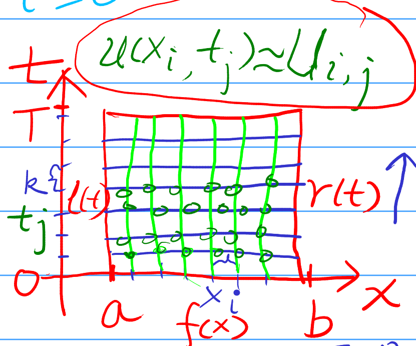
§8.1. Parabolic EQ: Heat EQ

$u(x,t)=?$

Diffusion EQ
$$\begin{cases} \text{EQ: } u_t = D\,u_{xx}, & a\le x\le b,\ t\ge 0 \\ \text{IC: } u(x,0)=f(x) \\ \text{BC: } u(a,t)=l(t) & \text{left} \\ \quad\ \ u(b,t)=r(t) & \text{right} \end{cases}$$

given. Dirichlet

Exists a unique solution



Want Find FD solution:

· Approximate Derivatives by FD formulas.

$h=\frac{b-a}{M}$, $k=\frac{T-0}{N}$

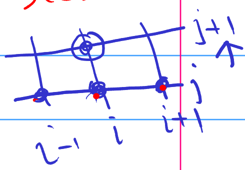
① Forward Euler scheme (heatfd.m)

$u_t(x_i,t_j)=D\,u_{xx}(x_i,t_j)$

FD ⇒
$$\frac{u(x_i,t_{j+1})-u(x_i,t_j)}{k}+\cancel{O(k)}=D\,\frac{u(x_{i-1},t_j)-2u(x_i,t_j)+u(x_{i+1},t_j)}{h^2}\ \cancel{+O(h^2)}$$

$$\Rightarrow \frac{U_{i,j+1}-U_{i,j}}{k}=D\,\frac{U_{i-1,j}-2U_{i,j}+U_{i+1,j}}{h^2},\quad 1\le i\le M-1,\ 0\le j\le N-1$$

stencil



Denote $\sigma=\frac{Dk}{h^2}$.

$$U_{i,j+1}-U_{i,j}=\sigma\left[U_{i-1,j}-2U_{i,j}+U_{i+1,j}\right]$$

$$\Rightarrow U_{i,j+1}=\sigma U_{i-1,j}+(1-2\sigma)U_{i,j}+\sigma U_{i+1,j},$$

with IC: $u(x_i,0)=f(x_i)\Rightarrow U_{i,0}=f(x_i)\equiv f_i$, $0\le i\le M-1$

BCs: $u(\underset{x_0}{a},t_j)=l(t_j)\Rightarrow U_{0,j}=l(t_j)\equiv l_j$, $0\le j\le N-1$.

$u(\underset{x_M}{b},t_j)=r(t_j)\Rightarrow U_{M,j}=r(t_j)=r_j$,

FE scheme.

Matrix-vector notations: only interior points as unknowns.

$$\begin{bmatrix} U_{1,j+1} \\ U_{2,j+1} \\ \vdots \\ U_{M-1,j+1} \end{bmatrix} = \begin{bmatrix} 1-2\sigma & \sigma & 0 & \cdots & 0 \\ \sigma & 1-2\sigma & \sigma & 0\cdots & 0 \\ & \sigma & 1-2\sigma & \sigma & \\ & & \ddots & \ddots & 0 \\ & & & \sigma & 1-2\sigma \end{bmatrix} \begin{bmatrix} U_{1,j} \\ U_{2,j} \\ \vdots \\ U_{M-1,j} \end{bmatrix} + \begin{bmatrix} \sigma l_j \\ 0 \\ \vdots \\ 0 \\ \sigma r_j \end{bmatrix} \begin{matrix} \leftarrow \text{BC} \\ \\ \\ \\ \leftarrow \text{BC} \end{matrix}$$

