Optimal control techniques for management strategies in biological models

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Outline

1. Modeling of Management of Forest Products in Benin
2. An Epidemic Model of Rabies in Raccoons
3. Formulation of PDE model for Zika virus in Brazil
4. Management of Fire Model

Objective Functional

System of ODEs (or PDEs) modeling situation
Decide on format and bounds on the controls
Design an appropriate objective functional
–balancing opposing factors in functional
–include (or not) terms at the final time

Examples:
1. minimize infecteds and cost of vaccination
including number of persons vaccinated over time

2. Maximize profit due to fishery and minimize cost of fishing effort

Big Idea

In optimal control applications, after formulating a problem appropriate to the scenario, my approach is :

(a) to prove the existence of an optimal control,
(b) to characterize the optimal control through states and adjoints,
(c) to prove the uniqueness of the control,
(d) to compute the optimal control numerically,
(e) to investigate how the optimal control depends on various parameters in the model.

Optimal control in a model of management of forest resources

GOAL Investigate management strategies in a model of harvesting forest products, motivated by forests in Benin

collaborators:
O. Gaoue (at UT, from Benin), Wandi Ding (MTSU), Jiang Jiang, Folashade Agusto (U of Kansas)
publication: J. Theoretical Ecology 2016

Where is Benin?

Khaya seneglansis: African Mahogany

Lethal harvesting

State System

x, the density of a species and r, its intrinsic growth rate

$$
\begin{aligned}
\frac{dx(t)}{dt} &= r(t)x(t)\left(1 - \frac{x(t)}{K}\right) - h_L(t)x(t) \qquad (1) \\
\tau\frac{dr(t)}{dt} &= r_e - r(t) - (\alpha h_N(t) + \beta h_L(t)) \qquad (2)
\end{aligned}
$$

Our species x has a Logistic-type growth function with a carrying capacity K but a time dependent intrinsic growth rate $r(t)$.

Control, $h_N(t)$, non-lethal harvesting rate, only affects $r(t)$ directly.

Control $h_L(t)$, lethal harvesting rate, affects $r(t)$ and directly pulls down the population in $x(t)$ DE.

r_e is the equilibrium growth rate without any harvesting.

τ average lifespan of the plant in years

Our goal is to find an optimal control pair, h_L and h_N , in order to maximize the objective functional

$$J(h_L, h_N) =$$

$$\int_0^T e^{-\delta t}(Ax(t) + B_1 h_L(t)x(t) + B_2 h_N(t)x(t) - C_1 h_L^2(t) - C_2 h_N^2(t))dt$$

The coefficients B_1 and B_2 represent prices from the two types of harvesting

Terms with $B_1 h_L(t)x(t) + B_2 h_N(t)x(t)$ give the corresponding revenue.

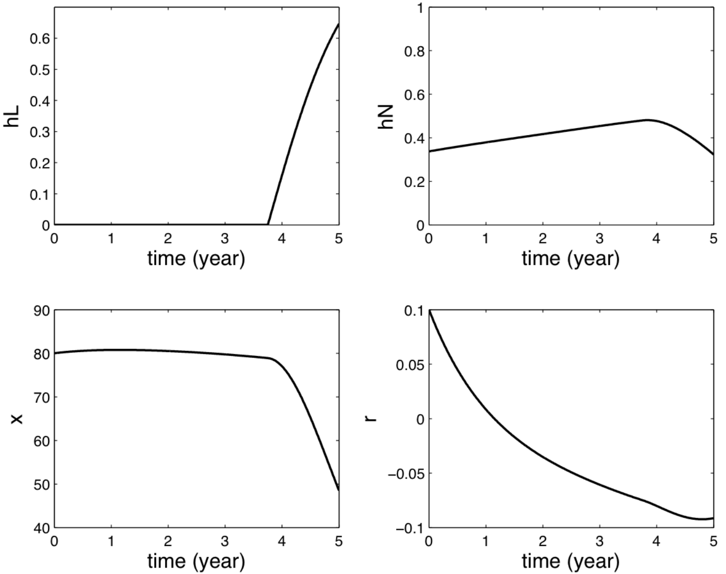
The weight coefficient A balances the relative importance of conservation of species x.

The quadratic terms with the controls give the costs.

