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Chapter 1

Classical Mechanics

Change in position

$$\Delta x = x_f - x_i \tag{1.1}$$

Velocity

$$v_x = \frac{\Delta x}{\Delta t} \to \lim_{\Delta t \to \infty} \frac{\Delta x}{\Delta t} \to \frac{dx}{dt} \tag{1.2}$$

Acceleration

$$a_x = \frac{\Delta v_x}{\Delta t} \to \lim_{\Delta t \to \infty} \frac{\Delta v_x}{\Delta t} \to \frac{dv_x}{dt} \tag{1.3}$$

1.1 Kinematics: 1-D, Constant Acceleration

Final velocity

$$v_{xf} = v_{xi} + a_x t \tag{1.4}$$

Average velocity

$$\bar{v}_x = \frac{v_{xi} + v_{xf}}{2} \tag{1.5}$$

Final position

$$x_f = x_i + \frac{1}{2}\left(v_{xi} + v_{xf}\right) t \tag{1.6}$$

$$x_f = x_i + v_{xi} t + \frac{1}{2} a_x t^2 \tag{1.7}$$

Final velocity

$$v_{xf}^2 = v_{xi}^2 + 2 a_x \Delta x \tag{1.8}$$

Displacement is the area under a velocity versus time graph

1.2 Kinematics: 2-D, Constant Acceleration

Position vector

$$\vec{r} = x\vec{\imath} + y\vec{\jmath} \tag{1.9}$$

Velocity

$$v = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} = v_x\vec{\imath} + v_y\vec{\jmath} \tag{1.10}$$

$$\vec{v}_f = \vec{v}_i + \vec{a}t \tag{1.11}$$

Projectile Motion

$$v_{xi} = v_i \cos\theta_i \tag{1.12}$$
$$v_{yi} = v_i \sin\theta_i \tag{1.13}$$

Y position

$$y = \left(\frac{g}{2v_i^2 \cos^2\theta_i}\right) x^2 \tag{1.14}$$

Range of projectile

$$Range = \frac{v_i^2 \sin(2\theta_i)}{g} \tag{1.15}$$

1.3 Uniform Circular Motion

Centripetal acceleration

$$a_c = \frac{v^2}{r} \tag{1.16}$$

Period

$$T = \frac{2\pi r}{v} \tag{1.17}$$

Net acceleration

$$a_{tot} = a_r + a_t \tag{1.18}$$

Tangental acceleration

$$a_t = \frac{d|v|}{dt} \tag{1.19}$$

Radial acceleration

$$a_r = -a_c = -\frac{v^2}{r} \tag{1.20}$$

1.4 Newton's Three Laws

1. Every object in a state of uniform motion tends to remain in that state of motion unless an external force is applied to it.
2. The relationship between an object's mass m, its acceleration a, and the applied force F is
$$\vec{F} = m\vec{a}. \tag{1.21}$$
Acceleration and force are vectors (as indicated by their symbols being displayed in slant bold font); in this law the direction of the force vector is the same as the direction of the acceleration vector.
3. For every action there is an equal and opposite reaction.
$$F_{12} = -F_{21} \tag{1.22}$$

1.5 Circular Motion

$$\sum F = ma_c = m\frac{v^2}{r} \tag{1.23}$$

1.6 Work

Work

$$W \equiv F\Delta r\cos\theta = F\cdot\Delta r \tag{1.24}$$

$$W = \int_{x_1}^{x_2} F_x \; dx \tag{1.25}$$

$$\sum W = K_f - K_i = \Delta K \tag{1.26}$$

where K_i is the initial kinetic energy and K_f is the final kinetic energy

Power

$$\mathcal{P} = \frac{W}{\Delta t} \to \frac{dE}{dt} \tag{1.27}$$

1.7 Energy

Kinetic Energy

$$K = \frac{1}{2}mv^2 = \frac{p^2}{2m} \tag{1.28}$$

where m is the mass, v is the velocity, and p is momentum

Potential Energy

$$P = mgh \tag{1.29}$$

where m is the mass, g is the acceleration due to gravity, and h is the height above the ground

Force on a spring

$$F = -kx \tag{1.30}$$

where k is the spring constant.

Potential energy of a spring

$$U_{spring} = \frac{1}{2}kx^2 \tag{1.31}$$

Mechanical Energy

$$M_E = K + U \tag{1.32}$$

Conservation of Energy

$$K_f + U_f = K_i + U_i \tag{1.33}$$

1.8 Momentum

Momentum

$$p = mv \tag{1.34}$$

where m is the mass and v is the velocity

$$\Delta p = \int_{t_i}^{t_f} F\ dt \tag{1.35}$$

Impulse

$$I = \int_{t_i}^{t_f} F\ dt \tag{1.36}$$

Perfectly Inelastic Collisions

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2)\, v_f \tag{1.37}$$

Perfectly Elastic Collisions

$$m_{1i} v_{1i} + m_{2i} v_{2i} = m_{1f} v_{1f} + m_{2f} v_{2f} \tag{1.38}$$

In two dimensions

$$m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx} \tag{1.39}$$

$$m_1 v_{1iy} + m_2 v_{2iy} = m_1 v_{1fy} + m_2 v_{2fy} \tag{1.40}$$

1.9 Center of Mass

x, y, and z coordinates of the center of mass

$$x_{cm} = \frac{\sum m_i x_i}{\sum_i^N m_i} \tag{1.41}$$

$$y_{cm} = \frac{\sum m_i y_i}{\sum_i^N m_i} \tag{1.42}$$

$$z_{cm} = \frac{\sum m_i z_i}{\sum_i^N m_i} \tag{1.43}$$

Vector form

$$\vec{r}_{cm} = \frac{\sum m_i \vec{r}_i}{M} \tag{1.44}$$

Where $M = \sum m_i$

Continuous Mass

$$\vec{r}_{cm} = \int \vec{r}\ dm \tag{1.45}$$

Velocity of the center of mass

$$\vec{v}_{cm} = \frac{d\vec{r}_{cm}}{dt} = \frac{1}{M} \sum m_i \frac{d\vec{r}_{cm}}{dt} = \frac{\sum m_i v_i}{M} \tag{1.46}$$

1.10 Rotational Motion

Arc length

$$s = r\theta \quad \rightarrow \quad \theta = \frac{s}{r} \tag{1.47}$$

Angular speed

$$\omega \equiv \lim_{t\rightarrow 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \tag{1.48}$$

Angular acceleration

$$\alpha = \lim_{t\rightarrow 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \tag{1.49}$$

1.11 Rotational Kinematics with Constant Angular Acceleration

Angular velocity

$$\omega_f = \omega_I + \alpha t \tag{1.50}$$

$$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta \tag{1.51}$$

Angular Position

$$\theta_f = \theta_i + \frac{1}{2}\left(\omega_i + \omega_f\right) t \tag{1.52}$$

$$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t \tag{1.53}$$

Tangential speed

$$v = r\omega \tag{1.54}$$

Tangential acceleration

$$a_t = r\alpha \tag{1.55}$$

Centripetal Acceleration

$$a_c = \frac{v^2}{r} = r\omega^2 \tag{1.56}$$

1.12 Moment of Inertia

$$I \equiv \sum_i^N m_i r_i^2 \tag{1.57}$$

Rotational Kinetic Energy

$$K = \frac{1}{2} I\omega^2 \tag{1.58}$$

Moment of Inertia of a continuous mass

$$I = \lim_{\Delta m\rightarrow 0} \sum_i^N r_i^2 \Delta m_i = \int r^2 \ dm \tag{1.59}$$

if $dm = \rho v$

$$I = \rho r^2 \ dV \tag{1.60}$$

1.13 Torque

$$\tau = r \times F \tag{1.61}$$

$$\tau \equiv rF \sin\phi = Fd \tag{1.62}$$

$$\sum \tau = I\alpha \tag{1.63}$$

Pulley

$$\tau = TR \tag{1.64}$$

where T is the tension on the string and R is the radius.

Acceleration

$$a = R\alpha = \frac{TR^2}{I} = \frac{mg - T}{m} = \frac{g}{1 + \frac{I}{mR^2}} \tag{1.65}$$

$$T = \frac{mg}{1 + \frac{mR^2}{I}} \tag{1.66}$$

$$\alpha = \frac{a}{R} = \frac{g}{R + \frac{I}{mR}} \tag{1.67}$$

Work done through rotational motion

$$\sum W = \int_{\omega_i}^{\omega_f} I\omega \; dw = \frac{1}{2} I\omega_f^2 + \frac{1}{2}\omega_i^2 \tag{1.68}$$

1.14 Rolling Cylinder

Velocity of the center of mass

$$v_{cm} = R\frac{d\theta}{dt} = R\omega \tag{1.69}$$

Acceleration of the center of mass

$$a_{cm} = \frac{dv_{cm}}{dt} = R\alpha \tag{1.70}$$

Kinetic energy

$$K = \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} M v_{cm}^2 \tag{1.71}$$

1.15 Angular momentum

$$\vec{L} \equiv \vec{r} \times \vec{p} \tag{1.72}$$

where p is the linear momentum.

Link between torque and angular momentum

$$\sum \tau = \frac{dL}{dt} \tag{1.73}$$

Magnitude of angular momentum

$$|L| = mvr \sin\phi \tag{1.74}$$

where ϕ is the angle between $\vec{r}$ and $\vec{p}$ For a rigid object

$$L_i = m_i r_i^2 \omega_i \tag{1.75}$$

Rigid disk

$$L_z = I\omega \tag{1.76}$$

if

$$\sum \tau_{ext} = 0, \quad \vec{L}_i = \vec{L}_f \tag{1.77}$$

1.16 Static Equilibrium

$$\sum F = 0, \ \& \ \sum \tau = 0 \tag{1.78}$$

1.17 Simple Harmonic Motion

Hooke's Law

$$F = -kx \tag{1.79}$$

Differential Equation

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \tag{1.80}$$

$\omega^2 = \frac{k}{m}$

$$\frac{d^2x}{dt^2} = -\omega^2 x \tag{1.81}$$

The solution to which is

$$x(t) = A\cos(\omega t + \phi) \tag{1.82}$$

ω is the angular frequency of the simple harmonic oscillator.

The period of a simple harmonic oscillator is

$$T = \frac{2\pi}{\omega} \tag{1.83}$$

The frequency is

$$f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{1.84}$$

Energy of a simple harmonic oscillator

$$E = \frac{1}{2}kA^2 \tag{1.85}$$

where A is the amplitude of the motion

1.18 Pendulum

Simple Pendulum

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \tag{1.86}$$

where L is the length of the pendulum and g is the acceleration due to gravity.

Physical pendulum

$$\omega = \sqrt{\frac{mgd}{I}} \tag{1.87}$$

where m is the mass, d is the distance from the pivot to the center of mass, and I is the moment of inertia.

Then the period would be

$$T = 2\pi\sqrt{\frac{I}{mgd}} \tag{1.88}$$

Torsional pendulum

$$\tau = -\kappa\theta \tag{1.89}$$

where κ is the torsion constant.

