Computational Methods in Optimal Control
Lecture 1. Introduction

William W. Hager

July 23, 2018

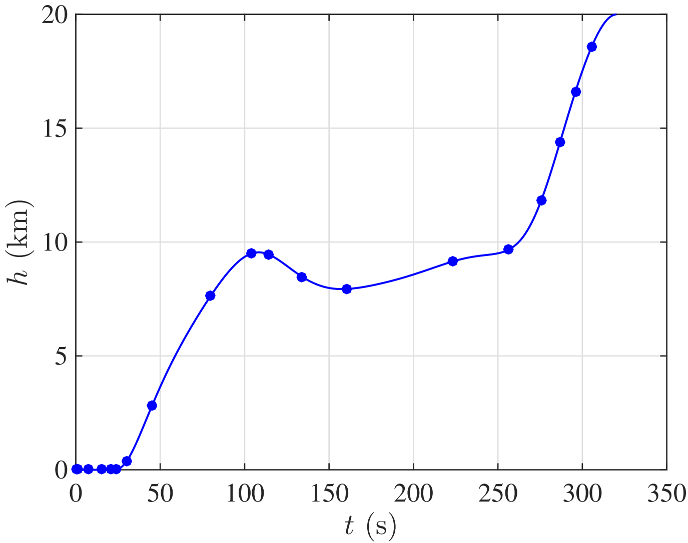
Benchmark Example: Minimum-Time Supersonic Climb

- Famous problem in supersonic aircraft performance optimization (F4H Phantom Aircraft)
- Table interpolation:
  - 1-D aerodynamic data (velocity)
  - 2-D Thrust (altitude/velocity)

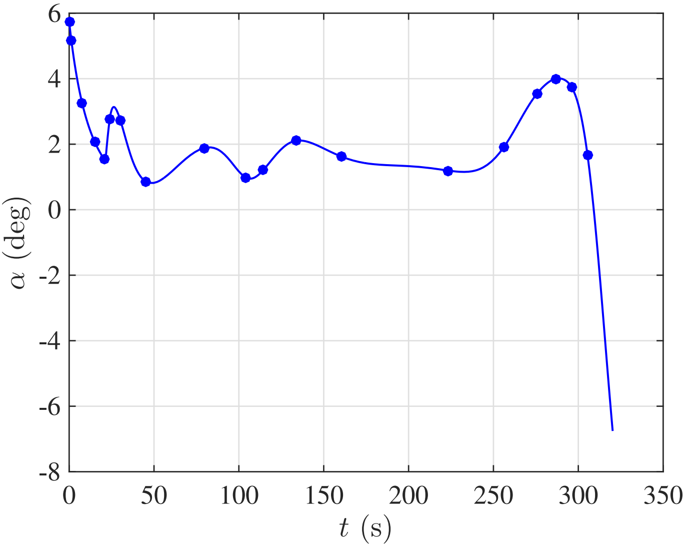
$$\text{Minimize } t_f \quad \text{subject to}$$

$$
\begin{aligned}
\dot{h} &= v \sin\gamma, & (h(0), h(t_f)) &= (0, 20) \text{ km} \\
\dot{v} &= \frac{T\cos\alpha - D}{m} - \frac{\mu}{r^2} sin\gamma, & (v(0), v(t_f)) &= (129, 295) \text{ m/s}, \\
\dot{\gamma} &= \frac{T\sin\alpha + L}{m} + \left(\frac{v}{r} - \frac{\mu}{r^2}\right)\cos\gamma, & (\gamma(0), \gamma(t_f)) &= (0, 0) \text{ rad}, \\
\dot{m} &= -\frac{T}{V_e}, & m(0) &= 19050 \text{ kg}, \\
h &\geq 0. & &
\end{aligned}
$$

Altitude Versus Time (GPOPS-II)