$\sigma=\frac{Dk}{h^2}>0$, ratio

$$\vec{U}^{\,j+1}=A\,\vec{U}^{\,j}+\sigma\vec{S}_j,\quad j=0,1,2,\ldots(N-1)\ \text{(FE scheme)}$$

$$\vec{U}^{\,0}=\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{M-1} \end{bmatrix}\leftarrow \text{IC}$$

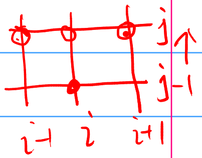
①. If $\rho(A)=\max|\lambda(A)|>1$, unstable.
spectral radius

when $\sigma=\frac{Dk}{h^2}>\frac{1}{2}\Rightarrow \rho(A)>1\Rightarrow$ unstable
$\Rightarrow k>\frac{h^2}{2D}$.

Explicit method "Conditionally stable".

require small k to get stable results, too much time steps!

② Backward Euler (BE) scheme (heatbd.m): unconditionally stable



FD

$$u_t = D u_{xx}$$

$$\frac{u_{i,j} - u_{i,j-1}}{k} = D\,\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

$\sigma = \frac{Dk}{h^2}$

$$-\sigma u_{i-1,j} + (1+2\sigma) u_{i,j} - \sigma u_{i+1,j} = u_{i,j-1}, \quad j = 1, 2, \cdots N-1, \quad i = 1, 2, \cdots (M-1)$$

Matrix-vector

$$\underbrace{\begin{bmatrix} 1+2\sigma & -\sigma & 0 & \cdots & 0 \\ -\sigma & 1+2\sigma & -\sigma & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -\sigma \\ 0 & & & -\sigma & 1+2\sigma \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{M-1,j} \end{bmatrix}}_{\vec{u}^j} = \begin{bmatrix} u_{1,j-1} \\ u_{2,j-1} \\ \vdots \\ u_{M-1,j-1} \end{bmatrix} + \begin{bmatrix} \sigma \ell_j \\ 0 \\ 0 \\ \vdots \\ 0 \\ \sigma r_j \end{bmatrix} \begin{matrix} \leftarrow BC \\ \\ \\ \\ \\ \leftarrow BC \end{matrix}$$

$$B \cdot \vec{u}^j = \vec{u}^{j-1} + \sigma \vec{s}^j$$

$$\Rightarrow \vec{u}^j = B^{-1}(\vec{u}^{j-1} + \sigma \vec{s}^j), \quad \vec{u}^0 = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{M-1} \end{bmatrix}.$$

Each time step need to solve a system of $B\backslash(\cdot\cdot)$.

① always stable since $\rho(B^{-1}) < 1$, no restriction on h & k, called unconditionally stable (Implicit scheme).

②N BE with Neumann BC: $\begin{cases} u_x(a,t) = 0 \\ u_x(b,t) = 0 \end{cases}$ (Natural BC)

use one-sided 2nd-order FD formulas:

$$0 = u_x(\overset{x_0}{a}, t_j) = \frac{-3u_{0,j} + 4u_{1,j} - u_{2,j}}{2h} \cancel{+ O(h^2)} = 0$$

$x_0 \quad x_M$

$$0 = u_x(b, t_j) = \frac{-u_{M-2,j} + 4u_{M-1,j} - 3u_{M,j}}{2h} \cancel{+ O(h^2)} = 0$$

Key: Include Boundary Nodes $u_{0,j}$, $u_{M,j}$ as unknowns

(BE-Neumann BC):

Matrix

$$\begin{bmatrix} -3 & 4 & -1 & 0 & 0 & \cdots & 0\\ -\sigma & 1+2\sigma & -\sigma & & & & \\ & -\sigma & 1+2\sigma & -\sigma & & 0 & \\ & & \ddots & \ddots & \ddots & & \\ & 0 & & & -\sigma & 1+2\sigma & -\sigma \\ 0 & \cdots & 0 & 0 & -1 & 4 & -3\end{bmatrix} \begin{bmatrix} u_{0,j}\\ u_{1,j}\\ u_{2,j}\\ \vdots\\ u_{M-2,j}\\ u_{M-1,j}\\ u_{M,j}\end{bmatrix} = \begin{bmatrix} 0\\ u_{1,j-1}\\ u_{2,j-1}\\ \vdots\\ u_{M-1,j-1}\\ 0\end{bmatrix}$$