Period

$$T = 2\pi\sqrt{\frac{I}{\kappa}} \tag{1.90}$$

1.19 Damped Oscillator

Equation of motion

$$\sum F = -kx - bv_x = ma_x \quad \rightarrow \quad -kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2} \tag{1.91}$$

The solution is

$$x(t) = A\mathrm{e}^{-\frac{b}{2m}t}\cos(\omega t + \phi) \tag{1.92}$$

Frequency of motion

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} \quad \rightarrow \quad \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2} \tag{1.93}$$

where $\omega_0 = \sqrt{\frac{k}{m}}$

1.20 Driven Oscillator

Equation of motion

$$\sum F = ma \quad \rightarrow \quad F_0\sin(\omega t) - b\frac{dx}{dt} - kx = m\frac{d^2x}{dt^2} \tag{1.94}$$

Solution

$$x(t) = A\cos(\omega t + \phi) \tag{1.95}$$

where

$$A = \frac{\frac{F_0}{m}}{\sqrt{\left(\omega^2 - \omega_0^2\right) + \left(\frac{b\omega}{m}\right)^2}} \tag{1.96}$$

1.21 Wave Motion

Frequency

$$f = \frac{1}{T} \tag{1.97}$$

Where T is the period of the motion

Wave number

$$k \equiv \frac{2\pi}{\lambda} \tag{1.98}$$

Angular frequency

$$\omega = \frac{2\pi}{T} \tag{1.99}$$

Velocity of a wave

$$v = \frac{\omega}{k} = \lambda f \tag{1.100}$$

Speed of a wave on a string

$$v = \sqrt{\frac{T}{\mu}} \tag{1.101}$$

where μ is the mass per unit length and T is the tension on the string

1.22 Sound Waves

Speed of sound

$$v = \sqrt{\frac{B}{\rho}} \tag{1.102}$$

where B is the Bulk modulus and ρ is the density of the medium

Intensity of a sound wave

$$I \equiv \frac{P}{A} \tag{1.103}$$

where P is the pressure and A is the area

Sound level

$$\beta = 10 \log \left(\frac{I}{I_0}\right) \tag{1.104}$$

where I_0 is a reference intensity

$$I_0 \equiv 1.00 \times 10^{-12} \ \frac{W}{m^2} \tag{1.105}$$

1.23 Doppler Effect

Observer moving towards source

$$f' = \left(\frac{v + v_0}{v}\right) f \tag{1.106}$$

Observer moving away from source

$$f' = \left(\frac{v - v_0}{v}\right) f \tag{1.107}$$

where v_0 is the observer's velocity.

If both source and observer are moving

$$f' = \left(\frac{v \pm v_0}{v \mp v_s}\right) f \tag{1.108}$$

Use the top signs if they are moving towards each other, and use the bottom signs if they are moving away from each other. where v_s is the source's velocity.

Fundamental frequency of a string

$$f_1 = \frac{1}{2L}\sqrt{\frac{T}{\mu}} \tag{1.109}$$

Pipe opened at both ends

$$f_n = n\frac{v}{2L}, \quad n = 1, 2, 3, \cdots \tag{1.110}$$

Pipe closed at one end

$$f_n = n\frac{v}{4L}n \quad n = 1, 3, 5, \cdots \tag{1.111}$$

where v is the speed of sound in air.

beat frequency

$$f_{beat} = |f_1 - f_2| \tag{1.112}$$

Chapter 2

Electromagnetism

2.1 Coulomb's Law

$$F = k\frac{q_1 q_2}{r^2} \tag{2.1}$$

where

$$k = \frac{1}{4\pi\epsilon_0} \approx 9 \times 10^9 \ \frac{Nm^2}{C^2} \tag{2.2}$$

Forces add

$$F_{tot} = F_1 + F_2 + F_3 + \cdots \tag{2.3}$$

2.2 Electric Field

$$\vec{E} = \frac{F}{q_0} \tag{2.4}$$

$$F = q\vec{E} \tag{2.5}$$

Electric field as a superposition

$$\vec{E} = k\sum \frac{q_i}{r_i^2}\hat{r}_1 \tag{2.6}$$

where r_i is the distance between the charges and $\hat{r}_i$ is the unit vector that points in the direction of the two charges.

For a continuous charge distribution

$$\vec{E} = k\int \frac{dq}{r^2}\hat{r} \tag{2.7}$$

where

$$dq = \rho \ dV \quad \text{or} \quad \sigma \ dA \quad \text{or} \quad \lambda \ d\ell \tag{2.8}$$

Force and acceleration

$$F = e\vec{E} = ma \quad \rightarrow \quad a = \frac{q\vec{E}}{m} \tag{2.9}$$

Electric Flux

$$\Phi_E = EA \tag{2.10}$$

if at an angle

$$\Phi_E = EA\cos\theta \tag{2.11}$$

Electric flux through a surface

$$\Phi_E = \int_{\text{surface}} \vec{E} \cdot d\vec{A} \tag{2.12}$$

2.3 Gauss's Law

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{in}}{\epsilon_0} \tag{2.13}$$

2.4 Potential

Potential energy

$$\Delta U = -q_0 \int_A^B \vec{E} \cdot d\vec{s} \tag{2.14}$$

Change in potential

$$\Delta V = \frac{\Delta U}{q_0} = -\int_A^B \vec{E} \cdot d\vec{s} \tag{2.15}$$

Work to move a chage

$$W = q\Delta V \tag{2.16}$$

2.5 Uniform Electric Field

$$\Delta V = -\int_A^B \vec{E} \cdot d\vec{s} = -Ed \tag{2.17}$$

where d is the distance the charge was moved.

Potential

$$V = k\sum \frac{q_i}{r_i} \tag{2.18}$$

Find the electric field from the potential

$$\vec{E} = -\nabla V \tag{2.19}$$

Potential from a continuous charge distribution

$$V = k\int \frac{dq}{r} \tag{2.20}$$

2.6 Capacitors

Capacitance

$$C \equiv \frac{Q}{\Delta V} \tag{2.21}$$

For a parallel plate capacitor,

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A} \tag{2.22}$$

$$\Delta V = Ed \tag{2.23}$$

$$C = \frac{Q}{\Delta V} = \frac{Q}{\frac{Qd}{\epsilon_0 A}} = \frac{\epsilon_0 A}{d} \tag{2.24}$$

Capacitors in parallel

$$C_{eq} = C_1 + C_2 + C_3 + \cdots \tag{2.25}$$

Capacitors in series

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots \tag{2.26}$$

Energy density

$$U_E = \frac{1}{2}\epsilon_0 E^2 \tag{2.27}$$

2.7 Dipoles

Dipole moment

$$p = qd \tag{2.28}$$

where d is the distance between the charges

Potential energy of a dipole

$$U = -\vec{p} \cdot \vec{E} \tag{2.29}$$

Torque on a dipole

$$\vec{\tau} = \vec{p} \times \vec{E} \tag{2.30}$$

2.8 Current

$$I = \frac{dq}{dt} \tag{2.31}$$

Total charge in a section of wire

$$\Delta Q = \text{number of carriers in section } \times \text{ charge per carrier}$$

$$\Delta Q = (nA\ \Delta x)\, q \tag{2.32}$$

where A is the cross sectional area, Δx is the length of the conductor, n is the number of mobile charge carriers per volume, and q is the charge on each carrier. If the carriers move with a speed v_d, the displacement they experience in the x direction in a time interval Δt is $\Delta x = v_d \Delta t$. We can then rewrite ΔQ in the form

$$\Delta Q = (nAv_d\Delta t)\, q \tag{2.33}$$

by dividing both sides by Δt, we get

$$I_{ave} = \frac{\Delta Q}{\Delta t} = nqv_d A \tag{2.34}$$

The current density J in the conductor is dened as the current per unit area

$$J \equiv \frac{I}{A} = nqv_d \tag{2.35}$$

This is only valid if A is perpendicular to direction of the current. In general

$$\vec{J} = nq\vec{v}_d \tag{2.36}$$

In some materials, the current density is proportional to the electric field

$$\vec{J} - \sigma\vec{E} \tag{2.37}$$

Materials that obey this are said to follow Ohm's Law

$$\Delta V = E\ell \tag{2.38}$$

$$J = \sigma E = \sigma\frac{\Delta V}{\ell} \tag{2.39}$$

$$\Delta V = \frac{ell}{\sigma}J = \left(\frac{\ell}{\sigma A}\right) I = RI \tag{2.40}$$

2.9 Resistance

$$R \equiv \frac{\Delta V}{I} \tag{2.41}$$

The inverse of conductivity (σ) is resistivity

$$\rho = \frac{1}{\sigma} \tag{2.42}$$

Because $R = \frac{\ell}{\sigma A}$, this can be rewritten as

$$R = \rho\frac{\ell}{A} \tag{2.43}$$

Resistance of a conductor varies with temperature.

$$\rho = \rho_0 \left[1 + \alpha\left(T - T_0\right)\right] \tag{2.44}$$

where ρ is the resistance at some temperature, T (in Celsius), ρ_0 is the resistivity at some reference temperature, T_0, and α is the temperature coefficient of resistivity, which by rearranging the previous equation, can be found to be

$$\alpha = \frac{1}{\rho_0}\frac{\Delta\rho}{\Delta T} \tag{2.45}$$

This can be rewritten in terms of resistance

$$R = R_0 \left[1 + \alpha\left(T - T_0\right)\right] \tag{2.46}$$

the power $\mathcal{P}$, representing the rate at which energy is delivered to the resistor, is

$$\mathcal{P} = I\Delta V \tag{2.47}$$

By using Ohm's Law, this can be rewritten as

$$\mathcal{P} = I^2 R = \frac{(\Delta V)^2}{R} \tag{2.48}$$

2.10 DC Circuits

The terminal voltage of the battery is

$$\Delta V = \varepsilon - Ir \tag{2.49}$$

where ε is the electromotive force (emf) of the battery and r is the internal resistance

By solving for the current, we get

$$I = \frac{\varepsilon}{R + r} \tag{2.50}$$

By multiplying both sides of equation (2.49) by I, we get

$$I\varepsilon = I^2 R + I^2 r \tag{2.51}$$

Resistors in series

$$R_{eq} = R_1 + R_2 + R_3 + \cdots \tag{2.52}$$

Resistors in parallel

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots \tag{2.53}$$

2.11 Kirchhoff's Rules

1. **Junction Rule**: The sum of the currents entering any junction in a circuit must equal the sum of the currents leaving that junction

$$\sum I_{in} = \sum I_{out} \tag{2.54}$$

2. **Loop Rule**: The sum of the potential differences across all elements around any closed circuit loop must be zero

$$\sum_{\text{closed loop}} \Delta V = 0 \tag{2.55}$$

2.12 RC Circuits

$$\varepsilon - \frac{q}{C} - IR = 0 \tag{2.56}$$

where $\frac{q}{C}$ is the potential difference across the capacitor and IR is the potential difference across the resistor

The initial current in a RC circuit is

$$I_0 = \frac{\varepsilon}{R} \tag{2.57}$$

The charge on the capacitor when it is charged to its maximum value is

$$Q = C\varepsilon \tag{2.58}$$

Since $I = \frac{dq}{dt}$

$$\begin{aligned}
\frac{dq}{dt} &= \frac{\varepsilon}{R} - \frac{q}{RC} \\
\frac{dq}{dt} &= \frac{C\varepsilon}{RC} - \frac{q}{RC} \\
\frac{dq}{dt} &= -\frac{q - C\varepsilon}{RC} \\
\frac{dq}{q - C\varepsilon} &= -\frac{1}{RC}dt \\
\int_0^q \frac{dq}{q - C\varepsilon} &= -\frac{1}{RC}\int_0^t dt \\
\ln\left(\frac{q - C\varepsilon}{-C\varepsilon}\right) &= -\frac{t}{RC}
\end{aligned}$$

By solving for q, we obtain

$$q(t) = C\varepsilon\left(1 - \mathrm{e}^{-\frac{t}{RC}}\right) \;=\; Q\left(1 - \mathrm{e}^{-\frac{t}{RC}}\right) \tag{2.59}$$

By taking the derivative with respect to time and using the relation $I = \frac{dq}{dt}$

$$I(t) = \frac{\varepsilon}{R}\mathrm{e}^{-\frac{t}{RC}} \tag{2.60}$$

The quantity RC is known as the time constant, τ, of the circuit.