Angle of Attach Versus Time



Altitude Versus Speed

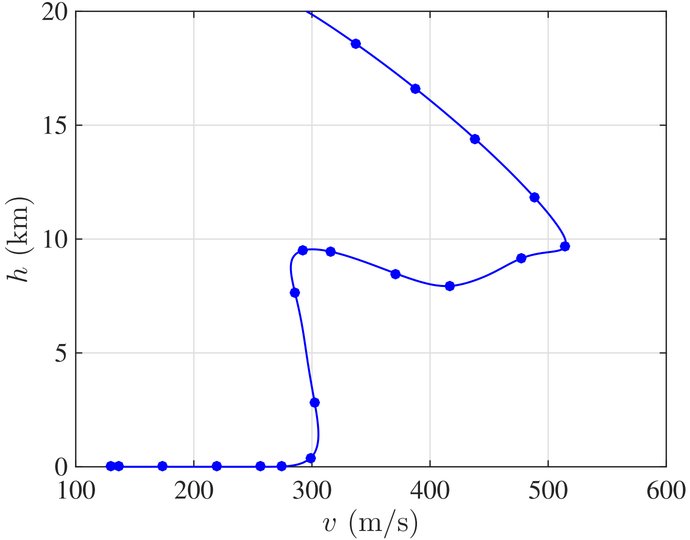


Table 1 Lift and drag coefficients as a function of angle of attack and Mach number for airplane 1

M	0	0.4	0.8	0.9	1.0	1.2	1.4	1.6	1.8
$C_{L\alpha}$	3.44	3.44	3.44	3.58	4.44	3.44	3.01	2.86	2.44
C_{D_0}	0.013	0.013	0.013	0.014	0.031	0.041	0.039	0.036	0.035
η	0.54	0.54	0.54	0.75	0.79	0.78	0.89	0.93	0.93

$C_L = C_{L\alpha}\alpha \qquad L = C_L \tfrac{1}{2}\rho V^2 S \qquad S = 530 \text{ ft}^2$

$C_D = C_{D_0} + \eta C_{L\alpha}\alpha^2 \qquad D = C_D \tfrac{1}{2}\rho V^2 S$

Arthur E. Bryson Jr., Mukund N. Desai, and William C. Hoffman, Energy-State Approximation in Performance Optimization of Supersonic Aircraft, Journal of Aircraft, 6 (1969), pp. 481–488.

Simple Example

$$\begin{aligned} &\text{minimize} \quad \frac{1}{2}\int_0^1 u(t)^2 + 2x(t)^2 \, dt \\ &\text{subject to} \quad \dot{x}(t) = .5x(t) + u(t), \quad x(0) = 1, \end{aligned}$$

with the optimal solution

$$x^*(t) = \frac{2e^{3t} + e^3}{e^{3t/2}(2 + e^3)}, \quad u^*(t) = \frac{2(e^{3t} - e^3)}{e^{3t/2}(2 + e^3)}.$$

Euler's Method (1st order accuracy for ODEs)

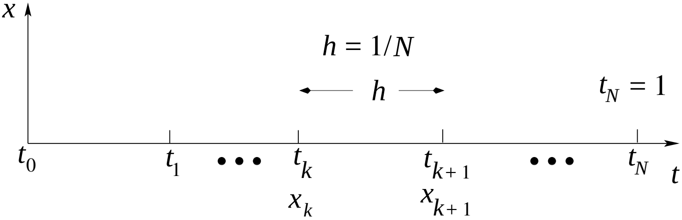
$$\text{Differential equation:} \quad \dot{x}(t) = f(x(t)), \quad x(0) = x_0$$

$$\text{Taylor expansion:} \quad x(t_{k+1}) = x(t_k + h) \approx x(t_k) + h\dot{x}(t_k)$$

$$\text{Euler's Method:} \quad x_{k+1} = x_k + hf(x_k), \quad x(t_k) - x_k = O(h)$$

$$\text{Integral:} \quad I = \int_0^1 g(x(t))\, dt$$

$$\text{Trapezoidal Rule:} \quad I_n = \sum_{k=0}^{N-1} hg(x_k), \quad I - I_n = O(h)$$



Euler's Method for the Example Problem

$$
\begin{aligned}
\text{minimize} \quad & \frac{h}{2} \sum_{k=0}^{N-1} u_k^2 + 2x_k^2 \\
\text{subject to} \quad & x_{k+1} = x_k + h(.5x_k + u_k), \quad x_0 = 1.
\end{aligned}
$$

