Computational Methods in Optimal Control
Lecture 3. More Methods

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Range-Kutta Versus Polynomial Approximation

The error associated with Runge-Kutta scheme is often of the form $O(h^p)$, where $p > 0$ is the order of the method (often ≤ 4). Range-Kutta methods achieve convergence as the mesh spacing tends to zero, and attaining a given error tolerance could require a very fine mesh. We now examine purely polynomial-based schemes, which can converge much faster when the solution is smooth. In particular, for a polynomial-based method, the error can can be $O(1/N^N)$ where N is the degree of the polynomials.

$$\begin{aligned} \text{minimize} \quad & C(\mathbf{x}(1)) \\ \text{subject to} \quad & \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \quad \mathbf{u}(t) \in \mathcal{U}, \quad t \in \Omega, \\ & \mathbf{x}(-1) = \mathbf{x}_0. \end{aligned}$$

- $\Omega = [-1, +1]$, $\mathbf{x}_0$ given, $\mathbf{x}(t) \in \mathbb{R}^n$,
- $\mathcal{U} \subset \mathbb{R}^m$ closed and convex,
- $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $C : \mathbb{R}^n \to \mathbb{R}$

Discrete Problem: Collocation at Gauss quadrature points

$$
\begin{array}{ll}
\text{minimize} & C(\mathbf{x}(1)) \\
\text{subject to} & \dot{\mathbf{x}}(\tau_i) = \mathbf{f}(\mathbf{x}(\tau_i), \mathbf{u}_i), \quad \mathbf{u}_i \in \mathcal{U}, \quad 1 \le i \le N, \\
& \mathbf{x}(-1) = \mathbf{x}_0, \quad \mathbf{x} \in \mathcal{P}_N^n.
\end{array}
$$

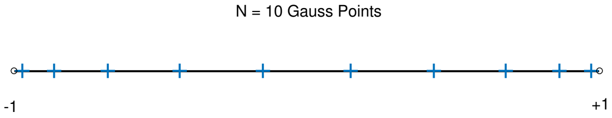
- $\mathcal{P}_N$ = polynomials of degree at most N,
- $\mathcal{P}_N^n$ = n-fold product $\mathcal{P}_N \times \ldots \times \mathcal{P}_N$.
- Gauss quadrature points:

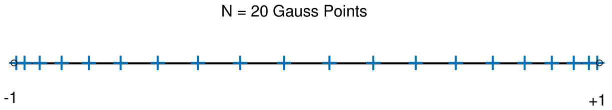
$$-1 < \tau_1 < \tau_2 < \ldots < \tau_N < +1.$$

- Additional points in analysis:

$$\tau_0 = -1 \quad \text{and} \quad \tau_{N+1} = +1.$$

The Gauss Points





Lagrange Interpolation and the Differentiation Matrix

- Lagrange interpolating polynomials: For $0 \leq j \leq N$,

$$\Phi_i(\tau) = \prod_{\substack{j=0 \\ j\neq i}}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \quad \Phi_i(\tau_j) = \begin{cases} 1 \text{ for } j = i \\ 0 \text{ otherwise.} \end{cases}$$

If $x \in \mathcal{P}_N$, then $x(\tau) = \sum_{j=0}^{N} x(\tau_j)\Phi_j(\tau)$

- Differentiation matrix $\mathbf{D} \in \mathbb{R}^{N\times(N+1)}$

$$\dot{x}(\tau_i) = \sum_{j=0}^{N} \dot{\Phi}_j(\tau_i)x(\tau_j) = \sum_{j=0}^{N} D_{ij}x(\tau_j), \quad D_{ij} = \dot{\Phi}_j(\tau_i)$$

Gauss Quadrature

The Gauss collocation points τ_i, $1 \leq i \leq N$, are the roots of the Legendre polynomial P_N of degree N. The associated Gauss quadrature weights ω_i, $1 \leq i \leq N$, are given by

$$\omega_i = \frac{2}{(1-\tau_i^2)P_N'(\tau_i)^2}. \tag{1}$$

For any $p \in \mathcal{P}_{2N-1}$, we have

$$\int_{-1}^{1} p(t)\,dt = \sum_{i=1}^{N} \omega_i p(\tau_i). \tag{2}$$

If $\mathbf{x} \in \mathcal{P}_N$ and $\mathbf{X}_i$ denotes $x(\tau_i)$, $0 \leq i \leq N+1$, then

$$\mathbf{X}_{N+1} = \mathbf{x}(1) = \mathbf{x}(-1) + \int_{-1}^{1} \dot{\mathbf{x}}(t)\,dt = \mathbf{X}_0 + \sum_{j=1}^{N} \omega_j \dot{\mathbf{x}}(\tau_j).$$

