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Algebraic Multigrid Preconditioner for a Finite Element Method in TM Electromagnetic Scattering

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Abstract — The finite element method (FEM) is applied on an arbitrarily shaped and perfectly conducted cylindrical scatterer whose cross section and material properties are uniform along its infinite axis, say the z axis here. The scatterer is enclosed within a fictitious boundary via a simple first order absorbing boundary condition (ABC). The Algebraic Multigrid (AMG) method is employed as a preconditioner in order to accelerate the convergence rate of the Krylov iterations. Our experimental results suggest much faster convergence compared to the non preconditioned Krylov subspace solver, and hence significant reduction to the overall computational time.

Keywords: Algebraic Multigrid; Finite elements; Absorbing boundary condition.

1 Introduction

It is widely known that the implementation of the standard finite element method (FEM) produces a sparse coefficient matrix. Due to significant growth of computer time and memory during the elimination process direct methods are not preferable for solving large sparse linear systems. Hence systems of this kind require iterative methods, such as Krylov subspace methods [15, 17], for their solution. The convergence rate of an iterative method depends on the spectral properties of the coefficient matrix. Preconditioning is a well-known technique used to accelerate the convergence of iterative methods [15, 17]. Hence, developing efficient preconditioners has been one of the major research interests in many applications [4, 8, 12, 13, 14]. In this work, scattering by a conducting cylinder is considered and the FEM is used to construct the resulting linear system. In our numerical experiments, BiCGSTAB (BiConjugate Gradient Stablized) is used as a choice of a linear solver and AMG (Algebraic Multigrid) method is employed as a preconditioner for accelerating the convergence.

AMG has been developed to solve large problems posed on unstructured grids since it doesn't require geometric grid information. To date, many AMG based on element interpolation and smooth aggregation have been proposed [3, 6, 18] and

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are shown to be effective and robust. A variant of smoothed aggregation AMG preconditioner was developed by Leem et al [13] to solve large, sparse, and real-valued saddle point systems from meshfree discretizations. In [14], the smoothed sggregation AMG preconditioner was modified to successfully accelerate the convergence of large, dense, and complex-valued linear systems encountered in scattering by dielectric objects.

It is important to mention here that since we are dealing with an open-region scattering problem, the infinite region exterior to the scatterer must be truncated with an artificial boundary. In order to obtain a unique finite element solution a boundary condition will be introduced. The purpose of this condition, which is called absorbing boundary condition (ABC), is to allow the scattered wave to propagate to infinity without causing any reflections back to the object. The (ABCs) are applied at the artificial boundary and hence their use does not introduce additional unknowns and retains sparsity. In this work, for simplicity, we will use first-order (ABCs) [9].

We organize our paper as follows. In section 2, we formulate the problem and summarize the well known finite elements steps required for the generation of the corresponding linear system. In section 3, we discuss the construction of the AMG preconditioner. Numerical experiments that yield a dramatic reduction to the number iterations and computational time are presented in section 4.

2 Formulation of the problem

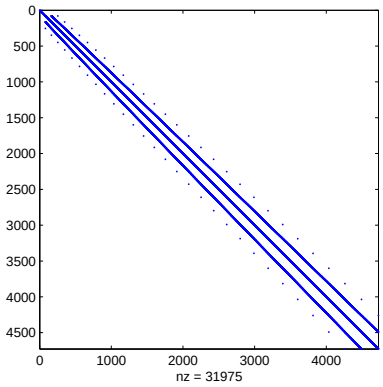
It is well known that the propagation of time-harmonic fields in a homogeneous medium, in the presence of a perfect conductor D, is modeled by the exterior boundary value problem (direct obstacle scattering problem)

$$\triangle_2 u_z(x) + k^2 u_z(x) = 0, \quad x \in \Re^2 \setminus \bar{D} \tag{2.1}$$