Solution Using Lagrange Multipliers

Form Lagrangian:

$$\mathcal{L}(x,u) = \sum_{k=0}^{N-1} \frac{h}{2}(u_k^2 + 2x_k^2) + \lambda_k[x_k + h(0.5x_k + u_k) - x_{k+1}]$$

Set derivative to zero:

$$\begin{aligned}
\nabla_u \mathcal{L}(x,u) &= hu_k + h\lambda_k = 0 \Longrightarrow \lambda_k = -u_k \\
\nabla_x \mathcal{L}(x,u) &= 2hx_k + \lambda_k(1 + 0.5h) - \lambda_{k-1} \\
&= 2hx_k - u_k(1 + 0.5h) + u_{k-1} = 0, \quad u_{N-1} = 0
\end{aligned}$$

Dynamics:

$$x_{k+1} - [x_k + h(0.5x_k + u_k)] = 0, \quad x_0 = 1$$

Linear System

Dynamics ($a = 1 + h$):

$$
\begin{array}{lclclcl}
x_1 & & & - & hu_0 & = & 1 + h/2 \\
x_2 & - & ax_1 & - & hu_1 & = & 0 \\
x_3 & - & ax_2 & - & hu_2 & = & 0 \\
\vdots & & & & & & \\
x_N & - & ax_{N-1} & & & = & 0
\end{array}
$$

Co-dynamics:

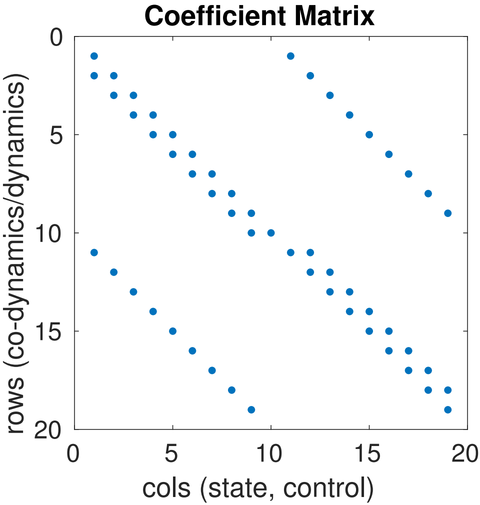
$$
\begin{array}{lclclcl}
2hx_1 & - & au_1 & + & u_0 & = & 0 \\
2hx_2 & - & au_2 & + & u_1 & = & 0 \\
\vdots & & & & & & \\
2hx_{N-1} & & & + & u_{N-2} & = & 0
\end{array}
$$

Coefficient Matrix

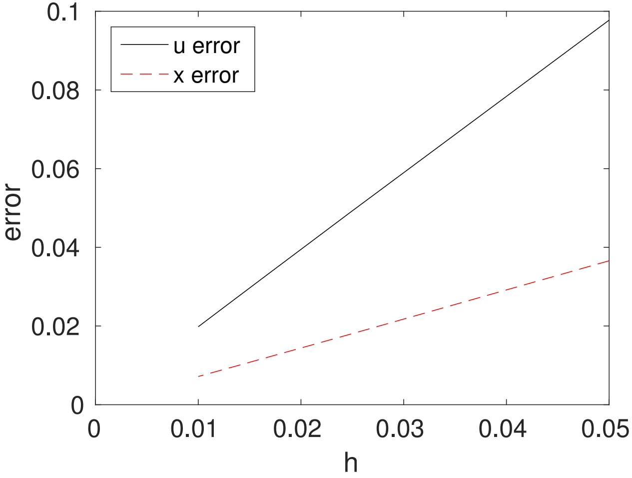
$$
\left(\begin{array}{ccccc|cccc}
1 & 0 & 0 & \dots & 0 & -h & 0 & \dots & 0 \\
a & 1 & 0 & \dots & 0 & 0 & -h & \dots & 0 \\
 & \ddots & \ddots & \ddots & \vdots & \vdots & & \ddots & \vdots \\
 & & a & 1 & 0 & 0 & & & -h \\
 & & & a & 1 & 0 & 0 & \dots & 0 \\
\hline
2h & 0 & \dots & 0 & 0 & 1 & a & & 0 \\
0 & 2h & \dots & 0 & 0 & 0 & 1 & \ddots & \\
\vdots & & \ddots & 0 & 0 & 0 & & \ddots & a \\
0 & & \dots & 2h & 0 & 0 & & \dots & 1
\end{array}\right)
$$

