Computational Methods in Optimal Control Lecture 5. Lipschitz Stability and Quadratic Programming

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Let us return to the optimization problem

$$\min \quad f(\mathbf{x}) \quad \text{subject to } \mathbf{x} \in \mathcal{U},$$

where $\mathcal{U} \subset \mathbb{R}^n$ is a convex set. Let us assume that a local minimizer $\mathbf{x}^*$ exists, f is twice continuously differentiable in a ball $B_r(\mathbf{x}^*)$, and the second derivative is Lipschitz continuous in the ball with Lipschitz constant κ. Moreover, it is assumed that $\mathbf{A} := \nabla^2 f(\mathbf{x}^*)$ is positive definite. Due to the convexity of $\mathcal{U}$, the first-order optimality condition satisfied by $\mathbf{x}^*$ is

$$\nabla f(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \geq 0 \quad \text{for all } \mathbf{x} \in \mathcal{U}.$$

The SQP Iteration

If $\mathbf{x}_k$ is the current iterate in the SQP method, then the next iterate $\mathbf{x}_{k+1}$ is given by

$$\begin{aligned}
\mathbf{x}_{k+1} &= \arg\min \{Q_k(\mathbf{x}) : \mathbf{x} \in \mathcal{U}\} \\
Q_k(\mathbf{x}) &= \nabla f(\mathbf{x}_k)(\mathbf{x} - \mathbf{x}_k) + \frac{1}{2}(\mathbf{x} - \mathbf{x}_k)^{\mathsf{T}} \nabla^2 f(\mathbf{x}_k)(\mathbf{x} - \mathbf{x}_k).
\end{aligned}$$

If $\mathbf{x}_{k+1}$ exists, then it satisfies the first-order optimality condition

$$\nabla Q_k(\mathbf{x}_{k+1})(\mathbf{x} - \mathbf{x}_{k+1}) \geq 0 \quad \text{for all } x \in \mathcal{U},$$

where $\nabla Q_k(\mathbf{x}) = \nabla f(\mathbf{x}_k) + (\mathbf{x} - \mathbf{x}_k)^{\mathsf{T}} \nabla^2 f(\mathbf{x}_k)$. Hence, the SQP iterate $\mathbf{x}_{k+1}$ is a solution of the problem: Find $\mathbf{x} \in \mathbb{R}^n$ such that

$$\mathcal{T}(\mathbf{x}) \in \mathcal{F}(\mathbf{x}), \quad \text{where } \mathcal{T}(\mathbf{x}) = \nabla Q_k(\mathbf{x}), \quad \mathcal{F}(\mathbf{x}) = -N_{\mathcal{U}}(\mathbf{x}).$$

A Perturbed Inclusion

If $\boldsymbol{\delta} = \nabla f(\mathbf{x}^*) - \mathcal{T}(\mathbf{x}^*)$, then

$$\mathcal{T}(\mathbf{x}^*) + \boldsymbol{\delta} \in \mathcal{F}(\mathbf{x}^*)$$

by the first-order optimality conditions for $\mathbf{x}^*$. Let us apply the abstract convergence theorem with $\mathcal{L} = \nabla^2 f(\mathbf{x}^*)$. As explained in Lecture 4, (C3) in the theorem amounts to showing that the following quadratic programming problem has a unique solution depending Lipschitz continuously on $\mathbf{y}$:

$$\min \ \{\frac{1}{2}\mathbf{z}^{\mathsf{T}}\mathbf{A}\mathbf{z} + \mathbf{y}^{\mathsf{T}}\mathbf{z} : \mathbf{z} \in \mathcal{U}\}, \tag{QP}$$

where $\mathbf{A} = \nabla^2 f(\mathbf{x}^*)$. Since $\mathbf{A}$ is positive definite, the objective function is strongly convex and there exists a unique solution.

