STAT 480b
Problem Set No. 6
Answer Key

Directions: Do the following problems completely. You are **expected** to work independently of each other. Show all your work. **No solution, no credit.**

1. Let $X_1, \ldots, X_{n_1}$ be a random sample from an exponential distribution with scale parameter α_1 and let $Y_1, \ldots, Y_{n_2}$ be a random sample from an exponential distribution with scale parameter α_2. Assume that X_i and Y_j are all independent, for $i = 1, \ldots, n_1$ and $j = 1, \ldots, n_2$. Find a $100(1-\alpha)\%$ confidence interval for α_2/α_1. (7 points)

Solution: Note that

$$\sum_{i=1}^{n_1} X_i \sim \mathrm{GAM}(\alpha_1, n_1) \quad \text{and} \quad \sum_{j=1}^{n_2} Y_j \sim \mathrm{GAM}(\alpha_2, n_2).$$

It follows that

$$\frac{2\sum_{i=1}^{n_1} X_i}{\alpha_1} \sim \chi^2(2n_1) \quad \text{and} \quad \frac{2\sum_{j=1}^{n_2} Y_j}{\alpha_2} \sim \chi^2(2n_2).$$

Since X_i and Y_j are independent, for $i = 1, \ldots, n_1$ and $j = 1, \ldots, n_2$, it follows that

$$F = \frac{\left(2\sum_{i=1}^{n_1} X_i/\alpha_1\right)/(2n_1)}{\left(2\sum_{j=1}^{n_2} Y_j/\alpha_2\right)/(2n_2)} \sim \mathrm{F}(2n_1, 2n_2).$$

One can simplify F in the form

$$F = \left(\frac{n_2}{n_1}\right)\left(\frac{\sum_{i=1}^{n_1} X_i}{\sum_{j=1}^{n_2} Y_j}\right)\left(\frac{\alpha_2}{\alpha_1}\right) = \left(\frac{\bar{X}}{\bar{Y}}\right)\left(\frac{\alpha_2}{\alpha_1}\right).$$

Hence, for $0 < \alpha < 1$, a $100(1-\alpha)\%$ confidence interval for α_2/α_1 is obtained from the equation

$$1 - \alpha = \mathbf{P}\left\{f_{\alpha/2}(2n_1, 2n_2) < F < f_{1-\alpha/2}(2n_1, 2n_2)\right\}$$

which results in the interval

$$\left(\left(\frac{\bar{Y}}{\bar{X}}\right) f_{\alpha/2}(2n_1, 2n_2), \left(\frac{\bar{Y}}{\bar{X}}\right) f_{1-\alpha/2}(2n_1, 2n_2)\right)$$

where $f_\gamma(a, b)$ is the 100γth percentile of the F distribution with degrees of freedom a and b.

2. Let $X_1, \dots, X_n$ be a random sample from a normal distribution with mean μ and standard deviation σ. For $n = 25$ suppose you wish to test $H_0 : \mu = 5$ at the significance level $\alpha = 0.01$ based on the sample mean $\bar{X}$.

(a) Determine the critical region $A = \{\bar{x} : -\infty < \bar{x} \leq a\}$. Assume that $\sigma = 2$. (4 points)

Solution: Since $\bar{X} \sim \mathrm{N}(\mu, \sigma^2/n)$, it follows that

$$0.01 = \mathbf{P}\left\{\bar{X} < a | \mu = 5\right\} = \mathbf{P}\left\{Z < \frac{a-5}{2/5}\right\}.$$

Hence, we obtain $a = 5 - z_{0.01}(0.4) = 5 - 2.326(0.4) = 4.0696$.

(b) Consider the critical region

$$B = \left\{\bar{x} : \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} < -z_{1-\alpha/2} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}} \quad \text{or} \quad \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} > z_{1-\alpha/2} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}}\right\}$$

and the critical region

$$C = \left\{\bar{x} : \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} > z_{1-\alpha} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}}\right\}.$$

Find the probability of Type II error, $\beta = \mathbf{P}\{\text{Type II error}\}$, for the critical regions given in part (a), B, and C for the alternative $H_a : \mu = 3$. Which critical region is the most reasonable for this alternative? Explain. (4 points)

Solution: For $\alpha = 0.01$, $z_{1-\alpha/2} = 2.576$. Thus, the probability of Type II error for critical region B is

$$\begin{aligned} \beta &= 1 - \left[\mathbf{P}\left\{Z < -2.576 + \frac{5-3}{0.4}\right\} + \mathbf{P}\left\{Z > -2.576 + \frac{5-3}{0.4}\right\}\right] \\ &= 1 - \left[\mathbf{P}\left\{Z < 2.424\right\} + \mathbf{P}\left\{Z > 7.576\right\}\right] \approx 1 - (0.9922 + 0) = 0.0078. \end{aligned}$$

On the other hand, the probability of Type II error for critical region C is

$$\beta = \mathbf{P}\left\{Z < 2.326 + \frac{5-3}{0.4}\right\} = \mathbf{P}\left\{Z < 7.326\right\} \approx 1.$$

Finally, the probability of Type II error for critical region A is

$$\begin{aligned} \beta &= \mathbf{P}\left\{\bar{X} > 4.0696 \,|\, \mu = 3\right\} = \mathbf{P}\left\{Z > -2.326 + \frac{5-3}{0.4}\right\} \\ &= \mathbf{P}\left\{Z > 2.674\right\} \approx 0.0038. \end{aligned}$$