Discharging a capacitor

$$q(t) = Q\mathrm{e}^{-\frac{t}{RC}} \tag{2.61}$$

$$I(t) = \frac{dq}{dt} = \frac{d}{dt}\left(Q\mathrm{e}^{-\frac{t}{RC}}\right) = -\frac{Q}{RC}\mathrm{e}^{-\frac{t}{RC}} \tag{2.62}$$

2.13 Magnetic Fields

Magnetic Force

$$\vec{F}_B = q\vec{v} \times \vec{B} \tag{2.63}$$

The magnitude of the magnetic force is then

$$F_b = |q|\, vB\sin\theta \tag{2.64}$$

The magnetic force on wire of length L is

$$\vec{F}_B = I\vec{L} \times \vec{B} \tag{2.65}$$

where $\vec{L}$ points in the direction of the current I and has a magnitude equal to the length of the segment Torque on a current loop

$$\vec{\tau} = I\vec{A} \times \vec{B} \tag{2.66}$$

where I is the current and A is the current of the loop

The product $I\vec{A}$ is defined to be the magnetic dipole moment, $\vec{\mu}$

$$\vec{\mu} = I\vec{A} \tag{2.67}$$

Then the torque on the loop can be rewritten as

$$\vec{\tau} = \vec{\mu} \times \vec{B} \tag{2.68}$$

The potential energy of a dipole in a magnetic field is

$$U = -\vec{\mu} \cdot \vec{B} \tag{2.69}$$

2.14 Charged Particle in a Magnetic Field

$$\begin{aligned} \sum F &= ma_c \\ F_B &= qvB = \frac{mv^2}{r} \\ r &= \frac{mv}{qB} \end{aligned}$$

The angular speed of the particle is

$$\omega = \frac{v}{r} = \frac{qB}{m} \tag{2.70}$$

The period of the motion is equal to the circumference of the circle divided by the linear speed of the paricle

$$T = \frac{2\pi r}{v} = \frac{2\pi}{\omega} = \frac{2\pi m}{qB} \tag{2.71}$$

The total force (called the Lorenz force) of a particle in an electric and magnetic field is

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B} \tag{2.72}$$

Kinetic energy of a charged particle when it exits a cyclotron of radius R is

$$K = \frac{1}{2}mv^2 = \frac{q^2B^2R^2}{2m} \tag{2.73}$$

2.15 Biot-Savart Law

A mathematical expression for for the magnetic field at some point in terms of the current that produces the field

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\ d\vec{s} \times \hat{r}}{r^2} \tag{2.74}$$

where μ_0 is the permeability of free space

$$\mu_0 = 4\pi \times 10^{-7}\ \frac{T \cdot m}{A} \tag{2.75}$$

By integrating $d\vec{B}$, we get

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{s} \times \hat{r}}{r^2} \tag{2.76}$$

The force between two parallel conductors

$$F_1 = I_1 \ell B_2 = I_1 \ell \left(\frac{\mu_0 I_2}{2\pi a} \right) = \frac{\mu_0 I_1 I_2}{2\pi a} \ell \tag{2.77}$$

where a is the distance between the wires, and ℓ is the length of the section of the wire we are interested in

Force per unit length

$$\frac{F_B}{\ell} = \frac{\mu_0 I_1 I_2}{2\pi a} \tag{2.78}$$

2.16 Ampère's Law

When the magnitude of $\vec{B}$ is constant

$$\oint \vec{B} \cdot d\vec{s} = B \oint ds = \frac{\mu_0 I}{2\pi r}(2\pi r) = \mu_0 I \tag{2.79}$$

Magnetic field of a Torid

$$\oint \vec{B} \cdot d\vec{s} = B \oint ds = B(2\pi r) = \mu_0 N I \tag{2.80}$$

$$B = \frac{\mu_0 N I}{2\pi r} \tag{2.81}$$

where r is radius of the amperian loop and N is the number of times the wire is wrapped around the torus

2.17 Magnetic Field of a Solenoid

$$\oint \vec{B} \cdot d\vec{s} = \int \vec{B} \cdot d\vec{s} = B \int ds = B\ell \tag{2.82}$$

$$\oint \vec{B} \cdot d\vec{s} = B\ell = \mu_0 N I \tag{2.83}$$

$$B = \mu_0 \frac{N}{\ell} I = \mu_0 n I \tag{2.84}$$

If N is the number of turns in the length ℓ, the total current through the amperian loop is NI. Then, $n = \frac{N}{\ell}$ is the number of turns per unit length

2.18 Magnetic Flux

The total magnetic flux through a surface is

$$\Phi_B = \int \vec{B} \cdot d\vec{A} \tag{2.85}$$

If plane of area A and $\vec{B}$ is uniform, then the magnetic flux through this plane is

$$\Phi_B = BA\cos\theta \tag{2.86}$$

Gauss's Law for magnetism

$$\oint \vec{B} \cdot d\vec{a} = 0 \tag{2.87}$$

2.19 Displacement Current

Maxwell found an error with Ampère's Law, so he added displacement current to to Ampère's Law

The displacement current is

$$I_d \equiv \epsilon_0 \frac{d\Phi_E}{dt} \tag{2.88}$$

Then, Ampère's Law can be rewritten more generally as

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \left(I + I_d\right) = \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \tag{2.89}$$

2.20 Magnetic Moment

We assume that an electron moves with constant speed v in a circular orbit of radius r about the nucleus. Because the electron travels a distance of $2r$ (the circumference of the circle) in a time interval T, its orbital speed is $v = \frac{2\pi r}{T}$. The current I associated with this orbiting electron is its charge e divided by T. Using $T = \frac{2\pi}{\omega}$ and $\omega = \frac{v}{r}$, we have

$$I = \frac{e}{T} = \frac{e\omega}{2\pi} = \frac{ev}{2\pi r} \tag{2.90}$$

The magnitude of the magnetic moment associated with this current loop is $\mu = IA$, where $A = \pi r^2$ is the area enclosed in the orbit. then, the magnetic moment can be written as

$$\mu = IA = \left(\frac{ev}{2\pi r}\right)\pi r^2 = \frac{1}{2}evr \tag{2.91}$$

The magnitude of the orbital angular momentum is $L = m_e vr$, and the magnetic moment can be written as

$$\mu = \left(\frac{e}{2m_e}\right)L \tag{2.92}$$

2.21 Faraday's Law of Induction

The emf induced in a circuit is directly proportional to the time rate of change of the magnetic flux through the circuit

$$\varepsilon = -\frac{d\Psi_B}{dt} \tag{2.93}$$

If the circuit is a coil consisting of N loops all of the same area, and Φ_B is the magnetic flux through one loop, and emf is induced in every loop. The total induced emf in the coil is

$$\varepsilon = -N\frac{d\Phi_B}{dt} \tag{2.94}$$

If the loop encloses an area A and lies in a uniform magnetic field $\vec{B}$, then the magnetic flux through the loop is $BA\cos\theta$, then the induced emf is

$$\varepsilon = -\frac{d}{dt}\left(BA\cos\theta\right) \tag{2.95}$$

2.22 Lenz's Law

The induced current in a loop is in the direction that creates a magnetic field that opposes the change in magnetic flux through the area enclosed by the loop

2.23 Induced emf and Electric Fields

The work done by the electric eld in moving a test charge, q, once around the loop is equal to $q\varepsilon$. Because the electric force acting on the charge is $q\vec{E}$, the work done by the electric eld in moving the charge once around the loop is $qE(2\pi r)$, where $2\pi r$ is the circumference of the loop. These two expressions for the work done must be equal; therefore, we see that circumference of the loop.

$$q\varepsilon = qE(2\pi r) \tag{2.96}$$

$$E = \frac{\varepsilon}{2\pi r} \tag{2.97}$$

Using this result, and the fact that $\Phi_B = BA = \pi r^2 B$ for a circular loop, the induced electric field is

$$E = -\frac{1}{2\pi r}\frac{d\Phi_B}{dt} = -\frac{r}{2}\frac{dB}{dt} \tag{2.98}$$

Faraday's law of induction can be rewritten in a general integral form

$$\oint \vec{E}\cdot d\vec{s} = -\frac{d\Phi_B}{dt} \tag{2.99}$$

2.24 Maxwell's Equations

Gauss's Law

$$\oint \vec{E}\cdot d\vec{s} = \frac{q}{\epsilon_0} \tag{2.100}$$

Gauss's Law in Magnetism

$$\oint \vec{B} \cdot d\vec{A} = 0 \tag{2.101}$$

Faraday's Law

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \tag{2.102}$$

Ampère-Maxwell Law

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I + \epsilon_0 \mu_0 \frac{d\Phi_E}{dt} \tag{2.103}$$

2.25 Inductance

$$\varepsilon = -L\frac{dI}{dt} \tag{2.104}$$

where L is a proportionality constant known as the inductance

$$L = \frac{N\Phi_B}{I} \tag{2.105}$$

Kirchhoff's Loop rule with an inductor and a resistor

$$\varepsilon - IR - L\frac{dI}{dt} = 0 \tag{2.106}$$

Current through a LR circuit

$$I = \frac{\varepsilon}{R}\left(1 - \mathrm{e}^{-\frac{t}{\tau}}\right) \tag{2.107}$$

where τ is given by

$$\tau = \frac{L}{R} \tag{2.108}$$

Energy stored in an inductor

$$U = \frac{1}{2}LI^2 \tag{2.109}$$

Magnetic field of an inductor

$$u_B = \frac{U}{A\ell} = \frac{B^2}{2\mu_0} \tag{2.110}$$

Charge in an LC circuit

$$Q = Q + max\cos(\omega t + \phi) \tag{2.111}$$

where Q_{max} is the maximum charge of the capacitor and the angular frequency, ω is given by

$$\omega = \frac{1}{\sqrt{LC}} \tag{2.112}$$

Charge in a RLC circuit

$$Q = Q_{max}\mathrm{e}^{-\frac{Rt}{2L}}\cos(\omega_d t) \tag{2.113}$$

where ω_d, the angular frequency at which the circuit oscillates is given by

$$\omega_d = \left[\frac{1}{LC} - \left(\frac{R}{2L}\right)^2\right]^{\frac{1}{2}} \tag{2.114}$$