Banded Linear System

Linear System $\mathbf{Ax} = \mathbf{b}$



Error Plot



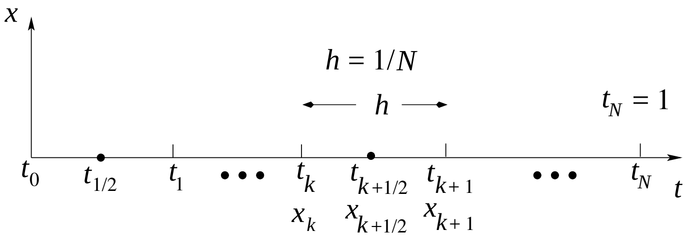
Euler's Improved Method (2nd order accuracy for ODEs)

Differential equation: $\dot{x}(t) = f(x(t)), \quad x(0) = x_0$

Taylor expansion: $x(t_{k+1}) = x(t_k + h) \approx x(t_k) + h\dot{x}(t_{k+1/2})$

Euler Improved Method:
$$\begin{cases} x_{k+1/2} & = & x_k + \frac{h}{2} f(x_k) \\ x_{k+1} & = & x_k + hf(x_{k+1/2}) \\ x(t_k) - x_k & = & O(h^2) \end{cases}$$

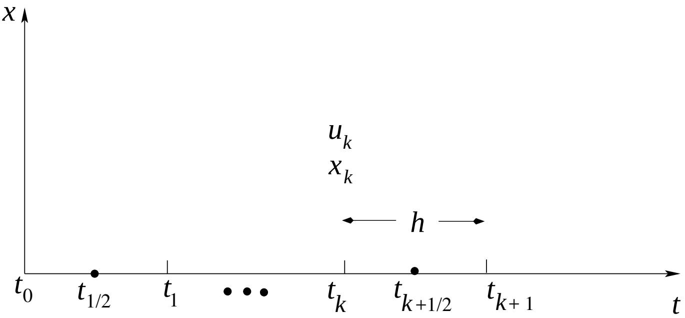
Mid point rule: $I_n = \sum_{k=0}^{N-1} hg(x_{k+1/2}), \quad I - I_n = O(h^2)$



Euler Improved Method for the Example Problem

$$\text{minimize} \quad \frac{h}{2}\sum_{k=0}^{N-1} u_{k+1/2}^2 + 2x_{k+1/2}^2$$

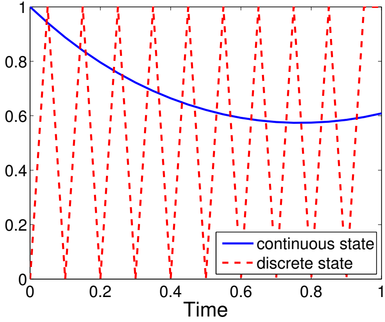
$$\text{subject to} \quad \begin{aligned} x_{k+1/2} &= x_k + \tfrac{h}{2}(.5x_k + u_k), \\ x_{k+1} &= x_k + h(.5x_{k+1/2} + u_{k+1/2}), \quad x_0 = 1. \end{aligned}$$

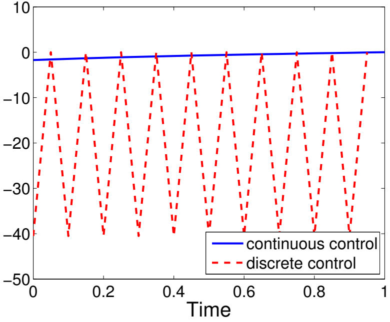


Optimal Solution to Discrete Problem

$$\begin{aligned} u_k &= -\left(\frac{4+h}{2h}\right) x_k, \\ u_{k+1/2} &= 0, \\ x_k &= 1, \\ x_{k+1/2} &= 0 \end{aligned}$$

Comparison Between Discrete and Continuous Solution





CONCLUSION: A more accuract numerical integration scheme may not yield a better approximation to the solution of the control problem. In fact with the second-order Euler Improved Method, the solution of the discretized problem diverged from the true solution as the mesh width approached zero.