Solve numerically using forward-backward iterative method for states and adjoints and optimal controls.

Using prices and growth rates from a variety of forest products in Benin.

Optimal control results for slow growth plant



Second Example: Model for Rabies in Raccoons

Collaborator: T. Clayton, S. Duke-Sylvester, L. Gross, L. A. Real, J. of Biological Dynamics, 2010.

GOAL: Model outbreaks of rabies in raccoons considering various features
MODEL with Birth Pulse and unusual feature of dynamic equation for vaccine, with systems of ordinary differential equations

- S susceptibles
- E exposeds
- I infecteds (able to transmit the disease)
- R immune (from vaccine or recovery from disease)
- V vaccine

State System

Note birth pulse and natural death rates are included.
control $u(t)$- input of vaccine baits.

$$
\begin{aligned}
S' &= -\left(\beta I + b + \frac{c_0 V}{K+V}\right) S + a(S+E+R)\chi_\Omega(t) \\
E' &= \beta I S - (\sigma + b) E \\
R' &= \sigma(1-\rho)E - bR + \frac{c_0 V S}{K+V} \\
I' &= \sigma \rho E - \alpha I \\
V' &= -V[c(S+E+R) + c_1] + u
\end{aligned}
\tag{3}
$$

The birth pulse acts only during the spring time of the year (March 20 to June 21) and χ_Ω is a characteristic function of the set Ω.

Goal with linear cost

Minimize infected population as well as cost of vaccine, the objective functional is

$$\min_u \int_0^T [I(t) + Bu(t)]dt,$$

where the set of all admissible controls is

$$U = \{u : [0, T] \to [0, M_1] | u \text{ is Lebesgue measurable}\}$$

where coefficient B is a weight factor balancing the two terms. When B is large, then the cost of implementing the control is high.

This problem is linear in the control, which implies that the optimal control is bang-bang, singular or combination. In this case, the optimal control is bang-bang.

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Infection with no birth pulse

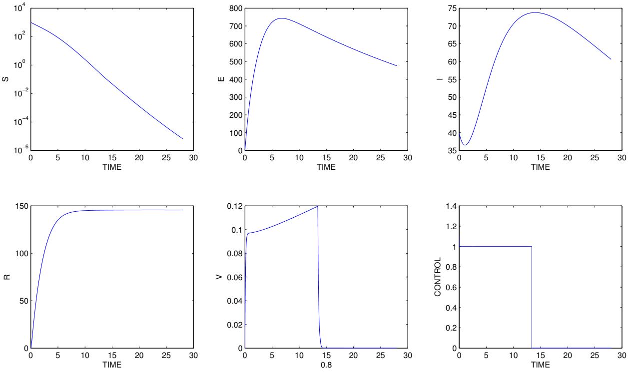


Figure: birth pulse not encountered

Infection, 3 weeks before birth pulse

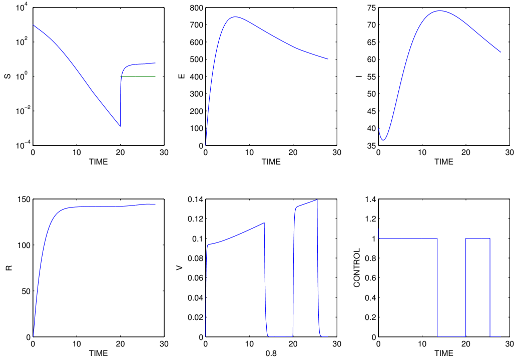


Figure: Notice two control pulses

Thrid Example: Zika and Vaccination

Optimal control of vaccination in a vector-borne reaction-diffusion model applied to Zika virus - Preliminary Report

collaborators: T. Miyaoka, J. Meyer
U of Campinas - Brazil

Modeling Zika Virus in Brazil

- Zika Virus is a Flavivirus and is primarily transmitted to humans mainly by *Aedes aegypti* mosquitoes
- Zika can also be transmitted by vertical transmisson, sexual relations and blood transfusions.
- Zika virus: concern about children born with neurological conditions (microcephaly)
- Vaccines still in development (clinical trial)
- How to balance the cost benefit in vaccination?

- Goal: Apply optimal control of vaccination in a partial differential equation model using data in a state in Brazil .

Spatial region for simulations

Figure: Rio Grande do Norte State.

