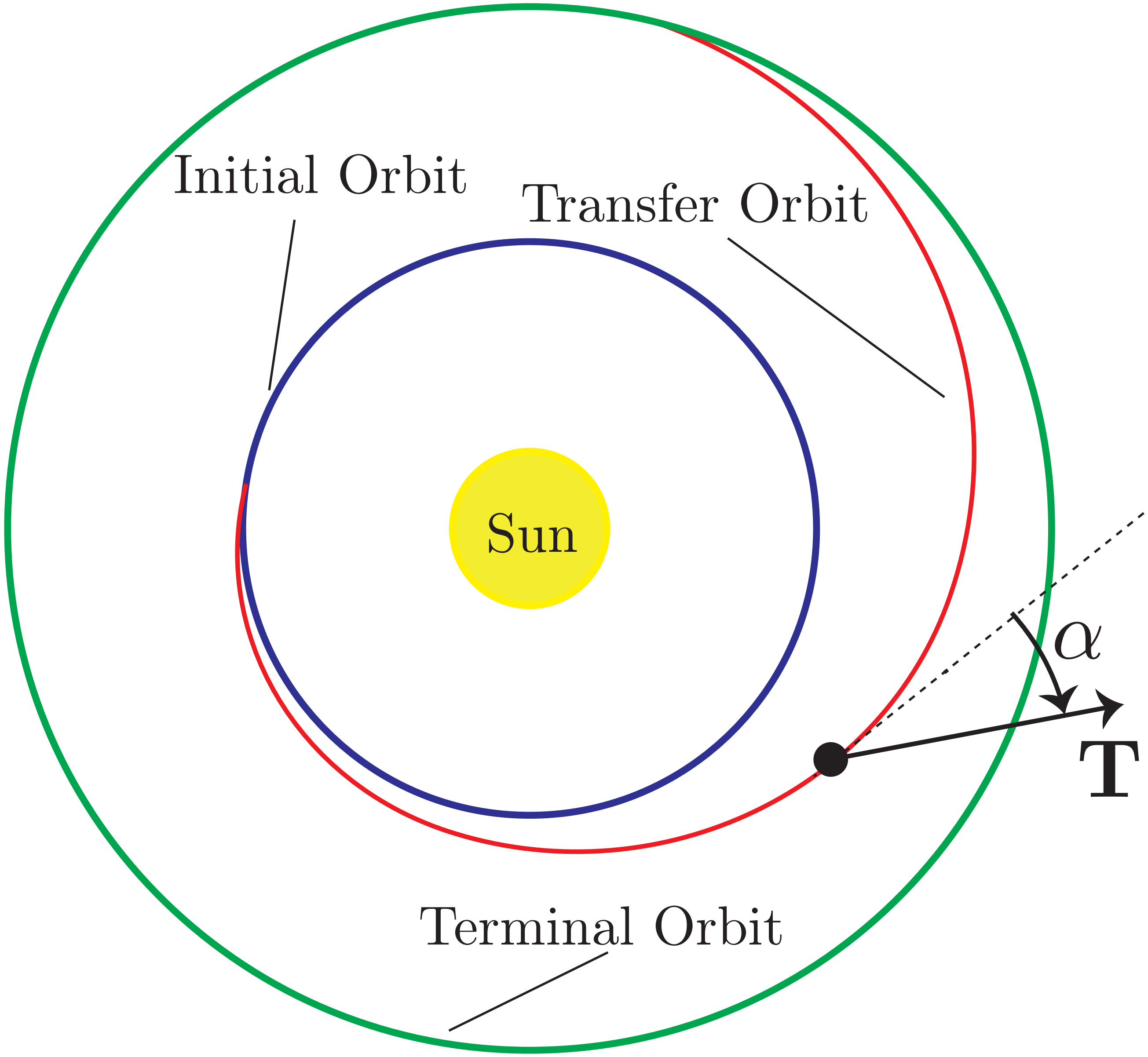
Earth-to-Mars Orbit Transfer



$$\max \; m(t_f)$$

subject to

$$\begin{aligned}
\dot{r} &= u &&, \quad (r(0), r(t_f)) = (r_0, r_f) = (1, 1.5) \\
\dot{u} &= \frac{v^2}{r} - \frac{\mu}{r^2} + \frac{T}{m}\sin\alpha &&, \quad (u(0), u(t_f)) = (u_0, u_f) = (0, 0) \\
\dot{v} &= -\frac{uv}{r} + \frac{T}{m}\cos\alpha &&, \quad (v(0), v(t_f)) = (v_0, v_f) = (\sqrt{\mu/r_0}, \sqrt{\mu/r_f}) \\
\dot{m} &= -\frac{T}{v_e} &&, \quad (m(0), m(t_f)) = (m_0, \text{Free}) = (1, \text{Free})
\end{aligned}$$

where

$$\begin{aligned}
\mu &= 1 \\
T &= 0.1405 \\
v_e &= 1.8758
\end{aligned}$$

Use the following bounds on the state and control:

$$
\begin{aligned}
r &\in [1, 10] \\
u &\in [-10, 10] \\
v &\in [-10, 10] \\
m &\in [0.1, 1] \\
\alpha &= [-2\pi, 2\pi]
\end{aligned}
$$

Use the following two different parameterizations of the control:

- Parameterization 1: use angle α as control (as shown above)
- Parameterization 2: replace α with the following two controls:

$$
w_1 = \sin\alpha \quad , \quad w_2 = \cos\alpha
$$

Note that

$$
w_i \in [-1, +1], \quad (i = 1, 2)
$$

In addition, because of the way w_1 and w_2 are defined, the following equality path constraint must be imposed:

$$w_1^2 + w_2^2 = 1$$

The angle α can then be obtained using a *four-quadrant* inverse tangent

$$\alpha = atan2(w_1, w_2)$$

Analyze the performance using both formulations.

Notes: it can be seen that the problem has been formulated using two different parameterizations of the control. The first parameterization could be thought of as the "natural" parameterization because an angle defines a direction in two dimensions. It is important to note, however, that using an angle for a control creates an issue computationally because angles are defined only up to a modulus of 2π. Thus, if the angle is defined as α, then $\alpha \pm 2\pi k$ (where k is any integer) is the same angle because $\cos\alpha = \cos(\alpha \pm 2\pi k)$ and $\sin(\alpha) = \sin(\alpha \pm 2\pi k)$. In order to avoid the ambiguity that can arise from using an angle for a control, the angle is replaced by the two controls w_1 and w_2, where w_1 and w_2 define the sine and cosine of the angle. Now, the degree of freedom that has been added by the two controls must be removed. Because

$(w_1, w_2$ defines a direction (that is, (w_1, w_2) must be a unit vector), a path constraint $w_1^2 + w_2^2 = 1$ must be imposed. Pay attention most carefully to the manner in which the mesh refinement proceeds using each parameterization. Also, plot angle α for each solution obtained and the ambuity using the first parameterization should become clearer.