$$u_z(x) + u^i(x) = 0, \quad x \in \partial D \tag{2.2}$$

where k is a real positive wavenumber and u^i is a given incident plane wave polarized along the $z-$direction, that in the presence of D will generate the scattered field u_z which will also be polarized along the $z-$ direction.
As indicated in the introduction the infinite region exterior to the scatterer needs to be truncated by the introduction of a fictitious boundary. Consequently, a boundary condition must be applied at this boundary so the field scattered by the cylinder must continue propagating toward infinity without disturbance. In other words, such a condition should minimize any reflections from the boundary. This kind of boundary conditions are called absorbing boundary conditions (ABC). As indicated in [10] an ideal boundary condition is one that possesses zero reflection for all angles of incidence. However, absorbing boundary conditions lead to localized relations between the boundary fields (i.e. they are approximate) hence zero reflection for all incident angles is impossible.
For simplicity in the present problem we will be using a first order ABC. According

Figure 1: Sparsity Pattern of the coefficient matrix K when $n = 4727$.



to Bayliss at al. [1, 2], the (ABC) employed here is given by

$$\frac{\partial u_z}{\partial n} + (ik + \frac{k(s)}{2\rho})u_z = 0 \tag{2.3}$$

where n denotes the outward unit vector normal to the artificial boundary, s is the arc length measured along the boundary, and $k(s)$ is the curvature of the boundary at s.

Details about the major steps of the solution of the problem above can be found in [9] and include discretization of the domain using linear triangular elements, generation of proper mesh data, construction of the elemental equations, assembly of these element matrices and vectors into the global matrix and right hand side vector, imposition of Dirichlet boundary conditions on the surface of the cylinder, and finally solution of the matrix system to obtain the total acoustic field at the nodes of the domain.

3 Smoothed Aggregation AMG Preconditioner

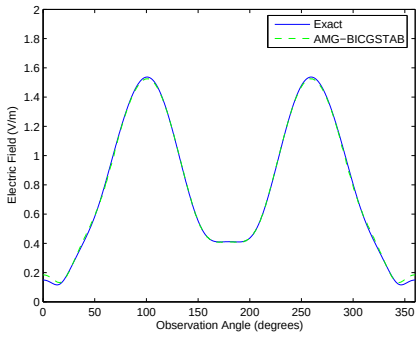
We begin our discussion by introducing the basic framework of AMG. Consider the complex linear system

$$K\mathbf{u} = \mathbf{b}, \tag{3.4}$$

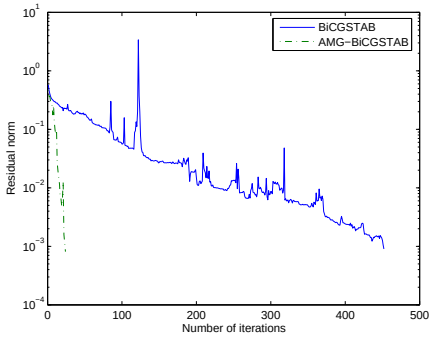
where K is a large, and sparse $n \times n$ matrix. AMG preconditioner will be constructed based on the coefficient matrix K. Note that AMG does not require to access the physical grids of problems. With "grids" we mean sets of indices of the unknown variables. Hence the grid set for (3.4) is $\Omega = \{1, 2, \cdots, n\}$, since the unknown

Table 1: Size of Coarse-grid Operator

n	K_0	K_1	K_2	K_3
4727	4727	659	133	
18563	18563	2454	349	
73569	73569	9036	952	361



(a) Comparison of exact and approximate solutions for $n = 4,727$



(b) Convergence history for $n = 4,727$

Figure 2: Solution Comparison (left) and Convergence History (right)

vector $\mathbf{u}$ in (3.4) has components $u_1, u_2, ..., u_n$. The graph of the matrix K can be defined as follows: A node in the graph represents a row and the normalized edge weights $\overline{\omega}_{ij}$ can be computed as follows:

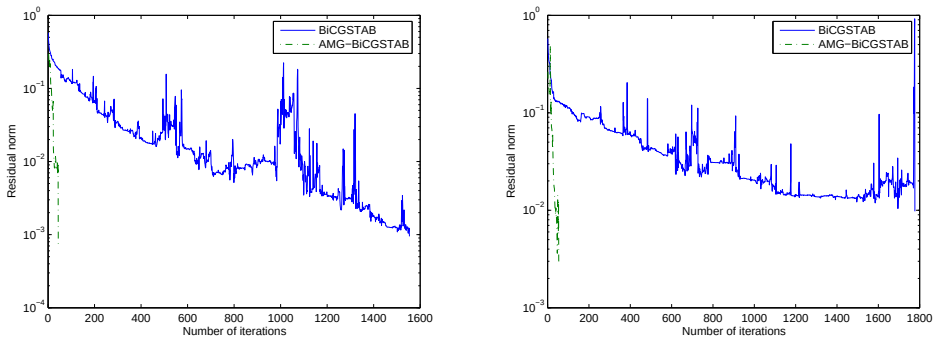
$$\overline{\omega}_{ij} = |k_{ij}|/\sqrt{|k_{ii}| \cdot |k_{jj}|}. \tag{3.5}$$

The main idea of AMG is to remove the smooth error by coarse grid correction, where smooth error is the error not eliminated by relaxation on the fine grid, which also can be characterized by small residuals [5].

In order to develop the multi-grid algorithm, we consider the sets of grids in each level. The number 0 stands for the finest-grid level. Then the numbers $1, 2, \cdots, l_{max}$ represent the corresponding coarse-grid levels. Hence, the original equation (3.4) can be written as $K_0\mathbf{u}_0 = \mathbf{b}_0$ and the set of finest grid set is $\Omega_0 = \Omega$.

AMG can be implemented in two main phases, so called the setup phase and the solve phase. The setup phase includes the following tasks:

- Create the coarse grid sets Ω_{l+1}.
- Construct interpolation operator I_{l+1}^{l}, and restriction operators I_{l}^{l+1}.
- Construct the coarse grid operator K_{l+1}.



(a) $n = 18,563$ (b) $n = 73,569$

Figure 3: Convergence Histories

In general, the restriction operator I_l^{l+1} is defined by the transpose of the interpolation operator I_{l+1}^l, i.e., $I_l^{l+1} = (I_{l+1}^l)^T$ and the coarse grid operator K_{l+1} is constructed from the fine grid operator K_l by the Galerkin approach:

$$K_{l+1} = I_l^{l+1} \, K_l \, I_{l+1}^l, \tag{3.6}$$

so that AMG satisfies the principle that the coarse-grid problem needs to provide a good approximation to fine-grid error in the range of interpolation [5].

We will now provide more information about the setup process. Two main tasks in the setup phase are to find a suitable coarsening strategy and an effective interpolation operator I_{l+1}^l. The creation of the coarse-grid sets Ω_l, where $l = 1, 2, \cdots, l_{max}$ is based on a combinatorial clustering algorithm developed by Vaněk, Mandel and Brezina in [18] with normalized edge weights, $\overline{\omega}_{ij}$.

The first step of their coarsening algorithm iterates through the nodes $\Omega_0 = \{1, 2, \cdots, n\}$ creating clusters $\{j \mid \overline{\omega}_{ij} \geq \eta_1\}$ for a given tolerance $\eta_1 > 0$, provided no node in $\{j \mid \overline{\omega}_{ij} \geq \eta_1\}$ is already a cluster. Two nodes i and j are said to be strongly connected if $\overline{\omega}_{ij} \geq \eta_1$. In the second step, unassigned nodes are assigned to a cluster from step one to which the node is strongly connected, if any. In the last step, the remaining nodes are assigned to clusters consisting of strong neighborhoods intersecting with the set of remaining nodes. This aggregation process will create the corresponding next coarse level grid set $\Omega_1 = \{C_1^l, C_2^l, \cdots, C_{n_l}^l\}$. Each C_k^l is called a cluster. By repeating same process, $\Omega_2, \Omega_3, \cdots, \Omega_{l_{max}}$ are obtained.