Convert from $\mathcal{P}_N^n$ to $\mathbb{R}^{nN}$

NOTE: If $\mathbf{x} \in \mathcal{P}_N^n$ is feasible in the discrete control problem, then $\dot{\mathbf{x}}(\tau_i) = \mathbf{f}(\mathbf{x}(\tau_i), \mathbf{u}_i) = \mathbf{f}(\mathbf{X}_i, \mathbf{U}_i)$; moreover,

$$\dot{\mathbf{x}}(\tau_i) = \sum_{j=0}^{N} D_{ij}\mathbf{x}(\tau_j) = \dot{\mathbf{x}}(\tau_i) = \sum_{j=0}^{N} D_{ij}\mathbf{X}_j.$$

Hence, the discrete control problem is equivalent to

$$\begin{aligned} \text{minimize} \quad & C(\mathbf{X}_{N+1}) \\ \text{subject to} \quad & \sum_{j=0}^{N} D_{ij}\mathbf{X}_j = \mathbf{f}(\mathbf{X}_i, \mathbf{U}_i), \quad \mathbf{U}_i \in \mathcal{U}, \quad 1 \le i \le N, \\ & \mathbf{X}_0 = \mathbf{x}_0, \quad \mathbf{X}_{N+1} = \mathbf{X}_0 + \sum_{j=1}^{N} \omega_j \mathbf{f}(\mathbf{X}_j, \mathbf{U}_j). \end{aligned}$$

Lagrangian and Stationarity

$$
\begin{aligned}
C(\mathbf{X}_{N+1}) \quad & + \sum_{i=1}^{N} \left\langle \boldsymbol{\mu}_i, \mathbf{f}(\mathbf{X}_i, \mathbf{U}_i) - \sum_{j=0}^{N} D_{ij} \mathbf{X}_j \right\rangle + \\
& + \left\langle \boldsymbol{\mu}_{N+1}, \mathbf{X}_0 - \mathbf{X}_{N+1} + \sum_{i=1}^{N} \omega_i \mathbf{f}(\mathbf{X}_i, \mathbf{U}_i) \right\rangle .
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{X}_j \quad & \Rightarrow \quad \sum_{i=1}^{N} D_{ij} \boldsymbol{\mu}_i = \nabla_x H(\mathbf{X}_j, \mathbf{U}_j, \boldsymbol{\mu}_j + \omega_j \boldsymbol{\mu}_{N+1}), \quad 1 \le j \le N, \\
\mathbf{X}_{N+1} \quad & \Rightarrow \quad \boldsymbol{\mu}_{N+1} = \nabla C(\mathbf{X}_{N+1}), \\
\mathbf{U}_i \quad & \Rightarrow \quad -\nabla_u H\left(\mathbf{X}_i, \mathbf{U}_i, \boldsymbol{\mu}_i + \omega_i \boldsymbol{\mu}_{N+1}\right) \in N_{\mathcal{U}}(\mathbf{U}_i), \quad 1 \le i \le N.
\end{aligned}
$$

First-order Optimality Conditions

Theorem. The multipliers $\boldsymbol{\mu} \in \mathbb{R}^{n(N+1)}$ satisfy the stationarity conditions if and only if the polynomial $\boldsymbol{\lambda} \in \mathcal{P}_N^n$ for which $\boldsymbol{\lambda}(1) = \boldsymbol{\mu}_{N+1}$ and $\boldsymbol{\lambda}(\tau_i) = \boldsymbol{\mu}_{N+1} + \boldsymbol{\mu}_i/\omega_i$, $1 \le i \le N$, also satisfies

$$\begin{aligned} \dot{\boldsymbol{\lambda}}(\tau_i) &= -\nabla_x H\left(\mathbf{x}(\tau_i), \mathbf{u}_i, \boldsymbol{\lambda}(\tau_i)\right), \quad 1 \le i \le N, \\ \boldsymbol{\lambda}(1) &= \nabla C(\mathbf{x}(1)), \\ N_{\mathcal{U}}(\mathbf{u}_i) &\ni -\nabla_u H\left(\mathbf{x}(\tau_i), \mathbf{u}_i, \boldsymbol{\lambda}(\tau_i)\right), \quad 1 \le i \le N. \end{aligned}$$