Direct Transcription

The process of discretizing the the differential equations and integrals in a control problem and then optimizing the resulting discrete problem is known as Direct Transcription.

Goal: Understand the errors that arise in direct transcription and approaches for solving the discretized problem.

Indirect Methods

In an Indirect Method, we discretize the first-order optimality conditions for the continuous control problem.

- Dynamics:

$$\dot{x}(t) = .5x(t) + u(t), \quad x(0) = 1$$

- Co-dynamics:

$$\dot{\lambda}(t) = -(.5\lambda(t) + 2x(t)), \quad \lambda(1) = 0$$

- Minimum principle:

$$\lambda(t) = -u(t)$$

Two-point Boundary-Value Problem:

$$\begin{aligned} \dot{x}(t) &= .5x(t) - \lambda(t), \quad x(0) = 1 \\ \dot{\lambda}(t) &= -2x(t) - .5\lambda(t), \quad \lambda(1) = 0 \end{aligned}$$

Comparisons

Direct transcription is generally preferred for the following reasons:

- Indirect methods have inherent stability problems
- For nonlinear problems, the system of equations are solved iteratively, and a starting guess is needed for both x and λ.
- Domain of convergence for nonlinear problem can be small with an indirect method.
- Incorporation of constraints on state and control is usually more difficult with an indirect method.
- For direct transcription, the objective function value can be used to guide the iterates to a solution; with indirect methods, the violation of the first-order conditions is more difficult to assess.

Model Optimal Control Problem

$$\begin{aligned} \text{minimize} \quad & C(\mathbf{x}(1)) \\ \text{subject to} \quad & \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \quad \mathbf{u}(t) \in \mathcal{U}, \quad t \in \Omega_0, \\ & \mathbf{x}(0) = \mathbf{x}_0 \end{aligned}$$

- $\Omega_0 = [0, 1]$, $\mathbf{x}_0$ given, $\mathbf{x}(t) \in \mathbb{R}^n$,
- $\mathcal{U} \subset \mathbb{R}^m$ closed and convex,
- $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $C : \mathbb{R}^n \to \mathbb{R}$

Terminal Versus Integral Cost

NOTE: The model problem contains a terminal cost, but not an integral cost. Since the state **x** is a vector in the model problem, an integral cost can be converted to a terminal cost by adding a component to **x**:

$$\min \int_0^1 g(x(t), u(t))\, dt \text{ subject to } \dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0$$

Equivalent problem:

$$\begin{aligned} \text{minimize} \quad & y(1) \\ \text{subject to} \quad & \dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0, \\ & \dot{y}(t) = g(x(t), u(t)), \quad y(0) = 0 \end{aligned}$$

$$y(1) = \int_0^1 \dot{y}(t)\, dt = \int_0^1 g(x(t), u(t))\, dt$$

Euler's Improved $\Longleftrightarrow$ Midpoint Rule

Euler's improved method for the system in (x, y):

$$
\begin{aligned}
x_{k+1/2} &= x_k + (h/2)f(x_k, u_k) \\
y_{k+1/2} &= y_k + (h/2)g(x_k, u_k) \\
x_{k+1} &= x_k + hf(x_{k+1/2}, u_{k+1/2}) \\
y_{k+1} &= y_k + hg(x_{k+1/2}, u_{k+1/2})
\end{aligned}
$$

THUS:

$$
\begin{aligned}
y_N &= h \sum_{k=0}^{N-1} g(x_{k+1/2}, u_{k+1/2}) \\
&= \text{midpoint rule for integrating } g
\end{aligned}
$$