Mathematical modeling

- Reaction–Diffusion PDE model.

- SIR dynamics for humans and SI for mosquitoes.
- S, I, R susceptible, infected, recovered (vaccinated) for humans
- S_v, I_v susceptible, infected for mosquitoes

- Vaccination rate u gives immunity to susceptible humans.

- Control using the vaccination rate $u(x, t)$.

Compartmental dynamics

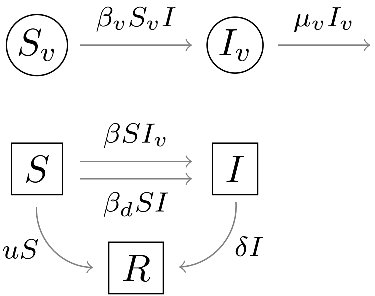


Figure: Flow chart for model (4).

Model in Weak solution sense

$$\begin{cases} \dfrac{\partial S}{\partial t} - \nabla \cdot (\alpha \nabla S) = -\beta S I_v - \beta_d S I - uS, \\ \dfrac{\partial I}{\partial t} - \nabla \cdot (\alpha_I \nabla I) = \beta S I_v + \beta_d S I - \delta I, \\ \dfrac{\partial R}{\partial t} - \nabla \cdot (\alpha \nabla R) = uS + \delta I, \\ \dfrac{\partial S_v}{\partial t} - \nabla \cdot (\alpha_v \nabla S_v) = -\beta_v S_v I + r_v (S_v + I_v) \left(1 - \dfrac{(S_v + I_v)}{\kappa_v}\right), \\ \dfrac{\partial I_v}{\partial t} - \nabla \cdot (\alpha_v \nabla I_v) = \beta_v S_v I - \mu_v I, \text{ in } Q = \Omega \times (0, T). \end{cases} \tag{4}$$

- Plus initial conditions and no flux boundary conditions.
- Logistic growth for mosquitoes
- In the simulations, sexual transmission coefficient was estimated to be 0.

Optimal control

- Goal: minimize cost of infecteds and administering the vaccine control:

$$J(u) = \int_Q (c_1 I(x,t) + c_2 u(x,t) S(x,t) + c_3 u(x,t)^2) dx dt.$$

- The control set

$$\{u \in L^2(Q), 0 \le u \le u_{max}\}$$

Necesary Condtions for Optimal control

- To derive necessary conditions for an optimal control, we need to differentiate the map $u \to J(u)$
- Since $J(u)$ depends on the states, we must differentiate the map, $u \to S, I, R, S_v, I_v$
- Using sensitivities (derivatives of states with respect to control), we derive an adjoint PDE system:

$$\mathcal{L}^*(\lambda) + M^T\lambda = (c_2 u, c_1, 0, 0, 0)^T,$$

$M^T =$

$$\begin{pmatrix} \beta I_v + \beta_d I + u & -\beta I_v - \beta_d I & -u & 0 & 0 \\ \beta_d S & \delta - \beta_d S & -\delta & \beta_v S_v & -\beta_v S_v \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r_v + \dfrac{2r_v}{\kappa_v}(S_v + I_v) + \beta_v I & -\beta_v I \\ \beta S & -\beta S & 0 & -r_v + \dfrac{2r_v}{\kappa_v}(S_v + I_v) & \mu_v \end{pmatrix},$$

- RHS of adjoint system is derivative of integrand of J w.r.t. states

- The derivative operators in L^* are $-(\lambda_1)_t - \nabla\dot(\alpha\nabla\lambda_1)$ ... $-(\lambda_5)_t - \nabla\dot(\alpha_v\nabla\lambda_5)$
- Plus no flux boundary conditions and transversality conditions $\lambda_i(x,T) = 0$ for $i = 1, ..., 5$
- Differentiating the map $u \to J(u)$ and using the sensitivity and adjoint systems, we can characterize our optimal control
- $u^*(x,t) = \min(\max(\frac{(\lambda_1 - \lambda_2 - c_2)S^*(x,t)}{2c_3}, 0))$

Numerical Simulations

- Forward-Backward sweep for optimality system with PDEs solved by finite elements.
- Data from Rio Grande do Norte state in Brazil
- Some parameters from literature and other parameters were estimated
- For estimation, used incidence from simulated system and data

$$\mathcal{Y}_{ij} = \int_{t_j}^{t_j+\tau} \int_{\omega_i} \left(\beta S I_v + \beta_d S I \right) \, dx dt.$$