2.26 Alternating Circuits

An AC circuit consists of circuit elements and a power source that provides an alternating voltage Δv. This time-varying voltage is described by

$$\omega = 2\pi f = \frac{2\pi}{T} \tag{2.115}$$

Resistors in an AC circuit the magnitude of the source voltage equals the magnitude of the voltage across the resistor

$$\Delta v = \Delta v_R = \Delta V_{max} \sin(\omega t) \tag{2.116}$$

where Δv_R is the instantaneous voltage across the resistor. From $R = \frac{V}{I}$, the instantaneous current in the resistor is

$$i_R = \frac{\Delta_R}{R} = \frac{\Delta V_{max}}{R} \sin(\omega t) = I_{max} \sin(\omega t) \tag{2.117}$$

where I_{max} is the maximum current in the circuit

$$I_{max} = \frac{\Delta V_{max}}{R} \tag{2.118}$$

What is of importance in an AC circuit is an average value of current, referred to as the rms current.

$$I_{rms} = \frac{I_{max}}{\sqrt{2}} = 0.7071 I_{max} \tag{2.119}$$

The average power delivered to a resistor that carries an alternating current is

$$\mathscr{P}_{av} = I_{rms}^2 R \tag{2.120}$$

Alternating voltage is also best discussed in terms of rms voltage, and the relationship is identical to that for current

$$\Delta V_{rms} = \frac{\Delta V_{max}}{\sqrt{2}} = 0.707 \Delta A_{max} \tag{2.121}$$

current in an inductive circuit

$$I_{max} = \frac{\Delta V_{max}}{\omega L} \tag{2.122}$$

we dene ωL as the inductive reactance

$$X_L \equiv \omega L \tag{2.123}$$

We can rewrite the maximum current as

$$I_{max} = \frac{\Delta V_{max}}{X_L} \tag{2.124}$$

Capacitors in an AC circuit. the magnitude of the source voltage is equal to the magnitude of the voltage across the capacitor

$$\Delta v = \Delta v_C = \Delta V_{max} \sin(\omega t) \tag{2.125}$$

the charge on the capacitor is given by

$$q = C \Delta V_{max} \sin(\omega t) \tag{2.126}$$

the instantaneous current in the circuit

$$i_C = \frac{dq}{dt} = \omega C \Delta V_{max} \cos(\omega t) \tag{2.127}$$

the current in the circuit reaches its maximum value

$$I_{max} = \omega C \Delta V_{max} = \frac{\Delta V_{max}}{\frac{1}{\omega C}} \tag{2.128}$$

we dene it as the capacitive reactance

$$X_C \equiv \frac{1}{\omega C} \tag{2.129}$$

We can rewrite the maximum current as

$$I_{max} = \frac{\Delta V_{max}}{X_C} \tag{2.130}$$

RLC Series Circuit Maximum current in a RLC circuit

$$I_{max} = \frac{\Delta V_{max}}{\sqrt{R^2 + (X_L - X_C)^2}} \tag{2.131}$$

The denominator of the fraction plays the role of resistance and is called the impedance, Z, of the circuit

$$Z \equiv \sqrt{R^2 + (X_L - X_C)^2} \tag{2.132}$$

Therefore, the maximum change in voltage is given by

$$\Delta V_{max} = I_{max} Z \tag{2.133}$$

the phase angle ϕ between the current and the voltage is

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) \tag{2.134}$$

We can express the average power as

$$\mathscr{P}_{av} = \frac{1}{2} I_{max} \Delta V_{max} \cos(\phi) \tag{2.135}$$

It is convenient to express the average power in terms of the rms current and rms voltage

$$\mathscr{P}_{av} = I_{rms} \Delta V_{rms} \cos(\phi) \tag{2.136}$$

A series RLCcircuit is said to be in resonancewhen the current has its maximum value. In general, the rms current can be written

$$I_{rms} = \frac{\Delta V_{rms}}{Z} = \frac{\Delta V_{rms}}{\sqrt{R^2 + (X_L - X_C)^2}} \tag{2.137}$$

The frequency ω_0 at which $X_L - X_C = 0$ is called the resonance frequency of the circuit. To nd ω_0, we use the condition $X_L = X_C$, from which we obtain

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{2.138}$$

2.27 Electromagnetic Waves

From Maxwell's equations, it can be shown that

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t} \tag{2.139}$$

$$\frac{\partial B}{\partial x} = -\mu_0\epsilon_0\frac{\partial E}{\partial t} \tag{2.140}$$

Combining these two equations results in

$$\frac{\partial^2 E}{\partial x^2} = \mu_0\epsilon_0\frac{\partial^2 E}{\partial t^2} \tag{2.141}$$

If one wants to combine the two equations in terms of the magnetic field

$$\frac{\partial^2 B}{\partial x^2} = \mu_0\epsilon_0\frac{\partial^2 B}{\partial t^2} \tag{2.142}$$

The wave speed v is replaced by c, where

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}} \tag{2.143}$$

The simplest solution to these two differential equations are

$$E = E_{max}\cos(kx - \omega t) \tag{2.144}$$

$$B = B_{max}\cos(kx - \omega t) \tag{2.145}$$

The angular wave number is $k = \frac{2\pi}{\lambda}$, where, λ is the wavelength. The angular frequency is $\omega = 2\pi f$, where f is wave frequency.The ratio $\frac{\omega}{k}$ equals the speed of an electromagnetic wave, c

$$\frac{\omega}{k} = \frac{2\pi f}{\frac{2\pi}{\lambda}} = \lambda f = c \tag{2.146}$$

Taking partial derivatives of our solutions of the differential equations (with respect to x and t), we nd that

$$\frac{\partial E}{\partial x} = -kE_{max}\sin(kx - \omega t) \tag{2.147}$$

$$\frac{\partial B}{\partial t} = \omega B_{max}\sin(kx - \omega t) \tag{2.148}$$

We can find that at instant

$$kE_{max} = \omega B_{max} \tag{2.149}$$

$$\frac{E_{max}}{B_{max}} = \frac{\omega}{k} = c \tag{2.150}$$

We see that the ratio of the amplitude of the electric field to the magnitude of the magnetic field

$$\frac{E_{max}}{B_{max}} = \frac{E}{B} = c \tag{2.151}$$

Electromagnetic waves carry energy, and as they propagate through space they can transfer energy to objects placed in their path. The rate of ow of energy in an electromagnetic wave is described by a vector $\vec{S}$, called the Poynting vector, which is dened by the expression

$$\vec{S} \equiv \frac{1}{\mu_0}\vec{E} \times \vec{B} \tag{2.152}$$

In the case of $\left|\vec{E} \times \vec{B}\right| = EB$, the intensity is the same as the average value of the S

$$I = S_{av} = fracE_{max}B_{max}2\mu_0 = \frac{E_{max}^2}{2\mu_0 x} = \frac{c}{2\mu_0}B_{max}^2 \tag{2.153}$$

if the surface absorbs all the incident energy Uin this time interval, the total momentum $\vec{p}$ transported to the surface has a magnitude

$$p = \frac{U}{c} \tag{2.154}$$

The pressure exerted on the surface is dened as force per unit area $\frac{F}{A}$. Let us combine this with Newtons second law

$$P = \frac{F}{A} = \frac{1}{A}\frac{dp}{dt} \tag{2.155}$$

If we replace p with the definition of p from above we have

$$P = \frac{1}{A}\frac{dp}{dt} = \frac{1}{A}\frac{d}{dt}\left(\frac{U}{c}\right) = \frac{1}{c}\frac{\frac{dU}{dt}}{A} \tag{2.156}$$

The radiation pressure exerted on a perfectly absorbing surface is

$$P = \frac{S}{c} \tag{2.157}$$

The momentum transported to a perfectly reflecting surface is

$$p = \frac{2U}{c} \tag{2.158}$$

The radiation pressure exerted on a perfectly reflecting surface is

$$P = \frac{2S}{c} \tag{2.159}$$

Chapter 3

Modern Physics

3.1 Principles of Relativity

Principle of Galilean relativity: The laws of mechanics must be the same in all inertial frames of reference.

Einstein's postulates of special relativity

1. **The principle of relativity**: The laws of physics must be the same in all inertial reference frames.

2. **The constancy of the speed of light**: The speed of light in vacuum has the same value, $c = 3.00 \times 10^8 m/s$, in all inertial frames, regardless of the velocity of the observer or the velocity of the source emitting the light.

The rst postulate asserts that all the laws of physicsthose dealing with mechanics, electricity and magnetism, optics, thermodynamics, and so onare the same in all reference frames moving with constant velocity relative to one another. This postulate is a sweeping generalization of the principle of Galilean relativity, which refers only to the laws of mechanics. From an experimental point of view, Einsteins principle of relativity means that any kind of experiment (measuring the speed of light, for example) performed in a laboratory at rest must give the same result when performed in a laboratory moving at a constant velocity with respect to the rst one. Hence, no preferred inertial reference frame exists, and it is impossible to detect absolute motion.

Note that postulate 2 is required by postulate 1: if the speed of light were not the same in all inertial frames, measurements of different speeds would make it possible to distinguish between inertial frames; as a result, a preferred, absolute frame could be identied, in contradiction to postulate 1.

In relativistic mechanics there is no such thing as an absolute length or absolute time interval. Furthermore, events at different locations that are observed to occur simultaneously in one frame are not necessarily observed to be simultaneous in another frame moving uniformly with respect to the rst.

3.1.1 Time Dilation

$$\Delta t = \frac{\Delta t_p}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma \Delta t_p \tag{3.1}$$

where

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{3.2}$$

The subscript p stands for proper (the proper time interval is the time interval between two events measured by an observer who sees the events occur at the same point in space)

3.1.2 Length Contraction

The proper length L_p of an object is the length measured by someone at rest relative to the object.

$$L = v\Delta t_p = v\frac{\Delta t}{\gamma} \tag{3.3}$$

Proper length is $L_p = v\Delta t$

$$L = \frac{L_p}{\gamma} = L_p\sqrt{1-\frac{v^2}{c^2}} \tag{3.4}$$

Note that length contraction takes place only along the direction of motion.

3.1.3 Relativistic Doppler Effect

If a light source and an observer approach each other with a relative speed v, the frequency f_{obs} measured by the observer is

$$f_{obs} = \frac{\sqrt{1+\frac{v}{c}}}{\sqrt{1-\frac{v}{c}}}f_{source} \tag{3.5}$$

3.1.4 Lorentz Transformation Equations

If an observer is moving solely in the x-direction, going from observer S to observer S' (S' is moving at velocity v)

$$x' = \gamma(x - vt) \tag{3.6}$$
$$y' = y \tag{3.7}$$
$$z' = z \tag{3.8}$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \tag{3.9}$$

If an observer is moving solely in the x-direction, going from observer S' to observer S (S' is moving at velocity v)

$$x = \gamma(x' + vt') \tag{3.10}$$
$$y = y' \tag{3.11}$$
$$z = z' \tag{3.12}$$
$$t = \gamma\left(t' + \frac{v}{c^2}x'\right) \tag{3.13}$$

3.1.5 Lorentz Velocity Transformation Equations

Suppose two observers in relative motion with respect to each other are both observing the motion of an object. Previously, we dened an event as occurring at an instant of time. Now, we wish to interpret the event as the motion of the object. We know that the Galilean velocity transformation is valid for low speeds. How do the observers measurements of the velocity of the object relate to each other if the speed of the object is close to that of light? Once again S' is our frame moving at a speed v relative to S. Suppose that an object has a velocity component u'_x measured in the S' frame, where

$$u'_x = \frac{dx'}{dt'} \tag{3.14}$$

by using the Lorentz transformations, we find

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \tag{3.15}$$

If the object has velocity components along the y and z axes, the components as measured by an observer in S' are

$$u'_y = \frac{u_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} \quad \text{and} \quad u'_z = \frac{u_z}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} \tag{3.16}$$

3.1.6 Relativistic Linear Momentum

The laws of physics are the same in all inertial frames, linear momentum of the system must be conserved in all frames. Assuming that the Lorentz velocity transformation equation is correct, we must modify the denition of linear momentum to satisfy the following conditions:

- The linear momentum of an isolated system must be conserved in all collisions.
- The relativistic value calculated for the linear momentum $\vec{p}$ of a particle must approach the classical value $m\vec{u}$ as $\vec{u}$ approaches zero.