Lipschitz Continuity

Let $\mathbf{x}_i$ be the solution of (QP) associated with $\mathbf{y} = \mathbf{y}_i$, $i = 1, 2$. The first-order optimality condition is

$$(\mathbf{A}\mathbf{x}_i + \mathbf{y}_i)^{\mathsf{T}}(\mathbf{x} - \mathbf{x}_i) \geq 0 \quad \text{for all } \mathbf{x} \in \mathcal{U}.$$

Taking $\mathbf{x} = \mathbf{x}_2$ and $i = 1$, and then $\mathbf{x} = \mathbf{x}_1$ and $i = 2$ yields

$$(\mathbf{A}\mathbf{x}_1 + \mathbf{y}_1)^{\mathsf{T}}(\mathbf{x}_2 - \mathbf{x}_1) \geq 0 \quad \text{and} \quad (\mathbf{A}\mathbf{x}_2 + \mathbf{y}_2)^{\mathsf{T}}(\mathbf{x}_1 - \mathbf{x}_2) \geq 0.$$

Add these inequalities together and rearrange to obtain

$$(\mathbf{y}_2 - \mathbf{y}_1)^{\mathsf{T}}(\mathbf{x}_1 - \mathbf{x}_2) \geq (\mathbf{x}_1 - \mathbf{x}_2)^{\mathsf{T}}\mathbf{A}_2(\mathbf{x}_1 - \mathbf{x}_2)$$

If $1/\gamma$ denotes the smallest eigenvalue of $\mathbf{A}$, then

$$\frac{1}{\gamma}\|\mathbf{x}_1 - \mathbf{x}_2\|^2 \leq \|\mathbf{y}_1 - \mathbf{y}_2\|\|\mathbf{x}_1 - \mathbf{x}_2\| \Longrightarrow \|\mathbf{x}_1 - \mathbf{x}_2\| \leq \gamma\|\mathbf{y}_1 - \mathbf{y}_2\|.$$

Thus the Lipschitz constant of (C3) is the reciprocal of the smallest eigenvalue of $\mathbf{A}$.

A Quadratic Convergence Result

Choose ϵ small enough that $\epsilon\gamma < 1$. Since

$$\mathcal{T}(\mathbf{x}) = \nabla Q_k(\mathbf{x}) = \nabla f(\mathbf{x}_k) + (\mathbf{x} - \mathbf{x}_k)^\mathsf{T} \nabla^2 f(\mathbf{x}_k),$$

it follows that $\nabla \mathcal{T} = \nabla^2 f(\mathbf{x}_k)$. In (C2) we want $\|\nabla^2 f(\mathbf{x}_k) - \nabla^2 f(\mathbf{x}^*)\| \leq \epsilon$. We require that $\mathbf{x}_k \in B_r(\mathbf{x}^*)$ where r is chosen smaller if necessary so that (C2) holds. Choose r smaller if necessary to ensure that $\nabla^2 f(\mathbf{x}_k)$ is positive definite when $\mathbf{x}_k \in B_r(\mathbf{x}^*)$. Finally, let us analyze $\boldsymbol{\delta}$:

$$\begin{aligned}
\boldsymbol{\delta} &= \nabla f(\mathbf{x}^*) - \mathcal{T}(\mathbf{x}^*) = \nabla f(\mathbf{x}^*) - \nabla Q_k(\mathbf{x}^*) \\
&= \nabla f(\mathbf{x}^*) - \nabla f(\mathbf{x}_k) + (\mathbf{x}^* - \mathbf{x}_k)^\mathsf{T} \nabla^2 f(\mathbf{x}_k) \\
&= (\mathbf{x}^* - \mathbf{x}_k)^\mathsf{T} \int_0^1 \left[\nabla^2 f(\mathbf{x}_k + s(\mathbf{x}^* - \mathbf{x}_k)) - \nabla^2 f(\mathbf{x}_k)\right] \, ds
\end{aligned}$$

Quadratic Convergence Result Continued . . .

