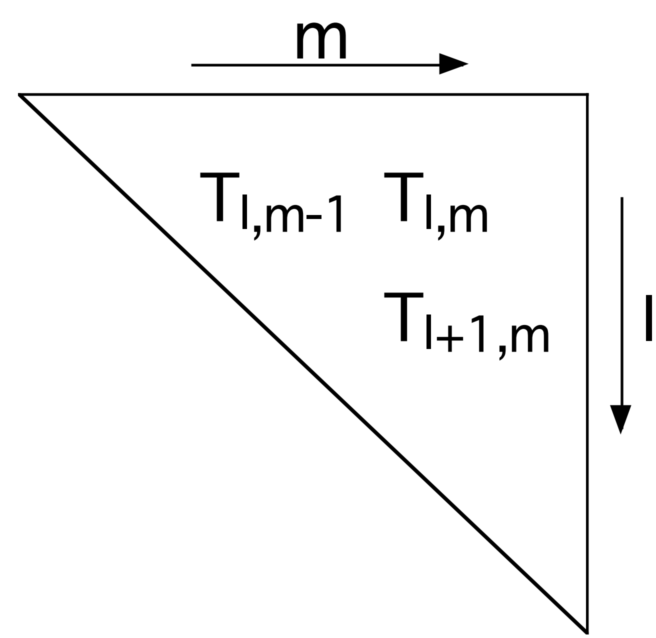
I have a matrix whose elements are defined as shown below. I want to calculate matrix elements along the diagonals.

$$T(\ell, m) = \frac{1}{\alpha^2 + 1} \, {}_2F_0\left(-\ell, -m; \frac{\alpha^2(\alpha^2+1)}{r^2} \right)$$

$${}_2F_0(-\ell, -m; \lambda) = \sum_{p=0}^{\min(\ell,m)} \frac{\ell! \; m!}{(\ell - p)! \; (m-p)!} \frac{\lambda^p}{p!}$$



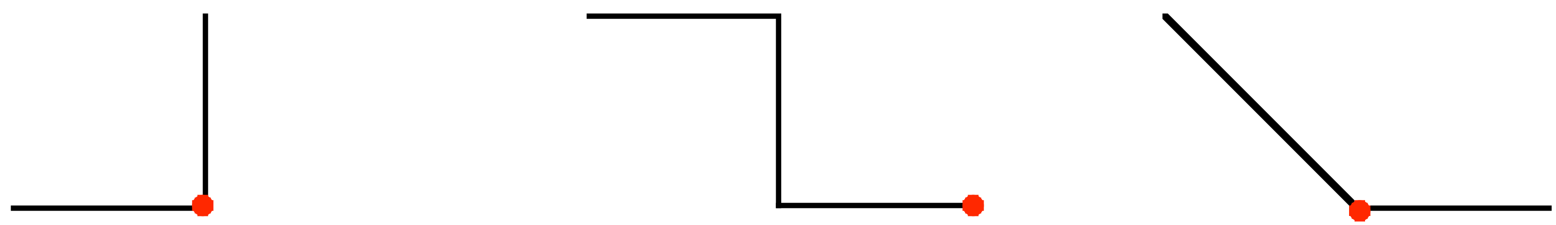
Below are the three standard contiguous relations that are defined for this generalized hypergeometric function. These provide a way of calculating the function from previously calculated functions as opposed to explicitly doing the sum above each time.

$$(\ell - \mathrm{m})\ {}_2F_0(-\ell, -\mathrm{m}; \lambda) - \ell\ {}_2F_0(-(\ell - 1), -\mathrm{m}; \lambda) + \mathrm{m}\ {}_2F_0(-\ell, -(\mathrm{m} - 1); \lambda) = 0$$

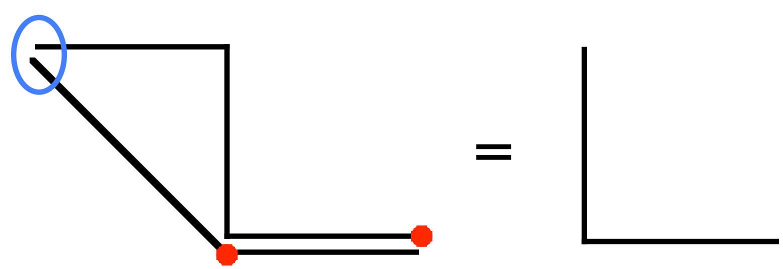
$$\mathrm{m}(\lambda(\ell + 1) - 1)\ {}_2F_0(-\ell, -\mathrm{m}; \lambda) - \ell(\ell + 1)\mathrm{m}\lambda^2\ {}_2F_0(-(\ell - 1), -(\mathrm{m} - 1); \lambda)$$
$$+ \mathrm{m}(1 - \lambda)\ {}_2F_0(-\ell, -(\mathrm{m} - 1); \lambda) - \ell\mathrm{m}(\mathrm{m} - 1)\lambda^2\ {}_2F_0(-(\ell - 1), -(\mathrm{m} - 2); \lambda) = 0$$

$$(1 + \lambda\ell)\ {}_2F_0(-\ell, -\mathrm{m}; \lambda)$$
$$- \ell\mathrm{m}\lambda^2\ {}_2F_0(-(\ell - 1), -(\mathrm{m} - 1); \lambda) + {}_2F_0(-\ell, -(\mathrm{m} + 1); \lambda) = 0$$

Below, from left to right, are representations of the three equations above, from top to bottom in l, m space. Notice that none of the relations lies along the diagonal so none is quite what I need for computing my matrix elements most efficiently.



Below is a graphical representation of combining two of these relations to produce a new contiguous relation. When the actual math is done, coefficients will be arranged so that the points circled in blue cancel each other out, leaving a relation among the remaining three functions.



Below I show how to combine the relations I now have into the relation I want, in the middle. The term on the left uses another intermediate relationship which I have not shown here. The term on the right uses only terms already shown here.