As explained earlier, the next important task in AMG is to construct an interpolation operator I_{l+l}^l. In smoothed aggregation AMG [13, 14], we solve a local linear system to obtain an interpolation vector that interpolates a value for a coarse-grid cluster onto its neighborhood. Assume that a set of grid points at level l is $\Omega_l = \{C_1^l, C_2^l, \cdots, C_{n_l}^l\}$. For each cluster C_k^l, we define a neighborhood N_k^l as

$$N_k^l = \{j \notin C_k^l \mid \overline{\omega}_{ij} \geq \eta_2, i \in C_k^l\},$$

Performing the coarsening process explained earlier yields the next coarse-grid set, $\Omega_{l+1} = \{C_1^{l+1}, C_2^{l+1}, \cdots, C_{n_{l+1}}^{l+1}\}$. Suppose that C_k^{l+1} is obtained by aggregating $\{C_i^l, C_j^l\}$ from the grid set Ω_l. Furthermore, assume that clusters C_i^l, and C_j^l have the neighborhoods $N_i^l = \{C_s^l, C_t^l\}$ and $N_j^l = \{C_u^l, C_v^l\}$, respectively.

The corresponding interpolation operator from level $l+1$ to l, I_{l+1}^l, is computed column-by-column by the following procedure: The kth column $\mathbf{p}_k$ of the interpolation operator $I_{l+1}^l = [\mathbf{p}_1|\mathbf{p}_2|\cdots|\mathbf{p}_{n_{l+1}}]$ is obtained by solving the following small local system:

$$L_{C_k^l}\ \mathbf{p}_k = \mathbf{e}, \tag{3.7}$$

where $\mathbf{e}$ is the vector given by

$$\mathbf{e}_i = \begin{cases} 1 & \text{if } i \in C_k^l, \\ 0 & \text{otherwise,} \end{cases} \tag{3.8}$$

and the local matrix $L_{C_k^l}$ is given by

$$L_{C_k^l} = \begin{bmatrix} K_{C_k^l C_k^l} & K_{C_k^l N_k^l} \\ K_{N_k^l C_k^l} & K_{N_k^l N_k^l} \end{bmatrix}. \tag{3.9}$$

Note that $K_{IJ} = [\ k_{ij} \mid i \in I, j \in J\ ]$ indicates an $|I| \times |J|$ matrix where I and J are sets of nodes.

After completion of the AMG setup, either V-, or W-cycle is typically used for the preconditioning step. In the sequel, the AMG V-cycle algorithm 3.1 is utilized as a preconditioner to the iterative methods used.

Algorithm 3.1 AMG V-Cycle

$$\mathbf{u}_l \leftarrow \mathbf{AMGV}(K_l, \mathbf{u}_l, \mathbf{b}_l)$$

if $\Omega_l =$ *coarsest grid,* **then**

$\mathbf{u}_l \leftarrow$ *Solve* $(K_l\mathbf{u}_l = \mathbf{b}_l)$ *using Direct Method*

else

$\mathbf{x}_l \leftarrow$ *Relax* ν_1 *times on* $K_l\mathbf{u}_l = \mathbf{b}_l$ *on* Ω_l *with initial guess* $\mathbf{u}_l$
$\mathbf{b}_{l+1} \leftarrow I_l^{l+1}(\mathbf{b}_l - K_l\mathbf{u}_l)$
$\mathbf{u}_{l+1} \leftarrow 0,$
$\mathbf{u}_{l+1} \leftarrow \mathbf{AMGV}(K_{l+1}, \mathbf{u}_{l+1}, \mathbf{b}_{l+1}).$
Correct $\mathbf{u}_l = \mathbf{u}_l + I_{l+1}^l\mathbf{u}_{l+1}.$
$\mathbf{u}_l \leftarrow$ *Relax* ν_1 *times on* $K_l\mathbf{u}_l = \mathbf{b}_l$ *on* Ω_l *with* $\mathbf{u}_l$.

endif

Note that we employ Gauss-Seidel iterations for the relaxation scheme. More details on classical AMG can be found in [5], and [7].