The $\mathbf{D}^\dagger$ Matrix

Let $\mathbf{W} = \text{diag}(\boldsymbol{\omega})$, let $\overline{\mathbf{D}} = \mathbf{D}_{1:N}$, and let $\mathbf{D}^\dagger$ be defined by

$$\overline{\mathbf{D}}^\dagger = -\mathbf{W}^{-1}\overline{\mathbf{D}}\mathbf{W}, \quad \mathbf{D}^\dagger_{N+1} = -\overline{\mathbf{D}}^\dagger \mathbf{1}.$$

If $\boldsymbol{\lambda} \in \mathcal{P}^n_N$ is a polynomial that satisfies the conditions $\boldsymbol{\lambda}(\tau_i) = \boldsymbol{\Lambda}_i$ for $1 \leq i \leq N+1$, then

$$\sum_{j=1}^{N+1} D^\dagger_{ij} \boldsymbol{\Lambda}_j = \dot{\boldsymbol{\lambda}}(\tau_i), \quad 1 \leq i \leq N.$$

Proof

Suppose p and $q \in \mathcal{P}_N$ with $p(-1) = q(1) = 0$. We have

$$\sum_{i=1}^{N} \omega_i \dot{p}_i q_i = \int_{-1}^{1} \dot{p}(\tau) q(\tau)\ d\tau = -\int_{-1}^{1} p(\tau) \dot{q}(\tau)\ d\tau = \sum_{i=1}^{N} \omega_i p_i \dot{q}_i,$$

where $p_i = p(\tau_i)$, $q_i = q(\tau_i)$, $\dot{p}_i = \dot{p}(\tau_i)$, and $\dot{q}_i = \dot{q}(\tau_i)$. In matrix notation,

$$(\mathbf{W}\overline{\mathbf{D}}\mathbf{p})^{\mathsf{T}}\mathbf{q} = -(\mathbf{W}\mathbf{p})^{\mathsf{T}}\dot{\mathbf{q}} \Longrightarrow \mathbf{p}^{\mathsf{T}}\overline{\mathbf{D}}^{\mathsf{T}}\mathbf{W}\mathbf{q} = -\mathbf{p}^{\mathsf{T}}\mathbf{W}\dot{\mathbf{q}}.$$

where $\mathbf{p}$, $\mathbf{q}$, and $\dot{\mathbf{q}}$ are N component vectors. Since this hold for all $\mathbf{p}$, it follows that

$$\overline{\mathbf{D}}^{\mathsf{T}}\mathbf{W}\mathbf{q} = -\mathbf{W}\dot{\mathbf{q}} \Longrightarrow \dot{\mathbf{q}} = -\mathbf{W}^{-1}\overline{\mathbf{D}}^{\mathsf{T}}\mathbf{W}\mathbf{q}.$$

Proof of Theorem

Define $\mathbf{\Lambda}_i = \boldsymbol{\mu}_{N+1} + \boldsymbol{\mu}_i/\omega_i$ for $1 \le i \le N$, $\mathbf{\Lambda}_{N+1} = \boldsymbol{\mu}_{N+1}$; Hence, we have $\boldsymbol{\mu}_i = \omega_i(\mathbf{\Lambda}_i - \mathbf{\Lambda}_{N+1})$ for $1 \le i \le N$. Substitute for $\mathbf{D}$ in terms of $\mathbf{D}^\dagger$ and for $\boldsymbol{\mu}_i$ to obtain

$$
\begin{aligned}
\sum_{j=1}^{N+1} D_{ij}^\dagger \mathbf{\Lambda}_j &= -\nabla_x H\left(\mathbf{X}_i, \mathbf{U}_i, \mathbf{\Lambda}_i\right), \quad 1 \le i \le N \\
\mathbf{\Lambda}_{N+1} &= \nabla C(\mathbf{X}_{N+1}), \\
N_{\mathcal{U}}(\mathbf{U}_i) &\ni -\nabla_u H\left(\mathbf{X}_i, \mathbf{U}_i, \mathbf{\Lambda}_i\right), \quad 1 \le i \le N
\end{aligned}
$$

Let $\boldsymbol{\lambda} \in \mathcal{P}_N^n$ be the polynomial that is given by $\boldsymbol{\lambda}(\tau_i) = \mathbf{\Lambda}_i$ for $1 \le i \le N+1$. Since $\mathbf{D}^\dagger$ is a differentiation matrix, we obtain the theorem.