For any particle, the correct relativistic equation for linear momentum that satises these conditions is

$$\vec{p} \equiv \frac{m\vec{u}}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma m\vec{u} \tag{3.17}$$

The relativistic force $\vec{F}$ acting on a particle whose linear momentum is $\vec{p}$ is dened as

$$\vec{F} \equiv \frac{d\vec{p}}{dt} \tag{3.18}$$

3.1.7 Relativistic Energy

The work done by the force $\vec{F}$ on the particle is (We assume that the particle is accelerated from rest to some nal speed u)

$$W = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} - mc^2 \tag{3.19}$$

Because we assumed that the initial speed of the particle is zero, we know that its initial kinetic energy is zero. We therefore conclude that the work W is equivalent to the relativistic kinetic energy K:

$$K = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} - mc^2 = \gamma mc^2 - mc^2 = (\gamma - 1)mc^2 \tag{3.20}$$

The constant term mc^2, which is independent of the speed of the particle, is called the rest energy, E_R of the particle

$$E_R = mc^2 \tag{3.21}$$

The term γmc^2, which does depend on the particle speed, is therefore the sum of the kinetic and rest energies. We dene $\gamma mc^2$to be the total energy E

$$E = K + mc^2 \tag{3.22}$$

or

$$E = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma mc^2 \tag{3.23}$$

In many situations, the linear momentum or energy of a particle is measured rather than its speed. It is therefore useful to have an expression relating the total energy E to the relativistic linear momentum p. This is accomplished by usingthe expressions $E = \gamma mc^2$and $p = \gamma mu$. By squaring these equations and subtracting, we can eliminate u. The result, after some algebra, is

$$E^2 = p^2c^2 + \left(mc^2\right)^2 \tag{3.24}$$

3.2 Quantum Mechanics

3.2.1 Bohr Model of the Atom

Bohr combined ideas from Planck's original quantum theory, Eisntein's concept of the photo, Rutherford's planetary model of the atom, and Newtonian mechanics to arrive at a semiclassical model of the atom. The basic idea of the Bohr theory as it applies to the hydrogen atom are as follows

1. The electron moves in circular orbits around the proton under the influence of the electric force of attraction

2. Only certain electron orbits are stable. When in one of these stationary states, as Bohr called them, the electron does not emit energy in the form of radiation. Hence, the total energy of the atom remains constant, and classical mechanics can be used to describe the electron's motion.

3. Radiation emitted by the atom when the electron makes a transition from a more energetic initial orbit to a lower energy orbit. This transition cannot be visualized or treated classically. In particular, the frequency, f, of the photon emitted in the transition is related to the change in the atom's energy and is independent of the frequency of the electron's orbital motion. The frequency of the emitted radiation is found from the energy conservation,

$$E_i - E_f = hf \tag{3.25}$$

Energy from an incident photon can be absorbed by the atom but only if the photon has an energy that exactly matches the difference in energy between allowed states of the atom.

4. The size of an allowed electron orbit is determined by a condition imposed on the electron's orbital angular momentum: the allowed orbits are those for which the electron's orbital angular momentum about the nucleus is quantized and equal to an integral multiple of $\hbar = h/2\pi$.

$$m_e v r = n\hbar \quad n = 1, 2, 3, \cdots \tag{3.26}$$

The electron potential energy of the system is

$$U = k_e \frac{q_1 q_2}{r} = -k_e \frac{e^2}{r}. \tag{3.27}$$

Where k_e is the Coulomb constant. Thus, the total energy of the atom which consists of the kinetic energy of the electron and the potential energy of the system is

$$E = K + U = \frac{1}{2} m_e v^2 - k_e \frac{e^2}{r} \tag{3.28}$$

By applying Newton's second law

$$k_e \frac{e^2}{r^2} = \frac{m_e v^2}{r} \tag{3.29}$$

Then, we can see that the kinetic energy of the electron is

$$K = \frac{1}{2} m_e v^2 = k_e \frac{e^2}{2r} \tag{3.30}$$

By using this expression in the expression for the total energy of the atom we get

$$E = -k_e \frac{e^2}{2r} \tag{3.31}$$

We can obtain an expression for the radius of the allowed orbits by

$$v^2 = \frac{n^2 \hbar^2}{m_e^2 r^2} = \frac{k_e e^2}{m_e r} \tag{3.32}$$

$$r_n = \frac{n^2 \hbar^2}{m_e k_e e^2} \quad n = 1, 2, 3, \cdots \tag{3.33}$$

The orbit with the smallest radius, called the Bohr radius, a_0 corresponding to $n = 1$ has the value

$$a_0 = \frac{\hbar^2}{m_e k_e e^2} = 0.0529 nm \tag{3.34}$$

The quantization of orbit radii immediately leads to energy quantization

$$E_n = -\frac{k_e e^2}{2a_0} \left(\frac{1}{n^2} \right) \quad n = 1, 2, 3, \cdots \tag{3.35}$$

By inserting numerical values into this expression, we find

$$E_n = -\frac{13.606}{n^2} eV \quad n = 1, 2, 3, \cdots \tag{3.36}$$

The frequency of an emitted photon emitted when an electron makes transition from an outer orbit to an inner orbit

$$f = \frac{E_i - E_f}{h} = \frac{k_e e^2}{2a_0 h}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \tag{3.37}$$

Because the quantity measured experimentally is the wavelength, it is convenient to use $c = f\lambda$ to find the wavelength of an emitted photon

$$\frac{1}{\lambda} = \frac{f}{c} = \frac{k_e e^2}{2a_o hc}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \tag{3.38}$$

For an atom that all but one of its electron removed orbiting a fixed nucleus of charge $+Ze$, where Z is the atomic number of the element, Bohr's theory gives

$$r_n = \left(n^2\right)\frac{a_0}{z} \tag{3.39}$$

$$E_n = -\frac{k_e e^2}{2a_0}\left(\frac{Z^2}{n^2}\right) \quad n = 1, 2, 3, \cdots \tag{3.40}$$

3.2.2 Wave properties of particles

In his 1923 doctoral dissertation, Louis de Broglie postulated that because photons have both wave and particle characteristics, perhaps all forms of matter have both properties. The momentum of a photon can be expressed as

$$p = \frac{h}{\lambda} \tag{3.41}$$

Because the magnitude of the momentum of a particle of mass m and speed v is $p = mv$, the de Broglie wavelength of a particle is

$$\lambda = \frac{h}{p} = \frac{h}{mv} \tag{3.42}$$

3.2.3 Wave functions

The amplitude of the wave associated with the particle, the probability amplitude, or wave function, it is usually symbolized by the symbol Ψ. In general, the complete wave function Ψ for a system depends on the positions of all the particles in the system and on time, and can be written as $\Psi(\vec{r}_1, \vec{r}_2, \vec{r}_3, \cdots, \vec{r}_j, \cdots, t)$, where $\vec{r}_j$ is the position vector of the jth particle in the system. For many situations, the wave function Ψ is mathematically separable in space and time and be written as a product of a space function ψ for one particle of the system and a complex time function:

$$\Psi(\vec{r}_1, \vec{r}_2, \vec{r}_3, \cdots, \vec{r}_j, \cdots, t) = \psi(\vec{r}_j)\mathrm{e}^{-i\omega t} \tag{3.43}$$

where $\omega = 2\pi f$ is the angular frequency of the wave function.

The wave function ψ for a free particle (a particle that is under no forces) moving along the x-axis can be written as

$$\psi(x) = A\mathrm{e}^{ikx} \tag{3.44}$$

where $k = 2\pi/\lambda$ in the angular wave number. Although we cannot measure ψ, we can measure the real quantity $|\psi|^2$,. If ψ represents a single particle, the $|\psi|^2$ − called the probability density − is the relative probability per unit volume that the particle will found at any given point in the volume. In another way, if dV is a small volume element surrounding some point, then the probability of finding the particle in that volume element is $|\psi|^2 \ dV$.

The probabilistic interpretation of the wave function was first suggested by Max Born in 1928. in 1928 Erwin Schrödinger proposed a wave equation that describes the manner in which the wave function changes in space and time. The Schrödinger wave equation represents a key element in the theory of quantum mechanics.

3.2.4 Normalization of Wave functions and Expectation Values

In one dimension, the probability of finding the particle in small section dx is $|\psi|^2 \ dx$. In this interpretation, the probability $P(x)\ dx$ that the particle will be found in the infinitesimal interval dx around the point x is

$$P(x)\ dx = |\psi|^2 \ dx \tag{3.45}$$

Although it is not possible to specify the position of a particle with complete certainty, it is a possible through $|\psi|^2$ to specify the probability of observing it in a region surrounding a given x. The probability of finding the particle in the arbitrary interval $a \leq x \leq b$ is

$$P_{ab} = \int_a^b |\psi|^2 \ dx \tag{3.46}$$

The probability P_{ab} is the area under the curve of $|\psi|^2$ versus x between the points $x = a$ and $x = b$. Because the particle must be somewhere along the x-axis, the sum of the probabilities over all values of x must be 1:

$$\int_{-\infty}^{+\infty} |\psi|^2 \ dx = 1 \tag{3.47}$$

Any wave function satisfying this relation is said to be normalized.

The average position is called the expectation value of x and is defined by the equation

$$\langle x \rangle \equiv \int_{-\infty}^{+\infty} \psi^* x \psi \ dx \tag{3.48}$$

Further, one can find the expectation value of any function $f(x)$ associated with the particle is

$$\langle f(x) \rangle \equiv \int_{-\infty}^{+\infty} \psi^* f(x) \psi \ dx \tag{3.49}$$

Some important mathematical features of a physically reasonable wave function $\psi(x)$ for a system

- $\psi(x)$ may be complex function or a real function, depending on the system

- $\psi(x)$ must be determined at all points in space and be single-valued
- $\psi(x)$ must be normalized
- $\psi(x)$ must be continuous on space − there must be no discontinuous jumps in the value of the wave function at any point

3.2.5 Heisenberg Uncertainty Principle

If a measurement of the position of a particle is made with uncertainty Δx and a simultaneous measurement of its x component of momentum is made with uncertainty Δp_x, the product of the two uncertainties can never be smaller than $\hbar/2$

$$\Delta x \Delta p_x \geq \frac{\hbar}{2} \tag{3.50}$$

that is, it is physically impossible to measure simultaneously the exact position and exact momentum of a particle. Heisenberg was careful to state the these uncertainties do not arise from imperfections in measuring instruments, rather the uncertainties arise from the quantum structure of matter.