- Least Squares Approach with normalized residual:

$$\mathcal{R} = \sqrt{\frac{\sum_{i,j} \left(\bar{\mathcal{Y}}_{ij} - \mathcal{Y}_{ij} \right)^2}{\sum_{i,j} \left(\bar{\mathcal{Y}}_{ij} \right)^2}}.$$

Data set

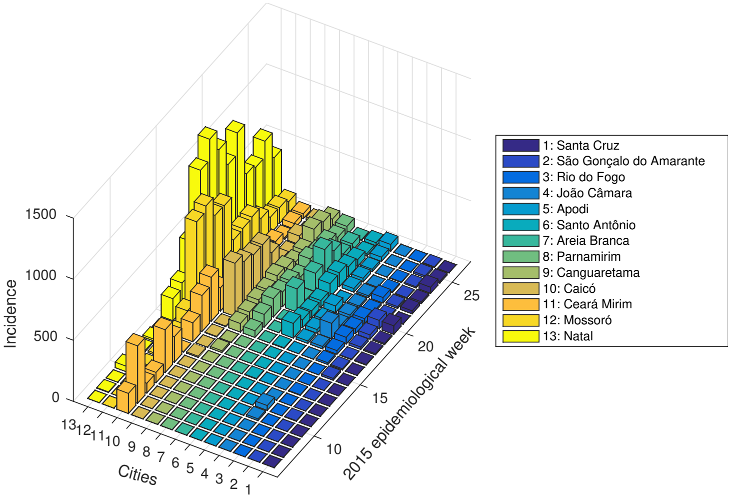


Figure: Incidence in selected cities by 2015 weeks, accounted for under-reporting of cases.

Spatial region for simulations

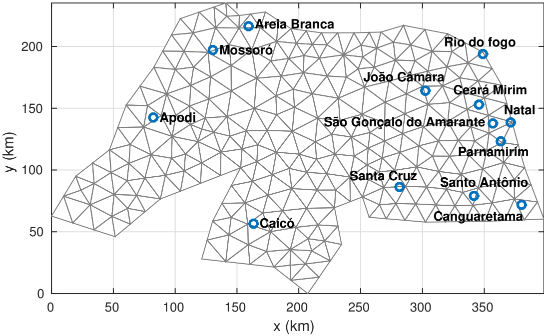


Figure: Finite elements mesh and city locations.

- Source term due to immigration added at 7 and 21 days in the human infected PDE

Initial conditions for simulations

- Initial conditions :
  - S : 3.4 million distributed over space.
  - I : small amount in one city.
  - R : none.
  - S_v : 17 million distributed over space.
  - I_v : none.
- Two sources of infecteds added in locations indicated by the data

No control

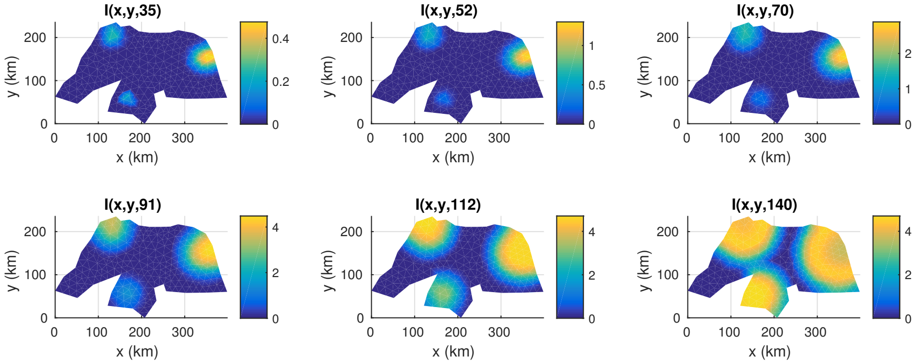


Figure: Solutions at different times, no control. The three infection sources spread over space. Difference in scales.