Review: Continuous Problem

$$\begin{array}{ll} \text{minimize} & C(\mathbf{x}(1)) \\ \text{subject to} & \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \quad \mathbf{u}(t) \in \mathcal{U}, \quad t \in [-1, 1], \\ & \mathbf{x}(0) = \mathbf{x}_0 \end{array}$$

First-order optimality conditions for a local minimizer:

$$\begin{array}{rcl} \dot{\mathbf{x}}(t) & = & \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \quad \mathbf{u}(t) \in \mathcal{U}, \\ \mathbf{x}(0) & = & \mathbf{x}_0 \\ \dot{\boldsymbol{\lambda}}(t) & = & -\nabla_x H(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t)), \\ \boldsymbol{\lambda}(1) & = & \nabla C(\mathbf{x}(1)) \\ N_{\mathcal{U}}(\mathbf{u}(t)) & \ni & -\nabla_u H(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t)) \quad \text{for all } t \in [-1, 1] \end{array}$$

Review: Pseudospectral Method

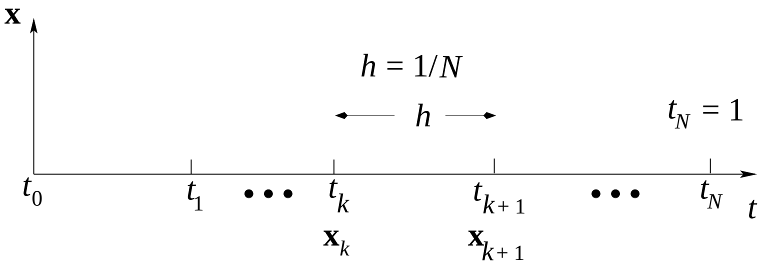
$$\begin{array}{ll} \text{minimize} & C(\mathbf{x}(1)) \\ \text{subject to} & \dot{\mathbf{x}}(\tau_i) = \mathbf{f}(\mathbf{x}(\tau_i), \mathbf{u}_i), \quad \mathbf{u}_i \in \mathcal{U}, \quad 1 \le i \le N, \\ & \mathbf{x}(-1) = \mathbf{x}_0, \quad \mathbf{x} \in \mathcal{P}_N^n. \end{array}$$

First-order optimality conditions for a local minimizer:

$$\begin{array}{rcl} \dot{\mathbf{x}}(\tau_i) & = & \mathbf{f}(\mathbf{x}(\tau_i), \mathbf{u}_i), \quad \mathbf{u}_i \in \mathcal{U}, \quad 1 \le i \le N, \\ \mathbf{x}(-1) & = & \mathbf{x}_0, \quad \mathbf{x} \in \mathcal{P}_N^n \\ \dot{\boldsymbol{\lambda}}(\tau_i) & = & -\nabla_x H\left(\mathbf{x}(\tau_i), \mathbf{u}_i, \boldsymbol{\lambda}(\tau_i)\right), \quad 1 \le i \le N, \quad \boldsymbol{\lambda} \in \mathcal{P}_N^n \\ \boldsymbol{\lambda}(1) & = & \nabla C(\mathbf{x}(1)), \\ N_{\mathcal{U}}(\mathbf{u}_i) & \ni & -\nabla_u H\left(\mathbf{x}(\tau_i), \mathbf{u}_i, \boldsymbol{\lambda}(\tau_i)\right), \quad 1 \le i \le N. \end{array}$$

Review: s-stage Runge-Kutta Discretization

$$
\begin{aligned}
\text{minimize} \quad & C(\mathbf{x}_N) \\
\text{subject to} \quad & \mathbf{y}_i = \mathbf{x}_k + h \sum_{j=1}^{s} a_{ij} \mathbf{f}(\mathbf{y}_j, \mathbf{u}_{kj}), \quad i = 1, \ldots, s \\
& \dot{\mathbf{x}}_k = \sum_{i=1}^{s} b_i \mathbf{f}(\mathbf{y}_i, \mathbf{u}_{ki}), \quad \mathbf{u}_{ki} \in \mathcal{U}
\end{aligned}
$$