Another version of the uncertainty principle relates wavelength and time. The corresponding variables wold be frequency and time. Because frequency is related to the energy of the particle by $E = hf$, the uncertainty principle in this form is

$$\Delta E \Delta t \geq \frac{\hbar}{2} \tag{3.51}$$

3.2.6 Schrödinger Equation

Erwin Schrödinger equation developed the wave equation in 1926. Solutions to the the wave equation give the allowed wave functions and energy levels of the system. The Schrödinger equation as it applies to a particle of mass m moving along the x axis and interacting through a potential energy function $U(x)$ is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U\psi = E\psi \tag{3.52}$$

where E is a constant equal to the total energy of the system. Because this equation is independent of time, it is often referred to as the time-independent Schrödinger equation.

3.2.7 Applications of the Schrödinger equation

Particle in a box

A particle that is confined to to a one-dimensional region of space. From a classical point of view, a particle is bouncing back and forth along the x-axis between impenetrable walls separated by a distance L. Because the walls are impenetrable, there is zero probability of finding the particle outside the box. Since the wave function must be continuous in space, if ψ is zero outside the wall, then ψ must be zero at the walls. The wave function that represents a particle in a box is

$$\psi(x) = A\sin\left(\frac{2\pi}{\lambda}x\right) \tag{3.53}$$

where λ is the de Broglie wavelength. This wave function must satisfy the boundary conditions at the walls. The boundary condition at $x = 0$ is already satisfied. For the boundary condition at $x = L$, we have

$$\psi(L) = 0 = A \sin\left(\frac{2\pi}{\lambda} L\right) \tag{3.54}$$

which can only be true if

$$\frac{2\pi}{\lambda} L = n\pi \;\rightarrow\; \lambda = \frac{2L}{n} \tag{3.55}$$

where $n = 1, 2, 3, \cdots$. Thus, only certain wavelength are allowed. Expressing the wave function in terms of the quantum numbr n, we have

$$\psi(x) = A \sin\left(\frac{n\pi}{L} x\right) \tag{3.56}$$

Since the wavelengths of the particle are restricted to certain values the momentum of the particle is also restricted to specific values. Using the de Broglie wavelength relation

$$p = \frac{h}{\lambda} = \frac{h}{\frac{2L}{n}} = \frac{nh}{2L} \tag{3.57}$$

Since the potential energy is zero inside the box, there are only certain allowed energies for the particle, which is simply the kinetic energy of the particle

$$E_n = \frac{1}{2} m v^2 = \frac{p^2}{2m} = \frac{\left(\frac{nh}{2L}\right)^2}{2m} \tag{3.58}$$

$$E_n = \left(\frac{h^2}{8mL^2}\right) n^2 \quad n = 1, 2, 3, \cdots \tag{3.59}$$

as we can see from this expression, the energy of the particle is quantized. The lowest allowed energy is called the ground state, which for the particle in a box is $E_1 = h^2/8mL^2$. Because $E_n = n^2 E_1$, the excited states corresponding to $n = 1, 2, 3 \cdots$ have energies $4E_1, 9E_1, 16E_1, \cdots$. According to quantum mechanics, the particle can never be at rest, the smallest energy it can have the ground state energy.

A Well of Finite Height

The Schrödinger equation for regions I and III may be written as

$$\frac{d^2\psi}{dx^2} = \frac{2m(U - E)}{\hbar^2} \psi \tag{3.60}$$

The general solution to the Schrödinger equation for this these regions is

$$\psi = A\mathrm{e}^{Cx} + B\mathrm{e}^{-Cx} \tag{3.61}$$

By applying the boundary conditions, the solutions in region I (ψ_{I})and in region II (ψ_{III}) are

$$\psi_{\mathrm{I}} = A\mathrm{e}^{Cx} \quad \text{for } x < 0 \tag{3.62}$$

$$\psi_{\mathrm{III}} = B\mathrm{e}^{-Cx} \quad \text{for } x > L \tag{3.63}$$

In region II, the wave function is

$$\psi_{\mathrm{II}} = F \sin(kx) + G \cos(kx) \tag{3.64}$$

Step Potential $E < V_0$

We consider today the case $E < V_0$ and treat the eigenfunction. Of course, a classical particle with energy $E < V_0$ and incident from the left would simply bounce back from the first wall with 100% probability. In quantum mechanics, as with the step potential, there is probability (but not 100%, as with the step) of reflection at the left barrier, but here there is, as we will see, finite probability that a traveling probability current will be excited in the region to the right of the barrier (region III in figure 19.1) in spite of the intervening classically forbidden region. As I mentioned to you, without the existence of this usually very, very small effect, called "quantum tunneling", the sun would not shine, and therefore, we would not live. Other applications of the effect abound, conduction of electrons in solids (tunneling of electrons through lattice ions, ammonia masers, radioactive decay, field emission process, new semiconductor devices, the "scanning tunneling microscope", etc.) so, in region I, the Schrödinger equation becomes

$$\psi_{\mathrm{I}}''(x) = -\frac{2mE}{\hbar^2}\psi_{\mathrm{I}}(x) \tag{3.65}$$

with solution

$$\psi_{\mathrm{I}}(x) = \mathcal{A}\mathrm{e}^{ikx} + \mathcal{B}\mathrm{e}^{-ikx} \tag{3.66}$$

where $k = \frac{\sqrt{2mE}}{\hbar}$. In region II, the Schrödinger equation is

$$\psi''(x) = +\frac{2m(v_0 - E)}{\hbar^2}\psi(x) \tag{3.67}$$

with solution

$$\psi_{\mathrm{II}}(x) = \mathcal{C}\mathrm{e}^{-\kappa x} + \mathcal{D}\mathrm{e}^{+\kappa x} \tag{3.68}$$

with $\kappa = \frac{\sqrt{2m(V_0-E)}}{\hbar}$ here we must keep $\mathcal{C}$ since x does not go to infinity in region II. In region III,

$$\psi_{\mathrm{III}}(x) = \mathcal{F}\mathrm{e}^{ikx} + \mathcal{G}\mathrm{e}^{-ikx} \tag{3.69}$$

where $k = \frac{\sqrt{2mE}}{\hbar}$. The second part of this a reflection; since there is nothing out at infinity to cause this, we must have $\mathcal{G} = 0$. Now we must use the boundary conditions to stitch together the three parts of the eigenfunction. The boundary conditions are, of course

- continuity of ψ at $x = 0$
- continuity of ψ' at $x = 0$
- continuity of ψ at $x = a$
- continuity of ψ' at $x = a$

As you can easily show, applications of these four conditions leads to four equations

1. $\mathcal{A} + \mathcal{B} = \mathcal{C} + \mathcal{D}$
2. $ik\mathcal{A} - ik\mathcal{B} = -\kappa\mathcal{C} + \kappa\mathcal{D}$
3. $\mathcal{C}\mathrm{e}^{-\kappa a} + \mathcal{D}\mathrm{e}^{\kappa a} = \mathcal{F}\mathrm{e}^{ika}$

4. $-\kappa\mathcal{C}e^{-\kappa a} + \kappa\mathcal{D}e^{\kappa a} = ik\mathcal{F}e^{ika}$

We have five unknowns and only four equations, so we could express $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{F}$ in terms of $\mathcal{A}$, which can later be set by the overall normalization condition[1]. However, since the algebraic situation is a little complicated here, it is best to focus on a more specific goal − of the greatest interest is determining the fraction of the probability current that leaks into region III, since that is a measure of the "tunneling". That fraction is the "transmission coefficient".

$$T = \frac{\text{current in region III}}{\text{current in region I}} = \frac{v_{\text{III}}\,|\mathcal{F}|^2}{v_{\text{I}}\,|\mathcal{A}|^2} = \frac{|\mathcal{F}|^2}{|\mathcal{A}|^2} \tag{3.70}$$

Wave Packet Incident on a Potential Step: Case $E > V_0$

The eigenfunction for energy E is of the form (as you should know)

$$\psi_{\text{I}}(x) = \mathcal{A}e^{ik_1x} + \mathcal{B}e^{-ik_1x} \tag{3.71}$$

$$\psi_{\text{II}}(x) = \mathcal{C}e^{ik_2x} + \mathcal{D}e^{-ik_2x} \tag{3.72}$$

where $k_1 = \frac{\sqrt{2mE}}{\hbar}$ and $k_2 = \frac{\sqrt{2m(E-V_0)}}{\hbar}$. For definiteness, we specify that a particle in "incident from the left", thus we set $\mathcal{D} = 0$ (there is nothing at $+\infty$ to cause back reflections). Thus, we have three unknowns ($\mathcal{A}$,, $\mathcal{B}$, and $\mathcal{C}$). Continuity of ψ and ψ' give two equations, thus one unknown is unspecified. We take this one to be $\mathcal{A}$ (arbitrary incident amplitude). The algebra then yields

$$\mathcal{B} = \frac{k_1 - k_2}{k_1 + k_2}\mathcal{A} \tag{3.73}$$

$$\mathcal{C} = \frac{2k_1}{k_1 + k_2}\mathcal{A} \tag{3.74}$$

We define a "reflection coefficient" and a "transmission coefficient" as

$$\mathcal{R} = \frac{S_{\text{reflected}}}{S_{\text{incident}}} = \frac{|\mathcal{B}|^2}{|\mathcal{A}|^2} \tag{3.75}$$

$$\mathcal{T} = \frac{S_{\text{transmitted}}}{S_{\text{indicent}}} = \frac{k_2\,|\mathcal{C}|^2}{k_1\,|\mathcal{A}|^2} = \frac{v_2\,|\mathcal{C}|^2}{v_1\,|\mathcal{A}|^2} \quad \text{(since } k \propto v\text{)} \tag{3.76}$$

$$\mathcal{R} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \tag{3.77}$$

$$\mathcal{T} = \frac{4k_1k_2}{(k_1 + k_2)^2} \tag{3.78}$$

As it must be, the sum $\mathcal{R} + \mathcal{T} = 1$ (a given particle is either transmitted or reflected). Note the definite break with classical physics − as we say, a given incident particle is either reflected or transmitted − it never splits. The probability of reflection is given by $\mathcal{R}$, and $\mathcal{R}$ decreases with increasing incident energy. The probability of transmission, $\mathcal{T}$ increases with increasing incident energy (see figure 18.4). Note that, in the case $E < V_0$, $\mathcal{R} = 1$ and $\mathcal{T} = 0$. This makes good sense, $\mathcal{R}$, since the reflected amplitude differs from the incident amplitude only by a phase factor, as we saw in the last class. $\mathcal{T} = 0$ for $E < V_0$ since then ψ_{II} is not a traveling wave, so $S_{\text{II}} = 0$.

[1] Thus, any value of $E < V_0$ is possible − the energy isn't quantized.

forming an incident wave packet for the case $E_{\text{inc}} > V_0$ leads to splitting of the packet − part is transmitted and part is reflected. Remember, however, that the reflected packet (or its modulus squared) represents the probability of reflection in a given case. If I send in a beam of identical particles, all in the same packet state,

$\mathcal{R}$ is the fraction of particles reflected and $\mathcal{T}$ is the fraction of particles transmitted. A given particle either reflects or transmits. How does a given particle "know" if it must reflect of transmit? Good question − this is quantum mechanics!

Finite potential barrier

In quantum mechanics, the finite potential barrier is a standard one-dimensional problem that demonstrates the phenomenon of quantum tunnelling. The problem consists of solving the time-independent Schrdinger equation for a particle with a finite size barrier potential in one dimension. Typically, a free particle impinges on the barrier from the left.