Left BC scheme

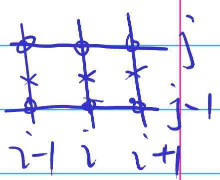
$j=1,2,\cdots N$.

right BC scheme.

$$B\,\vec{u}^{\,j} = \vec{v}^{\,j-1}$$

$O(h^2+\boxed{k})$

③ Crank-Nicolson (CN) Scheme: $\left\{\begin{array}{l}\text{unconditionally stable}\\ O(h^2+k^2)\end{array}\right.$



$$\frac{u_{i,j}-u_{i,j-1}}{k} = \frac{D}{2}\left[\overbrace{\frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{h^2}}^{BE} + \overbrace{\frac{u_{i-1,j-1}-2u_{i,j-1}+u_{i+1,j-1}}{h^2}}^{FE}\right]$$

Let $\sigma = \frac{Dk}{h^2}$. j terms to left, $(j-1)$ terms to right

$j=1,2,\cdots N$, $i=1,2,\cdots(M-1)$.

$$\Rightarrow -\sigma u_{i-1,j}+(2+2\sigma)u_{i,j}-\sigma u_{i+1,j} = \sigma u_{i-1,j-1}+(2-2\sigma)u_{i,j-1}+\sigma u_{i+1,j-1}$$

Matrix $\rightarrow$

$$\underbrace{\begin{bmatrix} 2+2\sigma & -\sigma & 0 & \cdots & 0\\ -\sigma & 2+2\sigma & -\sigma & \cdots & 0\\ & \ddots & \ddots & \ddots & \\ & & & & -\sigma\\ & & & -\sigma & 2+2\sigma\end{bmatrix}}_{A} \underbrace{\begin{bmatrix} u_{1,j}\\ u_{2,j}\\ \vdots\\ u_{M-1,j}\end{bmatrix}}_{\vec{u}^{\,j}} = \underbrace{\begin{bmatrix} 2-2\sigma & \sigma & 0 & \cdots & 0\\ \sigma & 2-2\sigma & \sigma & \cdots & 0\\ & \ddots & \ddots & \ddots & \\ & & & & \sigma\\ & & & \sigma & 2-2\sigma\end{bmatrix}}_{B} \begin{bmatrix} u_{1,j-1}\\ u_{2,j-1}\\ \vdots\\ u_{M-1,j-1}\end{bmatrix} + \sigma \begin{bmatrix} l_j+l_{j-1}\\ 0\\ 0\\ \vdots\\ 0\\ \sigma r_j+\sigma r_{j-1}\end{bmatrix}$$

$$A\,\vec{u}^{\,j} = B\,\vec{u}^{\,j-1} + \sigma(\vec{s}_j+\vec{s}_{j-1})$$

$$\Rightarrow \vec{u}^{\,j} = A^{-1}B\,\vec{u}^{\,j-1} + \sigma A^{-1}(\vec{s}_j+\vec{s}_{j-1}).$$

always stable since $\rho(A^{-1}B)<1$.

Ex. $\left\{\begin{array}{l} u_t = D u_{xx} + \boxed{cu.}\\ \text{BC/IC same.}\end{array}\right.$

$\Rightarrow$ reaction term

What BE/FE/C-N schemes look like?

$c u_{i,j}$, $c u_{i,j-1}$, $c\frac{(u_{i,j-1}+u_{i,j})}{2}$.

Ex 8.5. Page 410.

Modify the codes to get Fig 8.9.