Taking the norm and utilizing the Lipschitz continuity of $\nabla^2 f$ yields

$$\|\boldsymbol{\delta}\| \le (\kappa/2)\|\mathbf{x}_k - \mathbf{x}^*\|^2.$$

The theorem requires that $\|\boldsymbol{\delta}\| \le (1-\gamma\epsilon)r/\gamma$, which is satisfied when

$$(\kappa/2)r^2 \le (1-\gamma\epsilon)r/\gamma \Longrightarrow r \le 2(1-\gamma\epsilon)/\gamma,$$

since $\mathbf{x}_k \in B_r(\mathbf{x}^*)$. Thus all the conditions of the abstract convergence theorem hold when r satisfies this bound; it follows that for all $\mathbf{x}_k \in B_r(\mathbf{x}^*)$, the SQP problem has a unique solution $\mathbf{x}_{k+1}$ and

$$\|\mathbf{x}_{k+1} - \mathbf{x}^*\| \le \frac{\gamma\kappa}{2(1-\gamma\epsilon)}\|\mathbf{x}_k - \mathbf{x}^*\|^2.$$

Back to Pseudospectral Method $\boldsymbol{\theta} = (\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda})$

$$
\begin{aligned}
\mathcal{T}_0(\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda}) &= \mathbf{X}_0 - \mathbf{x}_0, \\
\mathcal{T}_{1i}(\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda}) &= \left(\sum_{j=0}^{N} D_{ij} \mathbf{X}_j \right) - \mathbf{f}(\mathbf{X}_i, \mathbf{U}_i), \quad 1 \le i \le N, \\
\mathcal{T}_2(\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda}) &= \mathbf{X}_{N+1} - \mathbf{X}_0 - \sum_{j=1}^{N} \omega_j \mathbf{f}(\mathbf{X}_j, \mathbf{U}_j), \\
\mathcal{T}_3(\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda}) &= \boldsymbol{\Lambda}_{N+1} - \boldsymbol{\Lambda}_0 + \sum_{i=1}^{N} \omega_i \nabla_x H(\mathbf{X}_i, \mathbf{U}_i, \boldsymbol{\Lambda}_i), \\
\mathcal{T}_{4i}(\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda}) &= \left(\sum_{j=1}^{N+1} D^{\dagger}_{ij} \boldsymbol{\Lambda}_j \right) + \nabla_x H(\mathbf{X}_i, \mathbf{U}_i, \boldsymbol{\Lambda}_i), \quad 1 \le i \le N, \\
\mathcal{T}_5(\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda}) &= \boldsymbol{\Lambda}_{N+1} - \nabla C(\mathbf{X}_{N+1}), \\
\mathcal{T}_{6i}(\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda}) &= -\nabla_u H(\mathbf{X}_i, \mathbf{U}_i, \boldsymbol{\Lambda}_i), \quad 1 \le i \le N.
\end{aligned}
$$

Abstract Set Up for Pseudospectral Method

$\mathcal{T}(\mathbf{X}, \mathbf{U}, \mathbf{\Lambda}) \in \mathcal{F}(\mathbf{U})$ where

$$\mathcal{F}_0 = \mathcal{F}_1 = \ldots = \mathcal{F}_5 = \mathbf{0}, \quad \text{while } \mathcal{F}_{6i}(\mathbf{U}) = N_{\mathcal{U}}(\mathbf{U}_i).$$

In the abstract theorem, take $\mathcal{L} = \nabla\mathcal{T}(\mathbf{X}^*, \mathbf{U}^*, \mathbf{\Lambda}^*)$. The $\mathcal{X}$ space containing $(\mathbf{X}, \mathbf{U}, \mathbf{\Lambda})$ is $\mathbb{R}^{n(N+2)} \times \mathbb{R}^{mN} \times \mathbb{R}^{n(N+2)}$ with the sup-norm. The $\mathcal{Y}$ space containing $\mathcal{T}(\mathbf{X}, \mathbf{U}, \mathbf{\Lambda})$ is

$$\mathbb{R}^n \times \mathbb{R}^{nN} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{nN} \times \mathbb{R}^n \times \mathbb{R}^{mN},$$

with the norm

$$\|\mathbf{y}\|_{\mathcal{Y}} = |\mathbf{y}_0| + \|\mathbf{y}_1\|_\omega + |\mathbf{y}_2| + |\mathbf{y}_3| + \|\mathbf{y}_4\|_\omega + |\mathbf{y}_5| + \|\mathbf{y}_6\|_\infty.$$

where

$$\|\mathbf{z}\|_\omega^2 = \left(\sum_{i=1}^{N} \omega_i |\mathbf{z}_i|^2\right)^{1/2}, \quad \mathbf{z} \in \mathbb{R}^{nN}.$$