Although classically the particle would be reflected, quantum mechanics states that there is a finite probability that the particle will penetrate the barrier and continue travelling through to the other side. The likelihood that the particle will pass through the barrier is given by the transmission coefficient, while the likelihood that it is reflected is given by the reflection coefficient.

The wave function for this case is

$$\psi_{\text{I}}(x) = A_r e^{ik_0 x} + A_l e^{-ik_0 x} \quad x <; 0 \tag{3.79}$$

$$\psi_{\text{II}}(x) = B_r e^{ik_1 x} + B_l e^{-ik_1 x} \quad 0 < x < a \tag{3.80}$$

$$\psi_{\text{III}}(x) = C_r e^{ik_0 x} + C_l e^{-ik_0 x} \quad x > a \tag{3.81}$$

where

$$k_0 = \sqrt{2mE/\hbar^2} \qquad x < 0 \quad or \quad x > a \tag{3.82}$$

$$k_1 = \sqrt{2m(E - V_0)/\hbar^2} \quad 0 < x < a \tag{3.83}$$

3.2.8 Quantum Model of the Hydrogen Atom

The potential energy function for the hydrogen atom is

$$U(r) = -k_e \frac{e^2}{r} \tag{3.84}$$

The wave function again is separable and can be rewritten as

$$\psi(r, \theta, \phi) = R(r) f(\theta) g(\phi) \tag{3.85}$$

The first quantum number associated with the radial function $R(r)$ of the full wave function is called the principle quantum number and is assigned the symbol n. The energies of the allowed states for the hydrogen atom are found to be

$$E_n = -\left(\frac{k_e e^2}{2a_0}\right)\frac{1}{n^2} = -\frac{13.606\ eV}{n^2} \quad n = 1, 2, 3, \cdots \tag{3.86}$$

The orbital quantum number, symbolized ℓ, is associated with the orbital angular momentum of the electron, as is the orbital magnetic quantum number m_ℓ. Both ℓ and m_ℓ are integers. The applications of boundary conditions on the three parts of the wave function leads to important relationships among the three quantum numbers

- The values of n can range from 1 to ∞
- The values of ℓ can range from 0 to $n-1$
- The values of m_{ell} can range from $-\ell$ to ℓ

The simplest wave function for hydrogen is one that describes the 1 s state designated $\phi_{1s}(r)$

$$\psi_{1s}(r) = \frac{1}{\sqrt{\pi a_0^3}} \mathrm{e}^{-\frac{r}{a_0}} \tag{3.87}$$

The probability density for the 1 s state is

$$|\psi_{1s}|^2 = \left(\frac{1}{\pi a_0^3}\right) \mathrm{e}^{-\frac{2r}{a_0}} \tag{3.88}$$

The probability of locating the electron in a volume element dV is $|\psi|^2 \ dV$ It is convenient to define the radial probability in a spherical shell of radius r and a thickness dr, Thus, $P(r)dr$ is the probability of finding the electron in this shell. The volume dV of such an infinitesimally thin shell equals its surface area $4\pi r^2$ multiplied by the shell's thickness dr. So we can write the probability as

$$P(r) \ dr = |\psi|^2 \ dV = |\psi|^2 \ 4\pi r^2 \ dr \tag{3.89}$$

Thus the radial probability density function is

$$P(r) = 4\pi r^2 \, |\psi|^2 \tag{3.90}$$

The radial probability density function for the hydrogen atom in the ground state is

$$P_{1s}(r) = \left(\frac{4r^2}{a_0^3}\right) \mathrm{e}^{-\frac{2r}{a_0}} \tag{3.91}$$

3.2.9 Zeeman effect

The Zeeman effect is the splitting of a spectral line into several components in the presence of a static magnetic field. If the atom is placed in a magnetic field,the energy

$$U = -\vec{\mu} \cdot \vec{B} \tag{3.92}$$

where $\vec{\mu}$ is the magnetic moment of the atom. U is an additional energy for the atom. If there is a transition between two atomic levels in the absence of a magnetic field, if a magnetic field is applied, the upper level (with $\ell = 1$,), splits into three levels corresponding to the different directions of $\vec{\mu}$.

3.2.10 Spin-orbit Coupling

when doing looking at spectral lines under high resolution, many spectral lines are observed to be doublets. The most famous of these are the two yellow lines in the spectrum of sodium, with wavelengths of 588.995 nm and 589.592 nm. This phenomenon was explained in 1925 by Goudsmit and Uhlenbeck, who postulated that an electron has intrinsic spin angular momentum. When the sodium atom is excited with it outermost electron in a $3p$ state, and the outermost electron creates a magnetic field. the atom's energy is slightly different depending on whether the electron is spin up or spin down in this field. Then the photon energy the atom radiates as it falls back into the ground state depends on the energy of the excited state. The magnitude of this internal magnetic field is called spin-orbit coupling.

3.2.11 Angular Momentum

The angular momentum vector, $\vec{L}$m is also quantized. This quantization means that L_z (the projection of $\vec{L}$ along the z axis) can only have discrete values. The orbital magnetic quantum number m_ℓ specifies the allowed values of the z component of the orbital angular momentum according to the expression

$$L_z = m_\ell \hbar \tag{3.93}$$

It can be shown that $\vec{L}$ does not point in one specific direction, even though its z component is fixed. If $\vec{L}$ were known exactly, then all three components, L_x, L_y, L_z would be specified, which is inconstant with the uncertainty principle.

3.3 Pauli Exclusion Principle

The question, how many electrons can be in a particular quantum state? Pauli answered this important question in 1925, in a statement known as the exclusion principle:

> No two electrons can ever be in the same quantum state; therefore, no two electrons in the same atom can have the same set of quantum numbers.

Before we discuss the electronic configure of various elements, it is convenient to define an orbital as the atomic state characterized by the quantum numbers n, ℓ, and m_ℓ. From the exclusion principle we see that only two electrons can be present in any orbital. One of these electrons has a spin magnetic quantum number $m_s = +\frac{1}{2}$ and the other has $m_s = -\frac{1}{2}$. The general rule governing how electrons fill orbitals is called Hund's rule

> When an atom has orbitals of equal energy, the order in which they are filled by electrons is such that a maximum number of electrons have unpaired spins

3.4 Black-Body Radiation

Any object at any temperature emits thermal radiation from its surface. A black body is an ideal system that absorbs all radiation incident on it. The electromagnetic radiation emitted by the black body is called black body radiation. The wavelength distribution of radiation from cavities was studied. Two consistent experimental findings were seen

1. The total power of emitted radiation increases with temperature. Stefan's law is

$$\mathscr{P} = \sigma A e T^4 \tag{3.94}$$

where $\mathscr{P}$ is the power in Watts radiated from the surface, σ is the Stefan-Boltzmann constant ($5.670\times10^{-8}\frac{W}{m^2\cdot K^4}$, A is the surface area, e is the emissivity of the surface, and T is the surface temperature in Kelvins. For a black body, $e = 1$. Recalling that $I \equiv \frac{\mathscr{P}}{A}$ and that $e = 1$ for the black body, we can rewrite Stefan's law in terms of intensity,

$$I = \sigma T^4 \tag{3.95}$$

at the surface of the object.

2. The peak of the wavelength distribution shifts to shorter wavelengths as the temperature increases. this behavior was found to be described by the following relationship, called Wien's displacement law

$$\lambda_{\max} T = 2.898 \times 10^{-3} m \cdot K \tag{3.96}$$

where $\lambda_{\max}$ is the wavelength at which the curve peaks and T is the absolute temperature.

One early attempt to describe the distribution of energy from a black body, it is used to define $I(\lambda, T)\ d\lambda$ to be the intensity, or power per unit area, emitted in the wavelength interval $d\lambda$. This is known as the Rayleigh-Jeans Law

$$I(\lambda, T) = \frac{2\pi c k_{\mathrm{B}} T}{\lambda^4} \tag{3.97}$$

where k_{B} is Boltzmann's constant. Notice that as λ approaches zero, $I(\lambda, T)$ approaches infinity, but the experimental evidence shows that as λ approaches zero, $I(\lambda, T)$ approaches zero. This is known as the ultraviolet catastrophe.

In 1900, Max Planck developed a theory of black body radiation that leads to an equation for $I(\lambda, T)$ that aggress with the experimental evidence. Planck made two bold and controversial assumptions concerning the nature of the oscillators in the cavity holes.

- The energy of an oscillator can have only certain discrete values, E_n:

$$E_n = nhf \tag{3.98}$$

where n is a positive integer called a quantum number, f is the frequency of oscillation, and h is a parameter that Planck introduced known as Planck's constant.

- The oscillators emit or absorb energy when making a transition from one quantum state to another. The energy emitted by in a transition from one state to an adjacent lower state is

$$E = hf \tag{3.99}$$

Planck was able to generate a theoretical expression for the wavelength distribution that agreed well with experimental curves

$$I(\lambda, T) = \frac{2\pi h c^2}{\lambda^5 \left(\mathrm{e}^{\frac{hc}{\lambda k_{\mathbf{B}} R}} - 1\right)} \tag{3.100}$$

3.5 Nuclear reactions

3.5.1 Radioactivity

Three types of radioactive decay occur in radioactive substances: alpha (α) decay, beta decay (β), and gamma (γ) decay.

If N is the number of undecayed radioactive nuclei present at some instant, the rate of change of N is

$$\frac{dN}{dt} = -\lambda N \tag{3.101}$$

where λ is the decay constant, is the probability of decay per nucleus per second. Solving this differential equation, we get

$$\frac{dN}{N} = -\lambda\ dt \tag{3.102}$$

$$N = N_0 \mathrm{e}^{-\lambda t} \tag{3.103}$$

The decay rate, R, which is the number of decays per second is

$$R = \left|\frac{dN}{dt}\right| = \lambda N = N_0 \mathrm{e}^{-\lambda t} = R_0 \mathrm{e}^{-\lambda t} \tag{3.104}$$

where $R_0 = N_0\lambda$ is the decay rate at $t = 0$.

Another parameter used in decay is the half-life, $T_{1/2}$

$$\frac{N_0}{2} = N_0 \mathrm{e}^{-\lambda T_{1/2}} \tag{3.105}$$

solving for $T_{1/2}$, we get

$$T_{1/2} = \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda} \tag{3.106}$$

Nuclear Fission

Nuclear fission occurs when a heavy nucleus splits into two smaller nuclei. In fission, the mass of the two atoms produced is less than the mass of the original atom. This difference is called the mass defect. Multiplying the mass defect by c^2 gives the numerical value for the energy released.

