}{\sqrt{\lambda_1 \lambda_2}} & \frac{\langle \xi_1 \rangle \langle \xi_3 \rangle}{\sqrt{\lambda_1 \lambda_3}} & \cdots \\ \frac{\langle \xi_1 \rangle \langle \xi_2 \rangle}{\sqrt{\lambda_1 \lambda_2}} & \frac{1}{\lambda_2}(1 + \langle \xi_2 \rangle^2) & \frac{\langle \xi_2 \rangle \langle \xi_3 \rangle}{\sqrt{\lambda_2 \lambda_3}} & \cdots \\ \frac{\langle \xi_1 \rangle \langle \xi_3 \rangle}{\sqrt{\lambda_1 \lambda_3}} & \frac{\langle \xi_2 \rangle \langle \xi_3 \rangle}{\sqrt{\lambda_2 \lambda_3}} & \frac{1}{\lambda_3}(1 + \langle \xi_3 \rangle^2) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

If for some  $\delta \in \mathbb{R}^+$ ,  $\lambda_j \geq \delta$ , i.e., bounded from below, then the operator

$$\begin{pmatrix} \frac{1}{\lambda_1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{\lambda_3} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \end{pmatrix}$$

is bounded. So

$$(7.20) \quad \left( \frac{1}{\sqrt{\lambda_j \lambda_k}} \langle \xi_j \rangle \langle \xi_k \rangle \right)$$

must be unbounded, i.e.,  $\|\cdot\|_{l^2(F) \rightarrow l^2(F)} \rightarrow \infty$ . But (7.20) is a rank-one operator;

$$|\rho \rangle \langle \rho|, \quad \rho = \rho^F; \quad F \subset G^0 \setminus (o) \text{ is fixed where } \rho = (e_j) \in l^2(1, 2, \dots, \#F),$$

i.e.,  $\rho = \rho^F$  and  $\rho_j^F = \frac{\langle \xi_j^F \rangle}{\sqrt{\lambda_j}}$ ,  $\lambda_j = \lambda_j^F$ .

Now,

$$\|\rho^F\|_{l^2(1, \dots, \#F)}^2 = \sum_{j=1}^{\#F} \frac{\langle \xi_j^F \rangle^2}{\lambda_j}.$$

So in conclusion

$$\lim_{F \rightarrow \infty} \sum_{j=1}^{\#F} \frac{\langle \xi_j^F \rangle^2}{\lambda_j(F)} = \infty.$$

Pick  $\delta \in \mathbb{R}_+$  and assume  $\lambda_j^F \geq \delta$ . Then we need

$$\lim_{F \rightarrow \infty} \sum_{j=1}^{\#F} \langle \xi_j^F \rangle^2 = \infty.$$

We have  $\xi_\lambda^F(j) = \xi_j$ ,  $M^F \xi_\lambda^F = \lambda_j^F \xi_\lambda^F$ ,  $\|\xi_\lambda\|_2 = 1$ ,  $\langle \xi_\lambda^F \rangle = \sum_{x \in F} \xi_\lambda^F(x)$ , and  $\sum_\lambda \langle \xi_\lambda^F \rangle^2 = \#F$ ; so indeed

$$\lim_{F \rightarrow \infty} \sum_\lambda \langle \xi_\lambda^F \rangle^2 = \lim_F \#F = \infty.$$

Conclusion:  $\text{spec}_{\mathcal{H}_E(N)}(\Delta_{\mathcal{D}_E}) \sim (\#F) \rightarrow \infty$

□

*Acknowledgements.* The authors are please to acknowledge helpful discussions, both recent and not so recent, with John Benedetto, B. Brenken, Ilwoo Cho, D. Dutkay, Keri Kornelson, Kathy Merrill, P. Muhly, Judy Packer, Erin Pearse, Steen Pedersen, Gabriel Picioroaga, Karen Shuman.

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