Review: *s*-stage Runge-Kutta Discretization

First-order optimality conditions for a local minimizer:

$$
\begin{aligned}
\mathbf{y}_{ki} &= \mathbf{x}_k + h\sum_{j=1}^{s} a_{ij}\mathbf{f}(\mathbf{y}_{kj},\mathbf{u}_{kj}),\\
\dot{\mathbf{x}}_k &= \sum_{i=1}^{s} b_i\mathbf{f}(\mathbf{y}_{ki},\mathbf{u}_{ki}), \quad \mathbf{x}_0 \text{ given}\\
\boldsymbol{\lambda}_{ki} &= \psi_k - h\sum_{j=1}^{s} \bar{a}_{ij}\boldsymbol{\lambda}_{kj}\nabla_x\mathbf{f}(\mathbf{y}_{kj},\mathbf{u}_{kj}), \quad \bar{a}_{ij} = \frac{b_i b_j - b_j a_{ji}}{b_i},\\
\dot{\psi}_k &= -\sum_{i=1}^{s} b_i\boldsymbol{\lambda}_{ki}\nabla_x\mathbf{f}(\mathbf{y}_{ki},\mathbf{u}_{ki}), \quad \psi_N = \nabla C(\mathbf{x}_N),\\
N_{\mathcal{U}}(\mathbf{u}_{ki}) &\ni -\boldsymbol{\lambda}_{ki}\nabla_u\mathbf{f}(\mathbf{y}_{ki},\mathbf{u}_{ki})
\end{aligned}
$$

New Model

$$\begin{aligned} \text{minimize} \quad & \int_0^1 g(\mathbf{x}(t), \mathbf{u}(t))\, dt \\ \text{subject to} \quad & \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{u}(t) \in \mathcal{U}, \quad t \in [0,1], \\ & \mathbf{x}(0) = \mathbf{x}_0, \end{aligned}$$

where $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{B} \in \mathbb{R}^{n \times m}$. Hamiltonian:

$$H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) = g(\mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}).$$

First-order optimality conditions for a local minimizer:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{u}(t) \in \mathcal{U}, \\ \dot{\boldsymbol{\lambda}}(t) &= -\left[\boldsymbol{\lambda}(t)\mathbf{A} + \nabla_x g(\mathbf{x}(t), \mathbf{u}(t))\right], \quad \boldsymbol{\lambda}(1) = \mathbf{0} \\ N_{\mathcal{U}}(\mathbf{u}(t)) &\ni -\left[\boldsymbol{\lambda}(t)\mathbf{B} + \nabla_u g(\mathbf{x}(t), \mathbf{u}(t))\right] \quad \text{for all } t \in [0,1] \end{aligned}$$

Second-order Taylor Expansion of Objective

$$g(\mathbf{x},\mathbf{u}) \approx g_k + \mathcal{L}_k(\mathbf{x}-\mathbf{x}_k,\mathbf{u}-\mathbf{u}_k) + \frac{1}{2}\mathcal{B}_k(\mathbf{x}-\mathbf{x}_k,\mathbf{u}-\mathbf{u}_k)$$

where $\langle\cdot,\cdot\rangle$ denotes L^2 inner product and

$$\begin{aligned}
\mathcal{L}_k(\mathbf{x}-\mathbf{x}_k,\mathbf{u}-\mathbf{u}_k) &= \langle\nabla_x g_k, \mathbf{x}-\mathbf{x}_k\rangle + \langle\nabla_u g_k, \mathbf{u}-\mathbf{u}_k\rangle \\
\mathcal{B}_k(\mathbf{x}-\mathbf{x}_k,\mathbf{u}-\mathbf{u}_k) &= \langle\mathbf{Q}_k(\mathbf{x}-\mathbf{x}_k),\mathbf{x}-\mathbf{x}_k\rangle + 2\langle\mathbf{S}_k(\mathbf{x}-\mathbf{x}_k),\mathbf{u}-\mathbf{u}_k\rangle + \\
&\quad \langle\mathbf{R}_k(\mathbf{u}-\mathbf{u}_k),\mathbf{u}-\mathbf{u}_k\rangle
\end{aligned}$$

$$\begin{aligned}
\nabla_x g_k(t) &= \nabla_x g(\mathbf{x}_k(t),\mathbf{u}_k(t)), \quad \nabla_u g_k(t) = \nabla_u g(\mathbf{x}_k(t),\mathbf{u}_k(t)), \\
\mathbf{Q}_k(t) &= \nabla_{xx} g(\mathbf{x}_k(t),\mathbf{u}_k(t)), \quad \mathbf{S}_k(t) = \nabla_{ux} g(\mathbf{x}_k(t),\mathbf{u}_k(t)) \\
\mathbf{R}_k(t) &= \nabla_{uu} g(\mathbf{x}_k(t),\mathbf{u}_k(t))
\end{aligned}$$