Nuclear Fusion

Nuclear fusion occurs when two light nuclei combine to form a heavier nucleus. When two light nuclei combine a heavier nucleus, the mass of the heavier nucleus is not equal to the masses of the two lighter nuclei, there is a release of energy. An example of fusion is the proton-proton cycle, believed to be one of the basic cycles by which energy is generated in the Sun and other stars.

$$\begin{aligned}
{}^1_1H + {}^1_1H &\rightarrow {}^2_1H + e^+ + \nu \\
{}^1_1H + {}^2_1H &\rightarrow {}^3_2He + \gamma \\
{}^1_1H + {}^3_2He &\rightarrow {}^4_2He + e^+ + \nu \\
{}^3_2He + {}^3_2He &\rightarrow {}^4_2He + {}^1_1H + {}^1_1H
\end{aligned}$$

3.6 Quark Model

3.6.1 Original Quark Model

In 1963, Gell-Mann and George Zweig independently proposed a model for the substrate of hardons. According to their model, all hardons are composed of two or three elementary constituents called quarks. The model has three types of quarks, designated by the symbols u, d, and s. these are given the arbitrary names, up, down, and strange. The various types of quarks are called flavors. An unusual property of quarks is that they carry a fractional electronic charge. the u, d, and s quarks have charges $+2e/3$, $-e/3$, and $-e/3$ respectively, where e is the elementary charge. The composition of all hardons known when Gell-Mann and Zweig presented their model can be completely specified by three simple rules:

- A meson consists of one quark and one antiquark, giving a baryon number of 0, as required
- A baryon consists of three quarks
- An antibaryon consists of three antiquarks

Although this model of quarks was highly successful in classifying particles, there were some discrepancies. Scientist proposed a fourth flavor of quark, designated c, and was assigned a property called charm. A charmed quark has a charge $+2e/3$, just like the up quark has. This forth quark introduces a new quantum number, C, The new quark has charm $C = +1$ and the antiquark has charm $C = -1$, and all other quarks have $C = 0$.

3.7 Experiments

3.7.1 The Michelson-Morley Experiment

experiment designed to detect small changes in the speed of light was rst performed in 1881 by Albert A. Michelson and later repeated under various conditions by Michelson and Edward W. Morley.

The experiment was designed to determine the velocity of the Earth relative to thatof the hypothetical ether. The experimental tool used was the Michelson interferometer. Arm 2 is aligned along the direction of the Earths motion through space. The Earth moving through the ether at speed v is equivalent to the ether owing past the Earth in the opposite direction with speed v. This ether wind blowing in the direction opposite the direction of Earths motion should cause the speed of light measured in the Earth frame to be $c - v$ as the light approaches mirror M2and $c - v$ after reection, where c is the speed of light in the ether frame.

The two light beams reect from M1and M2 and recombine, and an interference pattern is formed. The interference pattern is observed while the interferometer is rotated through an angle of 90°. This rotation interchanges the speed of the ether wind between the arms of the interferometer. The rotation should cause the fringe pattern to shift slightly but measurably. Measurements failed, however, to show any change in the interference pattern! The MichelsonMorley experiment was repeated at different times of the year when the ether wind was expected to change direction and magnitude, but the results were always the same: no fringe shift of the magnitude required was ever observed

3.7.2 Photoelectric effect

Experiments showed light incident on certain metallic surfaces causes electrons to be emitted from those surfaces. This phenomenon is known as the photoelectric effect. There are several features of the photoelectric effect.

1. Dependence of photoelectric kinetic energy on light intensity
   - Classical prediction: Electrons should absorb energy continuously from the electromagnetic waves. As the light intensity incident on a metal is increased, energy should be transferred into the metal at a higher rate and the electrons should be ejected at a higher kinetic energy
   - Experimental result: The maximum kinetic energy of photoelectrons is independent of light intensity. The maximum kinetic energy is proportional to the stopping potential

$$K_{\text{max}} = e\Delta V_s \tag{3.107}$$

2. Time interval between incidence of light and ejection of photoelectrons
   - Classical prediction: At low light intensities, a measurable time interval should pass between the instant the light is turned on and the time an electron is ejected from the metal. This time interval is required for the electron to absorb the incident radiation before it acquires enough energy to escape from the metal.
   - Experimental results: Electrons are emitted from the surface of the metal almost instantaneously (less than 10^{-9} s after the surface is illuminated), even at very low light intensities.
3. Dependence of ejection of electrons on light frequency
   - Classical prediction: Electrons should be ejected from the metal at any incident light frequency, as long as the light intensity is high enough, because energy is transferred to the metal regardless of the incident light frequency.
   - Experimental results: No electrons are emitted if the incident light frequency falls below some cutoff frequency, f_c, whose value is characteristic of the material being illuminated. No electrons are ejected below this cutoff frequency regardless of the light intensity.
4. Dependence of the photoelectron kinetic energy on light frequency
   - Classical prediction: there should be no relationship between the frequency of the light and the electron kinetic energy. The kinetic energy should be related to the intensity of the light
   - Experimental result: The maximum kinetic energy of the photoelectrons increases with increasing light frequency.

Notice how the experimental results contradict all for classical predictions. A successful explanation of the photoelectric effect was given by Albert Einstein in 1905. Einstein extended Planck's idea quantization to electromagnetic waves. He assumed that light of frequency f can be considered a stream of quanta, regardless of the source of the radiation. Today we call these quanta photons.

each photon has an energy E, given by $E = hf$. Each photon moves at the speed of light ($c = 3.00 \times 10^8 \frac{m}{s}$)

Electrons ejected from the surface before escaping posses the maximum kinetic energy $K_{\max}$. According to Einstein, the maximum kinetic energy for those liberated electron is

$$K_{\max} = hf - \phi \tag{3.108}$$

where ϕ is called the work function of the metal. The work function represents the minimum energy with which an electron is bound to the metal.

The cutoff frequency is related to the work function through the relationship $f_c = \frac{\phi}{h}$. The cutoff frequency corresponds to a cutoff wavelength, λ_c, where

$$\lambda_x = \frac{c}{\lambda_c} = \frac{c}{\frac{\phi}{h}} = \frac{hc}{\phi} \tag{3.109}$$

where c is the speed of light. Wavelength greater than λ_c incident on a material having a work function ϕ do not result in the emission of photoelectrons.

3.7.3 The Compton Effect

Prior to 1922, Arthur Compton accumulated evidence showing that the classical wave theory of light failed to explain the scattering of x-rays from electrons. According to the classical theory, electromagnetic theory, electromagnetic waves of frequency f_0 incident on electrons should have two effects:

1. Radiation pressure should cause the electrons to accelerate in the direction of propagation of the waves

2. The oscillating electric field of the incident radiation should set the the electrons into oscillations at the apparent frequency f' where f' is the frequency in the frame of the moving electrons.

Because different electrons will move at different speeds after the interaction depending on the amount of energy absorbed from the electromagnetic waves, the scattered wave frequency at a given angle to the incoming radiation should show a distribution of Doppler-shifter values. Contrary to this prediction, Compton's experiments showed that, at a given angle, only one frequency of radiation is observed. Compton discovered that they could explain these experiments by treating the photons not as waves but rather as point-like particles. Compton adopted a particle model for something that is known to be a wave, and today this scattering phenomenon is known as Compton scattering.

The shifted wavelength after Compton scattering based on the scattering angle is

$$\lambda' - \lambda_0 = \frac{h}{m_e c} (1 - \cos\theta) \tag{3.110}$$

where m_e is the mass of the electron. This is known as the Compton shift equation, and the factor $h/m_e c$ is called the Compton wavelength of the electron.

3.7.4 Thomson e/m Experiment

A mass spectrometer separates ions according to their mass-to-charge ratio. A beam of ions pass into a magnetic field B_0. Upon entering this magnetic field the ions move in a semicircle of radius r. The ratio m/q can be expressed as

$$\frac{m}{q} = \frac{rB_0}{v} \tag{3.111}$$

A variation on this was used by J.J. Thomson in 1897 to measure the ration e/m_e for electrons.

3.7.5 Millikan Oil-drop Experiment

During the period from 1909 to 1913, Robert Millikan performed a brilliant set of experiments in which he measured e, the magnitude of the elementary charge on an electron, and demonstrated the quantized nature of this charge. His apparatus, contains two parallel metallic plates. Oil droplets from an atomizer are allowed to pass through a small hole in the upper plate. Millikan used x-rays to ionize the air in the chamber, so that freed electrons would adhere to the oil drops, giving them a negative charge. A horizontally directed light beam is used to illuminate the oil droplets, which are viewed through a telescope whose long axis is perpendicular to the light beam. When the droplets are viewed in this manner, they appear as shining stars against a dark background, and the rate at which individual drops fall can be determined. Let us assume that a single drop having a mass $m$and carrying a charge q is being viewed and that its charge is negative. If no electric eld is present between the plates,the two forces acting on the charge are the gravitational force mg acting downward and a viscous drag force F_D acting upward. The drag force is proportional to the drops speed. When the drop reaches its terminal speed v, the two forces balance each other $(mg - F_D)$.

Now suppose that a battery connected to the plates sets up an electric eld between the plates such that the upper plate is at the higher electric potential. In this case, a third force qE acts on the charged drop. Because qis negative and Eis directed downward, this electric force is directed upward. If this force is sufciently great, the drop moves upward and the drag force F'_D acts downward. When the upward electric force qE balances the sum of the gravitational force and the downward drag force F'_D, the drop reaches a new terminal speed v'in the upward direction. With the eld turned on, a drop moves slowly upward, typically at rates of hundredths of a centimeter per second. The rate of fall in the absence of a eld is comparable. Hence, one can follow a single droplet for hours, alternately rising and falling, by simply turning the electric eld on and off.

After recording measurements on thousands of droplets, Millikan and his co-workers found that all droplets, to within about 1% precision, had a charge equal to some integer multiple of the elementary charge e

$$q = ne \quad n = 0, -1, -2, -3, \cdots \tag{3.112}$$

where $e = 1.60 \times 10^{-19}$ C. Millikans experiment yields conclusive evidence that charge is quantized. For this work, he was awarded the Nobel Prize in Physics in 1923.

3.7.6 Franck-Hertz Experiment

In 1914, James Franck and Gustav Hertz performed an experiment which demonstrated the existence of excited states in mercury atoms, helping to confirm the quantum theory which predicted that electrons occupied only discrete, quantized energy states. Electrons were accelerated by a voltage toward a positively charged grid in a glass envelope filled with mercury vapor. Past the grid was a collection plate held at a small negative voltage with respect to the grid. The values of accelerating voltage where the current dropped gave a measure of the energy necessary to force an electron to an excited state.

The Franck-Hertz experiment was a physics experiment that provided support for the Bohr model of the atom, a precursor to quantum mechanics. In 1914, the German physicists James Franck and Gustav Ludwig Hertz sought to experimentally probe the energy levels of the atom. The now-famous Franck-Hertz experiment elegantly supported Niels Bohr's model of the atom, with electrons orbiting the nucleus with specific, discrete energies. Franck and Hertz were awarded the Nobel Prize in Physics in 1925 for this work.

3.7.7 Davisson-Germer experiment

De Broglie's proposal that matter exhibits both wave and particle properties was regarded as pure speculation. In 1926, C. J. Davisson and L. H. Germer succeeded in measuring the wavelength of electrons. Their important discovery provided the first experimental confirmation of the matter waves proposed by de Broglie. The experiment involved scattering low-energy electrons from a nickel target in a vacuum. During one experiment, the nickel surface was badly oxidized because of a break in their vacuum system. After the target was heated in a flowing stream of hydrogen to remove the oxide coating, electron scattered by it exhibited intensity maxima and minima at specific angles. Shortly thereafter, Davisson and Germer performed more extensive measurements. Their results showed conclusively the wave nature of electrons and confirmed the de Broglie relationship $p = h/\lambda$. In the same year, G. P. Thomas also observed electrons diffraction patterns by passing electrons through thin gold foils.