Lipschitz Property (C3)

Given $\mathbf{Y} \in \mathcal{Y}$, find $(\mathbf{X}, \mathbf{U}, \mathbf{\Lambda})$ such that

$$\nabla \mathcal{T}(\mathbf{X}^*, \mathbf{U}^*, \mathbf{\Lambda}^*)[\mathbf{X}, \mathbf{U}, \mathbf{\Lambda}] + \mathbf{Y} \in \mathcal{F}(\mathbf{U}).$$

For each $\mathbf{Y}$, we wish to show that there is a unique $(\mathbf{X}, \mathbf{U}, \mathbf{\Lambda})$ satisfying the inclusion, and it depends Lipschitz continuously on $\mathbf{Y}$. Let us define the following matrices:

$$\begin{aligned}
\mathbf{A}(t) &= \nabla_x \mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t)), & \mathbf{B}(t) &= \nabla_u \mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t)) \\
\mathbf{Q}(t) &= \nabla_{xx} H\left(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t)\right), & \mathbf{S}(t) &= \nabla_{ux} H\left(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t)\right) \\
\mathbf{R}(t) &= \nabla_{uu} H\left(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t)\right), & \mathbf{T} &= \nabla^2 C(\mathbf{x}^*(1))
\end{aligned}$$

$\nabla\mathcal{T}(\mathbf{X}^*, \mathbf{U}^*, \mathbf{\Lambda}^*)[\mathbf{X}, \mathbf{U}, \mathbf{\Lambda}]$

$$
\begin{aligned}
\nabla\mathcal{T}_0^*[\mathbf{X}, \mathbf{U}, \mathbf{\Lambda}] &= \mathbf{X}_0, \\
\nabla\mathcal{T}_{1i}^*[\mathbf{X}, \mathbf{U}, \mathbf{\Lambda}] &= \left(\sum_{j=1}^{N} D_{ij}\mathbf{X}_j\right) - \mathbf{A}_i\mathbf{X}_i - \mathbf{B}_i\mathbf{U}_i, \quad 1 \le i \le N, \\
\nabla\mathcal{T}_2^*[\mathbf{X}, \mathbf{U}, \mathbf{\Lambda}] &= \mathbf{X}_{N+1} - \mathbf{X}_0 - \sum_{j=1}^{N} \omega_j(\mathbf{A}_j\mathbf{X}_j + \mathbf{B}_j\mathbf{U}_j), \\
\nabla\mathcal{T}_3^*[\mathbf{X}, \mathbf{U}, \mathbf{\Lambda}] &= \mathbf{\Lambda}_{N+1} - \mathbf{\Lambda}_0 + \sum_{j=1}^{N} \omega_j(\mathbf{A}_j^{\mathsf{T}}\mathbf{\Lambda}_j + \mathbf{Q}_j\mathbf{X}_j + \mathbf{S}_j\mathbf{U}_j), \\
\nabla\mathcal{T}_{4i}^*[\mathbf{X}, \mathbf{U}, \mathbf{\Lambda}] &= \left(\sum_{j=1}^{N+1} D_{ij}^{\dagger}\mathbf{\Lambda}_j\right) + \mathbf{A}_i^{\mathsf{T}}\mathbf{\Lambda}_i + \mathbf{Q}_i\mathbf{X}_i + \mathbf{S}_i\mathbf{U}_i, \quad 1 \le i \le N, \\
\nabla\mathcal{T}_5^*[\mathbf{X}, \mathbf{U}, \mathbf{\Lambda}] &= \mathbf{\Lambda}_{N+1} - \mathbf{T}\mathbf{X}_{N+1}, \\
\nabla\mathcal{T}_{6i}^*[\mathbf{X}, \mathbf{U}, \mathbf{\Lambda}] &= -(\mathbf{S}_i^{\mathsf{T}}\mathbf{X}_i + \mathbf{R}_i\mathbf{U}_i + \mathbf{B}_i^{\mathsf{T}}\mathbf{\Lambda}_i), \quad 1 \le i \le N.
\end{aligned}
$$