Sequential Quadratic Programming (SQP)

Let $(\mathbf{x}_k, \mathbf{u}_k)$ denote the current iterate. In the SQP method, the next iterate is obtained by solving the quadratic programming problem

$$\begin{array}{ll} \text{minimize} & \mathcal{L}_k(\mathbf{x}-\mathbf{x}_k, \mathbf{u}-\mathbf{u}_k) + \frac{1}{2}\mathcal{B}_k(\mathbf{x}-\mathbf{x}_k, \mathbf{u}-\mathbf{u}_k) \\ \text{subject to} & \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{u}(t) \in \mathcal{U}, \quad t \in [0,1]. \end{array}$$

At a solution, the first-order optimality conditions hold:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{u}(t) \in \mathcal{U} \\ \dot{\boldsymbol{\lambda}}(t) &= -\Big(\boldsymbol{\lambda}(t)\mathbf{A} + \nabla_x g_k(t) + (\mathbf{x}(t) - \mathbf{x}_k(t))^{\mathsf{T}}\mathbf{Q}_k(t) \\ &\quad +(\mathbf{u}(t) - \mathbf{u}_k(t))^{\mathsf{T}}\mathbf{S}_k(t)\Big), \quad \boldsymbol{\lambda}(1) = \mathbf{0} \\ N_{\mathcal{U}}(\mathbf{u}(t)) &\ni -\Big(\boldsymbol{\lambda}(t)\mathbf{B} + \nabla_u g_k(t) + [\mathbf{S}_k(t)(\mathbf{x}(t) - \mathbf{x}_k(t))]^{\mathsf{T}} + \\ &\quad [\mathbf{R}_k(t)(\mathbf{u}(t) - \mathbf{u}_k(t))]^{\mathsf{T}}\Big) \end{aligned}$$

SQP Versus Original Optimality Conditions

The optimality conditions for the SQP scheme and for the original control problem are identical except that $\nabla g(\mathbf{x}(t), \mathbf{u}(t))$ in the original first-order optimality conditions is replaced by the first-order Taylor expansion around $(\mathbf{x}_k(t), \mathbf{u}_k(t))$.

Abstractly, the discretizations and algorithms such as SQP amount to the problem:

$$\text{Find } \mathbf{w} \in \mathcal{X} \text{ such that } \mathcal{T}(\mathbf{w}) \in \mathcal{F}(\mathbf{w}), \tag{D}$$

where $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$, a normed linear space, and $\mathcal{F} : \mathcal{X} \to 2^{\mathcal{Y}}$. We are given a solution $\mathbf{w}^*$ to the control problem and we wish to bound the distance from $\mathbf{w}^*$ to a solution of (D).

Example: SQP

Take $\mathbf{w} = (\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) \in W^{1,\infty} \times W^{0,\infty} \times W^{1,\infty}$, and

$$\mathcal{T}(\mathbf{w}) = \begin{pmatrix} \dot{\mathbf{x}} - \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{u} \\ \mathbf{x}(0) - \mathbf{x}_0 \\ \dot{\boldsymbol{\lambda}} + \boldsymbol{\lambda}\mathbf{A} + \nabla_x g_k + (\mathbf{x} - \mathbf{x}_k)^{\mathsf{T}}\mathbf{Q}_k + (\mathbf{u} - \mathbf{u}_k)^{\mathsf{T}}\mathbf{S}_k \\ \boldsymbol{\lambda}(1) \\ -\left(\boldsymbol{\lambda}\mathbf{B} + \nabla_u g_k + [\mathbf{S}_k(\mathbf{x} - \mathbf{x}_k)]^{\mathsf{T}} + [\mathbf{R}_k(\mathbf{u} - \mathbf{u}_k)]^{\mathsf{T}}\right) \end{pmatrix}$$

$$\mathcal{F}(\mathbf{w}) = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ N_{\mathcal{U}}(\mathbf{u}) \end{pmatrix}$$

