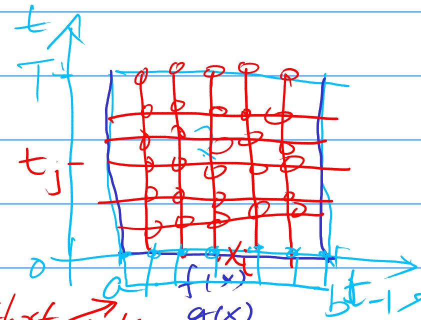
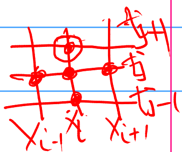
§8.2 Hyperbolic PDE: Sound/Wave Eq.

$$\begin{cases} u_{tt} = c^2 u_{xx}, & a \le x \le b, \quad t \ge 0 \\ u(x,0) = f(x) & \text{Displacement} \\ u_t(x,0) = g(x) & \text{Velocity} \\ u(a,t) = \ell(t) & \\ u(b,t) = r(t) & \end{cases}$$

$\left.\begin{matrix} u(x,0)=f(x) \\ u_t(x,0)=g(x)\end{matrix}\right\}$ IC

$\left.\begin{matrix} u(a,t)=\ell(t) \\ u(b,t)=r(t)\end{matrix}\right\}$ BC





$O(h^2+k^2)$

→ Central FD in u_{tt} and u_{xx} at (x_i,t_j): $h = \frac{b-a}{M}$, $k = \frac{T-0}{N}$.

$u_{i,j} \approx u(x_i,t_j)$

$O(h^2+k^2)$

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = c^2 \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \qquad \boxed{\sigma = \frac{ck}{h}}$$

$$(*) \quad \boxed{u_{i,j+1} = \sigma^2 u_{i-1,j} + (2-2\sigma^2)u_{i,j} + \sigma^2 u_{i+1,j} - \boxed{u_{i,j-1}}} \quad j = 0,1,2,\dots,(N-1)$$

For $j=0$, this formula need to be fixed: $u_{i,-1} = ?$

$u_{i,0} = u(x_i,0) = f(x_i) = f_i$, but we DO NOT have $u_{i,-1} = ?$

0, $t+k$, $t-k$, -1

$u_t(x_i,0) = g(x_i)$, $\xRightarrow{CFD}$ $\frac{u_{i,1} - u_{i,-1}}{2k} = g(x_i) = g_i \Rightarrow \underline{u_{i,-1} = u_{i,1} - 2kg_i}$

(I) For 1st time step, plug in $u_{i,-1} = u_{i,1} - 2kg_i$

$$(*) \xRightarrow{j=0} u_{i,1} = \sigma^2 u_{i-1,0} + (2-2\sigma^2)u_{i,0} + \sigma^2 u_{i+1,0} - u_{i,-1}$$

$$2u_{i,1} = \sigma^2 u_{i-1,0} + (2-2\sigma^2)u_{i,0} + \sigma^2 u_{i+1,0} + 2kg_i.$$

BC: $u_{0,j} = \ell_j$, $u_{M,j} = r_j$

IC: $u_{i,0} = f(x_i) = f_i$

$$u_{i,1} = \frac{1}{2}\sigma^2 u_{i-1,0} + (1-\sigma^2)u_{i,0} + \frac{1}{2}\sigma^2 u_{i+1,0} + kg_i,$$

→ Matrix-Vector:

$$\vec{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{m-1} \end{bmatrix}, \quad \vec{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_{m-1} \end{bmatrix}, \quad A = 2\begin{bmatrix} 1-\sigma^2 & \frac{1}{2}\sigma^2 & 0 & \cdots & \\ \frac{1}{2}\sigma^2 & 1-\sigma^2 & \frac{1}{2}\sigma^2 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \frac{1}{2}\sigma^2 \\ 0 & & & \frac{1}{2}\sigma^2 & 1-\sigma^2 \end{bmatrix}$$

(I) $j=0$:

$$\vec{u}^1 = \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ \vdots \\ u_{m-1,1} \end{bmatrix} = \frac{1}{2}A \begin{bmatrix} u_{1,0} \\ u_{2,0} \\ \vdots \\ u_{M-1,0} \end{bmatrix} + k \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_{m-1} \end{bmatrix} + \frac{1}{2}\sigma^2 \begin{bmatrix} u_{0,0} \\ 0 \\ \vdots \\ 0 \\ u_{M,0} \end{bmatrix}$$

$u_{0,0} \to \ell_0$, $u_{M,0} \to r_0$

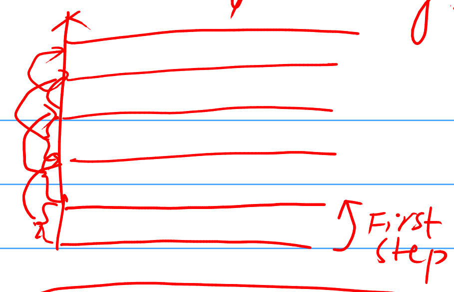
IC, BC →

$$\vec{u}^1 = \frac{1}{2}A\vec{f} + k\vec{g} + \frac{1}{2}\sigma^2 \vec{s}^0$$

$\vec{u}^1$ $j=1$ ——— $t=k$

$\vec{u}^0$ $j=0$ ——— $t=0$

time-stepping/marching



(II) for Later steps : $j \geq 1$.

$$\vec{u}^{j+1} = A\,\vec{u}^{j} - \vec{u}^{j-1} + \sigma^2 \vec{s}^{j}$$

Key : **Stable** if $\sigma = \frac{ck}{h} \leq 1$ $\boxed{\text{(CFL Condition)}}$

↑ take a lot of Mathematics $O(h^2+k^2)$

to prove $\underbrace{\text{stability}}_{???}$ + $\underbrace{\text{Consistent}}_{\checkmark\text{easy}}$ $\Leftrightarrow$ Convergence