Associated Linear-Quadratic Problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\mathcal{Q}(\mathbf{X},\mathbf{U}) + \mathcal{L}(\mathbf{X},\mathbf{U},\mathbf{Y}) \\ \text{subject to} & \sum_{j=1}^{N} D_{ij}\mathbf{X}_j = \mathbf{A}_i\mathbf{X}_i + \mathbf{B}_i\mathbf{U}_i - \mathbf{y}_{1i}, \quad \mathbf{U}_i \in \mathcal{U}, \quad 1 \le i \le N, \\ & \mathbf{X}_0 = -\mathbf{y}_0, \quad \mathbf{X}_{N+1} = \mathbf{X}_0 - \mathbf{y}_2 + \sum_{j=1}^{N} \omega_j \left(\mathbf{A}_j\mathbf{X}_j + \mathbf{B}_j\mathbf{U}_j\right). \end{array}$$

$$\begin{aligned} \mathcal{Q}(\mathbf{X},\mathbf{U}) &= \mathbf{X}_{N+1}^{\mathsf{T}}\mathbf{T}\mathbf{X}_{N+1} + \sum_{i=1}^{N} \omega_i \left(\mathbf{X}_i^{\mathsf{T}}\mathbf{Q}_i\mathbf{X}_i + 2\mathbf{X}_i^{\mathsf{T}}\mathbf{S}_i\mathbf{U}_i + \mathbf{U}_i^{\mathsf{T}}\mathbf{R}_i\mathbf{U}_i\right) \\ \mathcal{L}(\mathbf{X},\mathbf{U},\mathbf{Y}) &= \mathbf{X}_0^{\mathsf{T}}\left(\mathbf{y}_3 - \sum_{i=1}^{N} \omega_i \mathbf{y}_{4i}\right) - \mathbf{y}_5^{\mathsf{T}}\mathbf{X}_{N+1} + \sum_{i=1}^{N} \omega_i \left(\mathbf{y}_{4i}^{\mathsf{T}}\mathbf{X}_i - \mathbf{y}_{6i}^{\mathsf{T}}\mathbf{U}_i\right) \end{aligned}$$

New Stuff

Unlike the previous QP, the parameters $\mathbf{y}_i$ also appear in the constraints. We need to solve for $\mathbf{X}$ in terms of $\mathbf{U}$ and the parameter $\mathbf{y}$. When we substitute for $\mathbf{X}$ in the objective, the QP is a function of $\mathbf{U}$ alone and the constraint is imply $\mathbf{U}_i \in \mathcal{U}$. When the QP is in this form, Lipschitz continuity is derived as before.

Lemma If there exists a constant $0 \leq \beta < 1/2$ such that $\|\mathbf{A}_i\|_\infty \leq \beta$ for all i, then the linear system

$$\left(\sum_{j=1}^{N} D_{ij} \mathbf{X}_j \right) - \mathbf{A}_i \mathbf{X}_i \quad = \quad \mathbf{p}_i \quad 1 \leq i \leq N,$$

has a unique solution $\mathbf{X} \in \mathbb{R}^{nN}$, and

$$\|\mathbf{X}\|_\infty \leq \left(\frac{\sqrt{2}}{1 - 2\beta} \right) \|\mathbf{p}\|_\omega.$$

