A Numerical Study of the Delta Function

Matthew S. Norton

September 10, 2008

The Delta function has the following properties

$$\int_{-\infty}^{+\infty} \delta(x-a)\,dx = 1 \tag{1}$$

$$\int_{-\infty}^{+\infty} f(x)\delta(x-a)\,dx = f(a) \tag{2}$$

Also, the delta function has the property that it is zero everywhere except at x = 0. There are three functions that are commonly used to approximate the delta function in numerical work.

$$\delta(x) = \frac{1}{\alpha\sqrt{\pi}} e^{\frac{-x^2}{\alpha^2}} \lim_{\alpha\to 0} \tag{3}$$

$$\delta(x) = \frac{1}{\pi}\frac{\alpha}{x^2+\alpha^2} \lim_{\alpha\to 0} \tag{4}$$

$$\delta(x) = \frac{1}{\pi}\frac{\sin(\frac{x}{\alpha})}{x} \lim_{\alpha\to 0} \tag{5}$$

The trapezoidal method for numerically approximating the integral was used to approximate an integral that involved a delta function. A value of $\alpha = 0.005$ was used in the calculations. The integral that was calculated was

$$\int_{-\infty}^{+\infty} (6-5x^5)\delta(x)\,dx \tag{6}$$

Since a computer cannot do limits at infinity, a large value of for the bounds is usually chosen. However, for the instance of the delta function, which is supposed to be zero everywhere except at x = 0, smaller bounds need to be chosen. This will allow the intervals between the sampling of the function to be closer together so that the finer points of the function can come out. If bounds of ± 1000 are chosen, the value of equation (6) using equation (3) to approximate the Delta function with 10,000 sampling points in the interval is 1354. With the same conditions, except for bounds of ± 100, the value for equation (6) is 135.4. Again, with the same condition except for the bounds of 10, the value for equation (6) is 13.5. For bounds of ± 5, the value of equation 6 is 7.02. If the bounds are brought to ± 1, the value for equation 6 is 6.000000058917321, which is extremely close the actual value of equation (6), 6.

When equation (4) is used to approximate the Delta function with still 10,000 sampling points in the interval, bounds of ± 1000 gives a value for equation (6) of 764. Bounds of 100 gives a value for equation (6) of 76.6. Bounds of ± 10 gives a value of equation (6) of 9.15, while bounds of ± 5 produces the value 6.54. If the bounds are brought to 1, the value for equation (6) is 5.998092003388308, which again is close to the actual value of equation (6), 6.