Control starting at $t = 35$ days

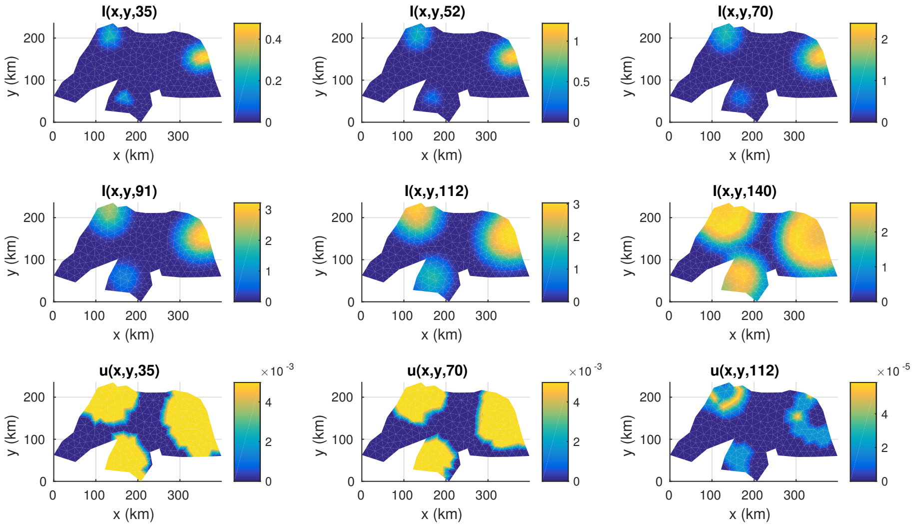


Figure: Solutions at different times, control starting at $t = 35$ days. Difference in scales.

Time plots

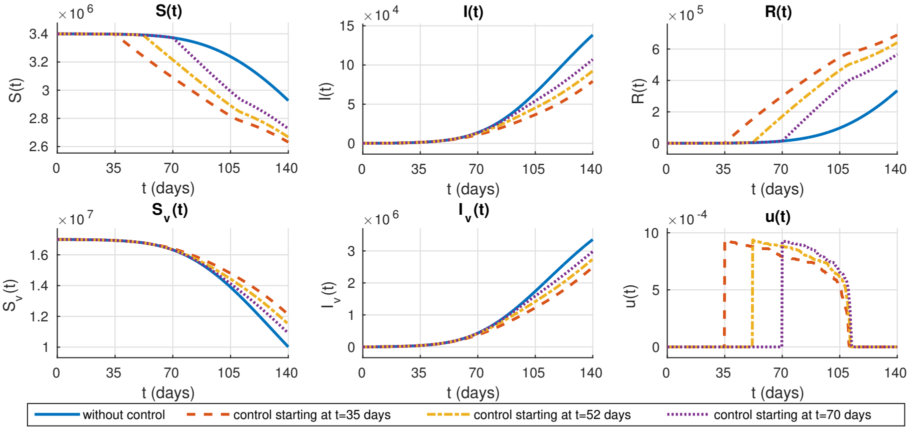


Figure: Solutions integrated over space.

Quantities of interest

Table: Optimal control results compared to the scenario without control

	without control	control starting at $t = 35$ days	control starting at $t = 52$ days	control starting at $t = 70$ days
Total incidence	4.719×10^5	2.746×10^5	3.271×10^5	3.811×10^5
% of no control		58.18%	69.32%	80.75%
Total vaccinated		4.951×10^5	4.072×10^5	2.939×10^5
$J(u^*)$	3.341×10^8	2.454×10^8	2.759×10^8	3.040×10^8
% of no control		73.44%	82.58%	90.98%
$J(u_{\max})$		3.093×10^8	3.326×10^8	3.536×10^8
% of no control		92.56%	99.53%	105.83%
$J(u_{max})/J(u^*)$		126.03%	120.53%	116.33%

Conclusions

- Successful application of vaccination using optimal control.
- Model has been applied to real data from Brazil.
- Global sensitivity analysis performed with optimal control $J(u^*)$ value as output.
- Other kinds of control could be applied and we are currently working on that.
- Same model can be adapted to other similar diseases.
- In future work, we are including better spatial varying ICs and efficacy of vaccine.

Assessing the Economic Tradeoffs Between Prevention and Suppression of Forest Fires

collaborators: Betsy Heines,Charles Sims, paper in Natural Resource Modelng, 2018

GOAL
Combine optimal control, ecology, and economics in order to determine the optimal fire prevention and suppression spending by maximizing the value of a forest under threat of fire using Pontryagin's Maximum Principle.