Example: Pseudospectral Method

Take $\mathbf{w} = (\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) \in \mathcal{P}_N^n \times \mathbb{R}^{mN} \times \mathcal{P}_N^n$ and

$$\mathcal{T}(\mathbf{w}) = \begin{pmatrix} \dot{\mathbf{x}}(\tau_i) - \mathbf{f}(\mathbf{x}(\tau_i), \mathbf{u}_i), \quad 1 \le i \le N, \\ \mathbf{x}(-1) - \mathbf{x}_0 \\ \dot{\boldsymbol{\lambda}}(\tau_i) + \nabla_x H\left(\mathbf{x}(\tau_i), \mathbf{u}_i, \boldsymbol{\lambda}(\tau_i)\right), \quad 1 \le i \le N \\ \boldsymbol{\lambda}(1) - \nabla C(\mathbf{x}(1)) \\ -\nabla_u H\left(\mathbf{x}(\tau_i), \mathbf{u}_i, \boldsymbol{\lambda}(\tau_i)\right), \quad 1 \le i \le N \end{pmatrix}$$

$$\mathcal{F}(\mathbf{w}) = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ N_{\mathcal{U}}(\mathbf{u}_i), \quad 1 \le i \le N \end{pmatrix}$$

Test Point $\mathbf{w}^l$ for Pseudospectral Method

Suppose that $(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*)$ satisfies the first-order optimality conditions for the continuous control problem. Let us consider the point $\mathbf{w}^l = (\mathbf{x}^l, \mathbf{u}^l, \boldsymbol{\lambda}^l)$ where $\mathbf{x}^l$ and $\boldsymbol{\lambda}^l \in \mathcal{P}_N^n$, and $\mathbf{u}^l \in \mathbb{R}^{mN}$ satisfy

$$
\begin{aligned}
\mathbf{x}^l(\tau_i) &= \mathbf{x}^*(\tau_i), \quad 0 \le i \le N \\
\boldsymbol{\lambda}^l(\tau_i) &= \lambda^*(\tau_i), \quad 1 \le i \le N+1 \\
\mathbf{u}_i^l &= \mathbf{u}(\tau_i), \quad 1 \le i \le N
\end{aligned}
$$

By the first-order optimality conditions for the continuous control problem, we have for $1 \le i \le N$:

$$
\begin{aligned}
\mathbf{f}(\mathbf{x}^l(\tau_i), \mathbf{u}_i^l) &= \mathbf{f}(\mathbf{x}^*(\tau_i), \mathbf{u}^*(\tau_i)) &&= \dot{\mathbf{x}}^*(\tau_i) \\
\nabla_x H(\mathbf{x}^l(\tau_i), \mathbf{u}_i^l, \boldsymbol{\lambda}^l(\tau_i)) &= \nabla_x H(\mathbf{x}^*(\tau_i), \mathbf{u}^*(\tau_i), \boldsymbol{\lambda}^*(\tau_i)) &&= \dot{\boldsymbol{\lambda}}^*(\tau_i), \\
\nabla_u H(\mathbf{x}^l(\tau_i), \mathbf{u}_i^l, \boldsymbol{\lambda}^l(\tau_i)) &= \nabla_u H(\mathbf{x}^*(\tau_i), \mathbf{u}^*(\tau_i), \boldsymbol{\lambda}^*(\tau_i))
\end{aligned}
$$

Residual for Pseudospectral Method

With these substitutions, we have

$$\mathcal{T}(\mathbf{w}^l) = \begin{pmatrix} \dot{\mathbf{x}}^l(\tau_i) - \dot{\mathbf{x}}^*(\tau_i), & 1 \le i \le N, \\ \mathbf{0} & \\ \dot{\boldsymbol{\lambda}}^l(\tau_i) - \dot{\boldsymbol{\lambda}}^*(\tau_i), & 1 \le i \le N \\ \boldsymbol{\lambda}^*(1) - \nabla \mathbf{x}^l(1) & \\ -\nabla_u H(\mathbf{x}^*(\tau_i), \mathbf{u}^*(\tau_i), \boldsymbol{\lambda}^*(\tau_i)), & 1 \le i \le N \end{pmatrix}$$

Hence, $\mathcal{T}(\mathbf{w}^l) + \boldsymbol{\delta} \in \mathcal{F}(\mathbf{w}^l)$ where

$$\boldsymbol{\delta} = \begin{pmatrix} \dot{\mathbf{x}}^*(\tau_i) - \dot{\mathbf{x}}^l(\tau_i), & 1 \le i \le N, \\ \mathbf{0} & \\ \dot{\boldsymbol{\lambda}}^*(\tau_i) - \dot{\boldsymbol{\lambda}}^l(\tau_i), & 1 \le i \le N \\ \nabla \mathbf{x}^l(1) - \nabla \mathbf{x}^*(1) & \\ \mathbf{0} & \end{pmatrix}$$

Residual Size

Thus the size of the residual $\boldsymbol{\delta}$ depends on difference between the derivative of a polynomial interpolant of either $\mathbf{x}^*$ or $\boldsymbol{\lambda}^*$ and the true derivative of either $\mathbf{x}^*$ or $\boldsymbol{\lambda}^*$.