Proof

Assume for convenience that $n = 1$ (otherwise Kronecker products needed to reformulate and manipulate the linear system). In this case, the linear system is $\mathbf{DX} - \mathbf{AX} = \mathbf{p}$ where $\mathbf{A}$ is diagonal with i-th diagonal element $\mathbf{A}_i$ and $\mathbf{D}$ is N by N. Will show that $\mathbf{D}^{-1}$ exists and $\|\mathbf{D}^{-1}\|_\infty \leq 2$. Hence, we have

$$(\mathbf{I} - \mathbf{D}^{-1}\mathbf{A})\mathbf{X} = \mathbf{D}^{-1}\mathbf{p}.$$

Since $\|\mathbf{D}^{-1}\mathbf{A}\|_\infty \leq 2\beta < 1$, the matrix $(\mathbf{I} - \mathbf{D}^{-1}\mathbf{A})$ is invertible and $\|(\mathbf{I} - \mathbf{D}^{-1}\mathbf{A})^{-1}\|_\infty \leq 1/(1 - 2\beta)$. This gives

$$\|\mathbf{X}\|_\infty \leq \frac{1}{1 - 2\beta}\|\mathbf{D}^{-1}\mathbf{p}\|_\infty = \frac{1}{1 - 2\beta}\|(\mathbf{W}^{1/2}\mathbf{D})^{-1}(\mathbf{W}^{1/2}\mathbf{p})\|_\infty.$$

The components of $(\mathbf{W}^{1/2}\mathbf{D})^{-1}(\mathbf{W}^{1/2}\mathbf{p})$ are the dot product between the rows of $(\mathbf{W}^{1/2}\mathbf{D})^{-1}$ and $\mathbf{W}^{1/2}\mathbf{p}$. The Euclidean length of $\mathbf{W}^{1/2}\mathbf{p}$ is $\|\mathbf{p}\|_\omega$. We will show that the rows of $(\mathbf{W}^{1/2}\mathbf{D})^{-1}$ have length bounded by $\sqrt{2}$.

More on $\mathbf{D}$

Invertibility of $\mathbf{D}$: Let $p \in \mathcal{P}_N$ be the polynomial that satisfies $p(-1) = 0$ and $p(\tau_i) = p_i$, $1 \le i \le N$. If $\mathbf{Dp} = \mathbf{0}$, then $\dot{p}(\tau_i) = 0$, $1 \le i \le N$. Since $\dot{p}$ has degree $N-1$, we deduce that $\dot{p} = 0$. Since $p(-1) = 0$, $p = 0$ which implies that $\mathbf{p} = \mathbf{0}$.

A formula for the inverse of $\mathbf{D}$: $p \in \mathcal{P}_N$ as above and $\dot{\mathbf{p}}$ denotes vector with components $\dot{p}(\tau_i)$. Expand $\dot{p}$ in a Lagrange basis:

$$\dot{p}(\tau) = \sum_{j=1}^{N} \ell_j(\tau)\dot{p}_j, \quad \ell_j(\tau) = \prod_{\substack{j=1 \\ j \ne i}}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j}$$

Integrate to obtain

$$p_i = \int_{-1}^{\tau_i} \dot{p}(\tau)\, d\tau = \sum_{j=1}^{N} \left(\int_{-1}^{\tau_i} \ell_j(\tau)\, d\tau \right) \dot{p}_j$$

Comparing to $\mathbf{p} = \mathbf{D}^{-1}\dot{\mathbf{p}}$, we see that $(\mathbf{D}^{-1})_{ij} = \displaystyle\int_{-1}^{\tau_i} \ell_j(\tau)\, d\tau$.