Introduction

- The total number of forested acres burning in the US is increasing, despite fewer fires total.
- Federal suppression spending is increasing.
- Strict fire exclusion policies have produced overgrown forests, leading to larger and more severe fire events.
- Active prevention management of forests has the potential to mitigate these effects.

Introduction

Fire Prevention

- Actions before a fire.
- Prescribed burning, mechanical thinning,...
- Decreases severity and probability of ignition.

Fire Suppression

- Actions to control fire.
- Aerial spraying, boots on the ground,...

Motivation

Reed, William J., and Hector Echavarria Heras. ”The conservation and exploitation of vulnerable resources.” *Bulletin of Mathematical Biology* 54.2-3 (1992): 185-207.

- Resource management models include risk of catastrophic collapse at an unknown time.
- Reed’s Method allows us to convert a stochastic problem into a deterministic optimal control problem.

Formulating the OC Problem

Some Assumptions:

- At most one fire occurs in $[0, T]$. We determine the optimal prevention schedule up to the time of fire.
- The fire event and all associated costs are taken to be instantaneous.
- The spread of fire is not modeled. However, the uncertainty in the timing of a fire is captured through our application of Reed's Method.

Let $A(t)$ represent the number of unburned acres in an $\bar{A}$ acre forest. Suppose a fire occurs at time $\tau \in [0, T]$.

Net Value of Forest Before Fire

$$\int_0^\tau \Big[B\big(A(t)\big) - h(t) \Big] e^{-rt} dt \tag{5}$$

- Flow of benefits B
- Prevention management spending rate h

where $A(t)$ is given by the solution to

$$A'(t) = \delta\big(\bar{A} - A(t)\big) \text{ with } A(0) = A_0. \tag{6}$$

Formulating the OC Problem

Value of Forest After Fire

Suppose $K\big(h(\tau), x(\tau)\big)$ acres are burned in the fire at time τ and $\hat{A}(t)$ represents the number of unburned acres after the fire.

Net Value of Forest After a Fire

$$\int_{\tau}^{T} B\big(\hat{A}(t)\big) e^{-rt} dt - \left[D\Big(K\big(h(\tau), x(\tau)\big)\Big) + x(\tau) \right] e^{-r\tau} \quad (7)$$

- Flow of benefits B
- Nontimber damages (instantaneous) D
- Suppression costs (instantaneous) h

where $\hat{A}(t)$ is given by the solution to

$$\hat{A}'(t) = \delta\big(\bar{A} - \hat{A}(t)\big) \text{ with } \hat{A}(\tau) = A(\tau) - K\big(h(\tau), x(\tau)\big). \quad (8)$$

Define the optimal value of the forest after a fire by

$$JW^*\big(\tau, A(\tau), h(\tau)\big) =$$
$$\max_{x(\tau)} \int_{\tau}^{T} B\big(\hat{A}(t)\big)e^{-r(t-\tau)}dt - \Big[D\Big(K\big(h(\tau), x(\tau)\big)\Big) + x(\tau)\Big]$$
subject to $x(\tau) \geq 0$
where $$\hat{A}(t) = \bar{A} - \Big(\bar{A} - \Big(A(\tau) - K\big(h(\tau), x(\tau)\big)\Big)\Big)e^{-\delta(t-\tau)}. \qquad (9)$$

We solve the problem above using scalar optimization with respect to x for a given τ, $A(\tau)$, and $h(\tau)$.

Note: $JW^*\big(\tau, A(\tau), h(\tau)\big)$ will be a function with an explicit closed form due to our functional choices.

Suppose that a fire occurs at time $\tau \in [0, T]$ and that suppression spending is optimal, then the value of the forest over $[0, T]$ is

$$\int_0^{\tau} \Big[B\big(A(t)\big) - h(t)\Big] e^{-rt} dt + JW^*\big(\tau, A(\tau), h(\tau)\big)e^{-r\tau} \tag{10}$$

- Net value before fire
- Net value after fire w/ optimal suppression expenditures x^*

where

$$A(t) = \bar{A} - (\bar{A} - A_0)e^{-\delta t}. \tag{11}$$

Suppose a fire does *not* occur in $[0, T]$. Then the value of the forest over $[0, T]$ is

$$\int_0^{T} \Big[B\big(A(t)\big) - h(t)\Big] e^{-rt} dt. \tag{12}$$

- Benefits minus prevention over full time horizon

where

$$A(t) = \bar{A} - (\bar{A} - A_0)e^{-\delta t}. \tag{13}$$

Formulating the OC Problem

Time of Fire as a Random Variable

To capture the uncertainty of the time of fire $\tau \in [0, \infty)$, we treat it as a random variable $\mathcal{T}$.