Residual in SQP

Suppose that $\mathbf{w}^* = (\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*)$ satisfies the first-order optimality conditions for the continuous control problem. Observe that components 1, 2, and 4 of $\mathcal{T}(\mathbf{w}^*)$ are zero. By the first-order conditions for the continuous problem,

$$
\begin{aligned}
\dot{\boldsymbol{\lambda}}^* &= -\nabla_x H(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*) \\
&= \nabla_x g(\mathbf{x}^*, \mathbf{u}^*) + \boldsymbol{\lambda}^* \mathbf{A}
\end{aligned}
$$

Hence, we have

$$
\mathcal{T}_3(\mathbf{w}^*) = \nabla_x g_k - \nabla_x g_* + (\mathbf{x}^* - \mathbf{x}_k)^{\mathsf{T}} \mathbf{Q}_k + (\mathbf{u}^* - \mathbf{u}_k)^{\mathsf{T}} \mathbf{S}_k,
$$

which implies that $\mathcal{T}_3(\mathbf{w}^*) + \delta_3 = \mathbf{0}$ where

$$
\delta_3 = -\left(\nabla_x g_k - \nabla_x g_* + (\mathbf{x}^* - \mathbf{x}_k)^{\mathsf{T}} \mathbf{Q}_k + (\mathbf{u}^* - \mathbf{u}_k)^{\mathsf{T}} \mathbf{S}_k \right).
$$

Similary, observe that

$$\mathcal{T}_5(\mathbf{w}^*) = -\big(\boldsymbol{\lambda}^*\mathbf{B} + \nabla_u g_* + (\nabla_u g_k - \nabla_u g_*) + [\mathbf{S}_k(\mathbf{x}^* - \mathbf{x}_k)]^{\mathsf{T}} + [\mathbf{R}_k(\mathbf{u}^* - \mathbf{u}_k)]^{\mathsf{T}}\big)$$

By the first-order optimality conditions for $\mathbf{w}^*$, we have

$$-[\boldsymbol{\lambda}^*(t)\mathbf{B} + \nabla_u g(\mathbf{x}^*(t), \mathbf{u}^*(t))] \in N_{\mathcal{U}}(\mathbf{u}^*(t)) \quad \text{for all } t \in [0,1]$$

Hence, if the trailing part of $\mathcal{T}_5(\mathbf{w}^*)$ is deleted, we are left with a vector contained in $N_{\mathcal{U}}(\mathbf{u}^*(t))$. More precisely,
$\mathcal{T}_5(\mathbf{w}^*) + \delta_5 \in N_{\mathcal{U}}(\mathbf{u}^*)$ where

$$\delta_5 = \nabla_u g_k - \nabla_u g_* + [\mathbf{S}_k(\mathbf{x}^* - \mathbf{x}_k)]^{\mathsf{T}} + [\mathbf{R}_k(\mathbf{u}^* - \mathbf{u}_k)]^{\mathsf{T}}.$$

IN SUMMARY: $\mathcal{T}(\mathbf{w}^*) + \boldsymbol{\delta} \in \mathcal{F}(\mathbf{w}^*)$ where

$$\boldsymbol{\delta} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ -\left(\nabla_x g_k + (\mathbf{x}^* - \mathbf{x}_k)^\mathsf{T}\mathbf{Q}_k + (\mathbf{u}^* - \mathbf{u}_k)^\mathsf{T}\mathbf{S}_k - \nabla_x g_*\right) \\ \mathbf{0} \\ \nabla_u g_k + [\mathbf{S}_k(\mathbf{x}^* - \mathbf{x}_k)]^\mathsf{T} + [\mathbf{R}_k(\mathbf{u}^* - \mathbf{u}_k)]^\mathsf{T} - \nabla_u g_* \end{pmatrix}$$

Note that $\nabla_x g_k + (\mathbf{x}^* - \mathbf{x}_k)^\mathsf{T}\mathbf{Q}_k + (\mathbf{u}^* - \mathbf{u}_k)^\mathsf{T}\mathbf{S}_k$ is the first-order Taylor expansion of $\nabla_x g_*$ around $(\mathbf{x}_k, \mathbf{u}_k)$.

GOAL: Obtain bounds for the residual and convert these bounds into error estimates and convergence results