Table 2: Convergence Results

n	ϵ	No preconditioner		AMG preconditioner			
		# of iter	total time	AMG setup	# of iter	iter time	total time
4,727	1.0e-03	452	2.86	0.40	24	2.40	2.80
18,563	1.0e-03	1554	40.89	2.56	44	17.65	20.21
73,569	1.0e-02	1777	187.42	17.86	38	61.34	79.20

Table 3: Thresholds

n	η_1	η_2
4,727	0.01	0.05
18,563	0.01	0.1
73,569	0.0001	0.1

4 Numerical Results

A two dimensional FEM algorithm [9] is used to solve the scattering problem described in section 2. In our numerical experiments the systems were solved via the BiConjugate Gradient Stabilized method (BiCGSTAB). The radius of the circular conducting cylinder, i.e., the inner circular boundary, is $\lambda/2$ whereas the radius of the ABC boundary, i.e., the outer cylinder boundary, is $3\lambda/2$. In other words, the first-order ABC is imposed at a distance of one wavelength away from the outer surface of the object. If the discretization size is $h = 0.4$ wavelengths, then the size of the corresponding coefficient matrix K is $n = 4727$ unknowns. Figure 3 shows the comparison between the total field calculated using the analytical solution and the field computed via the AMG preconditioned BiCGSTAB algorithm while figure 1 displays the sparsity pattern of the coefficient matrix K when the number of unknowns is $n = 4727$. Note that K is very sparse with density of only 1.4%. In addition, figure 2(b) shows that the number of iterations has been significantly reduced from 452 (without preconditioning) to 24(with AMG preconditioning).

Moreover, as h reduces to 0.2 and 0.1 wavelengths, the size of the coefficient matrix increases to $n = 18,563$ and $n = 73,569$ unknowns respectively. Figure 3(a) shows that for the case of $18,563$ unknowns, the number of iterations is reduced by 97%. Moreover, figure 3(b) shows that even when the number of unknowns is increased to 73,569, the number of iterations is still reduced by 97%. For a summary of the above results see table 2. The parameter ϵ indicates our stoping criterion. The reason why ϵ in the case of $n = 73,569$ is set to 1.0e-02 is that BiCGSTAB without preconditioning does not converge for a tolerance of 1.0e-03 even after 4,000 iterations. However, the BiCGSTAB with AMG preconditioning did converge in less than 100 iterations for the tolerance set above.

Table 4: Convergence Rates

n	No Preconditioner	AMG Preconditioner
4,727	6.22e-03	1.20e-01
18,563	1.79e-03	6.05e-02
73,569	9.97e-04	3.77e-02

In AMG, thresholds for the coarsening process, η_1 and for the interpolation process, η_2, are also needed. Table 4 shows the thresholds used in each case. Table 3 shows the size of the coarse-grid operators in each level. For instance, K_0 is the original matrix and K_3 is the coarsest level matrix in which the direct solver was used as indicated in the algorithm 3.1 for $n = 73,569$. It is also observed that the size of the coarsest level matrix, K_3 can't be reduced to even smaller size. This is due to the sparsity of the matrices on finer levels.

The computational complexity of the AMG preconditioner depends on the aggregations and the size of the neighborhoods of clusters at each level, which vary from problem to problem. In our experiments, it was observed that despite the fact that the AMG preconditioner requires some setup time, reduction to the overall CPU time was significant as shown in table 2. Hence the convergence rates have been improved with AMG preconditioning as shown in table 4. Note that the convergence rates are computed by the log(1/(convergence factor)) where the convergence factor is the geometric average of successive residual norms [13]. For our numerical experiments, Meschach [16] was used as a matrix library to speed software development.

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