The rows of $(\mathbf{W}^{1/2}\mathbf{D})^{-1}$ have length $\leq \sqrt{2}$ (proof by Xiang-Sheng Wang): Define

$$d_j(\tau) = \int_{-1}^{\tau} \ell_j(\tau)\, d\tau \quad \text{and} \quad R(\tau) = \sum_{j=1}^{N} \frac{d_j(\tau)^2}{\omega_j}.$$

Observe that $(\mathbf{D}^{-1})_{ij} = d_j(\tau_i)$ and $R(\tau_i)$ is the square of the Euclidean length of row i in $(\mathbf{W}^{1/2}\mathbf{D})^{-1}$. Choose any $s \in [-1, 1]$ and let $q \in \mathcal{P}_{N-1}$ be the polynomial defined by

$$q(\tau) = \sum_{j=1}^{N} \frac{d_j(s)\ell_j(\tau)}{\omega_j}.$$

Hence,

$$R(s) = \int_{-1}^{s} q(\tau)\, d\tau \leq \int_{-1}^{1} |q(\tau)|\, d\tau \leq \sqrt{2} \left(\int_{-1}^{1} q(\tau)^2\, d\tau \right)^{1/2}.$$

Continue . . .

Since q^2 has degree $2N-2$, Gaussian quadrature is exact, and

$$\int_{-1}^{1} q(\tau)^2 \, d\tau = \sum_{i=1}^{N} \omega_i q(\tau_i)^2 = \sum_{i=1}^{N} \frac{d_i(s)^2}{\omega_j} = R(s)$$

since $q(\tau_i) = d_i(s)/\omega_i$. Equating these two expressions for $R(s)$ yields

$$\int_{-1}^{1} q(\tau)^2 \, d\tau \le 2.$$

It follows that $R(s) \le 2$, which shows that the rows of $(\mathbf{W}^{1/2}\mathbf{D})^{-1}$ have length $\le \sqrt{2}$.

$\|\mathbf{D}^{-1}\|_\infty \le 2$: This follows from the Schwarz inequality and the fact that the quadrature weights sum to 2. If $\mathbf{r}$ is a row from $\mathbf{D}^{-1}$, then

$$\sum_{i=1}^{N} |r_i| = \sum_{i=1}^{N} \sqrt{\omega_i}\,(|r_i|/\sqrt{\omega_i}) \le \left(\sum_{i=1}^{N} \omega_i\right)^{1/2} \left(\sum_{i=1}^{N} r_i^2/\omega_i\right)^{1/2} \le 2$$

Lipschitz Continuity (C3)

Lemma. Suppose the following hold:

(A1) For some $\alpha > 0$, the smallest eigenvalue of the Hessian matrices $\nabla^2 C(\mathbf{x}^*(1))$ and $\nabla^2_{(x,u)} H(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t))$ are greater than α, uniformly for $t \in [0, 1]$.

(A2) For some $\beta < 1/2$, the Jacobian of the dynamics satisfies

$$\|\nabla_x \mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t))\|_\infty \le \beta \quad \text{and} \quad \|\nabla_x \mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t))^{\mathsf{T}}\|_\infty \le \beta$$

for all $t \in \Omega$ where $\| \cdot \|_\infty$ is the matrix sup-norm (largest absolute row sum), and the Jacobian $\nabla_x \mathbf{f}$ is an n by n matrix whose i-th row is $(\nabla_x f_i)^{\mathsf{T}}$.

Then there exists a constant c, independent of N, such that the change $(\Delta \mathbf{X}, \Delta \mathbf{U}, \Delta \boldsymbol{\Lambda})$ in the solution of linearized problem corresponding to a change $\Delta \mathbf{Y}$ in $\mathbf{Y} \in \mathcal{Y}$ satisfies

$$\max \{\|\Delta \mathbf{X}\|_\infty, \|\Delta \mathbf{U}\|_\infty, \|\Delta \boldsymbol{\Lambda}\|_\infty\} \le c \|\Delta \mathbf{Y}\|_{\mathcal{Y}}.$$

Analysis of the Residual δ

Recall that the residual had the form

$$\delta = \begin{pmatrix} \dot{\mathbf{x}}^*(\tau_i) - \dot{\mathbf{x}}^I(\tau_i), \quad 1 \le i \le N, \\ \mathbf{0} \\ \dot{\boldsymbol{\lambda}}^*(\tau_i) - \dot{\boldsymbol{\lambda}}^I(\tau_i), \quad 1 \le i \le N \\ \nabla \mathbf{x}^I(1) - \nabla \mathbf{x}^*(1) \\ \mathbf{0} \end{pmatrix}$$

where $\mathbf{x}^I \in \mathcal{P}_N^n$ satisfies $\mathbf{x}^I(\tau_i) = \mathbf{x}^*(\tau_i)$, $0 \le i \le N$, and where $\boldsymbol{\lambda}^I \in \mathcal{P}_N^n$ satisfies $\boldsymbol{\lambda}^I(\tau_i) = \boldsymbol{\lambda}^*(\tau_i)$, $1 \le i \le N+1$,

Interpolation Error

Lemma. If $u \in \mathcal{H}^\eta(\Omega)$ for some $\eta \geq 1$, then there exists a constant c, independent of N and η, such that

$$|u - u^I|_{\mathcal{H}^1(\Omega)} \leq (c/N)^{p-3/2} |u|_{\mathcal{H}^p(\Omega)}, \quad p = \min\{\eta, N+1\},$$

where $u^I \in \mathcal{P}_N$ is the interpolant of u satisfying $u^I(\tau_i) = u(\tau_i)$, $0 \leq i \leq N$, and $N > 0$.
Bernardi and Maday

Apply Interpolation Error to Residual

A bound is needed for $\|\dot{\mathbf{x}}^* - \dot{\mathbf{x}}^I\|_\omega$. Let $(\dot{\mathbf{x}})^J \in \mathcal{P}_{N-1}$ denote the polynomial that interpolates $\dot{\mathbf{x}}^*$ at the Gauss points. Then we have

$$\|\dot{\mathbf{x}}^* - \dot{\mathbf{x}}^I\|_\omega = \|(\dot{\mathbf{x}}^*)^J - \dot{\mathbf{x}}^I\|_\omega = \|(\dot{\mathbf{x}}^*)^J - \dot{\mathbf{x}}^I\|_{\mathcal{L}^2(\Omega)}$$

since Gauss quadrature is exact for polynomials of degree up to $2N-1$. Now

$$\begin{aligned}
\|(\dot{\mathbf{x}}^*)^J - \dot{\mathbf{x}}^I\|_{\mathcal{L}^2(\Omega)} &\leq \|(\dot{\mathbf{x}}^*)^J - \dot{\mathbf{x}}^*\|_{\mathcal{L}^2(\Omega)} + \|\dot{\mathbf{x}}^* - \dot{\mathbf{x}}^I\|_{\mathcal{L}^2(\Omega)} \\
&\leq (c/N)^{p-3/2}|\mathbf{x}^*|_{\mathcal{H}^p(\Omega)}.
\end{aligned}$$

Final Result

Theorem. Suppose $(\mathbf{x}^*, \mathbf{u}^*)$ is a local minimizer for the continuous problem with $(\mathbf{x}^*, \boldsymbol{\lambda}^*) \in \mathcal{H}^\eta(\Omega; \mathbb{R}^n)$ for some $\eta \geq 2$. If the convexity conditions hold as well as the bounds for $\nabla_x \mathbf{f}$, then for N sufficiently large, the discrete problem has a local minimizer $\mathbf{x}^N \in \mathcal{P}_N^n$ and $\mathbf{u} \in \mathbb{R}^{mN}$, and an associated multiplier $\boldsymbol{\lambda}^N \in \mathcal{P}_N^n$; moreover, there exists a constant c independent of N and η such that

$$\max\left\{\|\mathbf{X}^N - \mathbf{X}^*\|_\infty, \|\mathbf{U}^N - \mathbf{U}^*\|_\infty, \|\mathbf{\Lambda}^N - \mathbf{\Lambda}^*\|_\infty\right\}$$
$$\leq \left(\frac{c}{N}\right)^{p-3/2} \left(|\mathbf{x}^*|_{\mathcal{H}^p(\Omega;\, \mathbb{R}^n)} + |\boldsymbol{\lambda}^*|_{\mathcal{H}^p(\Omega;\, \mathbb{R}^n)}\right), \quad p := \min\{\eta, N+1\}.$$