Hazard Function:

$$\psi\big(h(t)\big) = \lim_{\Delta t \to 0} \left\{ Pr\big(\text{fire in } [t, t+\Delta t) | \text{ no fire up to } t\big) / \Delta t \right\} \quad (14)$$

Survivor Function:

$$S(t) = e^{-\int_0^t \psi(h(z))dz} \quad (15)$$

Cumulative Distribution Function:

$$F(t) = 1 - S(t) \quad (16)$$

We are considering a finite time interval $[0, T]$ and thus consider a truncated random variable $\mathcal{T}_M$:

$$\mathcal{T}_M = \begin{cases} \mathcal{T} & \text{if } \tau \leq T \\ T & \text{if } \tau > T. \end{cases} \tag{17}$$

Cumulative Distribution Function:

$$F_{\mathcal{T}_M}(\tau_M) = \begin{cases} 1 - S(\tau_M) & \text{if } \tau_M < T \\ 1 & \text{if } \tau_M = T \end{cases} \tag{18}$$

The expected value of the forest $J(h)$ is given by the expectation, with respect to the RV $\mathcal{T}_M$, of the piecewise function for the value of the forest:

$$J(h) = E_{\mathcal{T}_M} \begin{cases} \displaystyle\int_0^{\tau_M} \Big[B\big(A(t)\big) - h(t)\Big] e^{-rt} dt \\ \qquad +e^{-r\tau_M} JW^*\big(\tau_M, A(\tau_M), h(\tau_M)\big) & \text{if } \tau_M < T \\ \displaystyle\int_0^{T} \Big[B\big(A(t)\big) - h(t)\Big] e^{-rt} dt & \text{if } \tau_M = T. \end{cases} \tag{19}$$

After a little calculus we arrive at

$$J(h) = \int_0^T \Big[B\big(A(t)\big) - h(t) + \psi\big(h(t)\big) JW^*\big(t, A(t), h(t)\big) \Big] e^{-rt-y(t)} dt \tag{20}$$

where we have introduced a new state variable y defined by

$$y'(t) = \psi\big(h(t)\big) \text{ with } y(0) = 0. \tag{21}$$

This allows us to write $S(t) = e^{-y(t)}$.

The stochastic problem has been converted to deterministic.

The Optimal Control Problem:

$$\max_h \int_0^T \Big[B\big(A(t)\big) - h(t) + \psi\big(h(t)\big) JW^*\big(t, A(t), h(t)\big)\Big] e^{-rt-y(t)} dt \quad (22)$$

$$\text{subject to } y'(t) = \psi\big(h(t)\big) \text{ with } y(0) = 0 \quad (23)$$

$$h(t) \geq 0 \quad (24)$$

$$\text{where } A(t) = \bar{A} - (\bar{A} - A_0)e^{-\delta t}. \quad (25)$$

Next, we present the conditional current-value Hamiltonian and optimality system and introduce our chosen functional forms.

Conditional Current-Value Hamiltonian & Optimality System

Let **H** be the Hamiltonian with adjoint λ. Then the conditional current-value Hamiltonian is $\bar{\mathbf{H}} = e^{rt+y(t)}\mathbf{H}$ with corresponding adjoint equation $\rho(t) = e^{rt+y(t)}\lambda(t)$.

Conditional Current-Value Hamiltonian

$$\bar{\mathbf{H}} = B\big(A(t)\big) - h(t) + \psi\big(h(t)\big)JW^*\big(t, A(t), h(t)\big) + \rho(t)\psi\big(h(t)\big) \quad (26)$$

Optimality Condition, in interior of control set

$$\frac{\partial \bar{\mathbf{H}}}{\partial h} = -1 + JW^*\big(t, A(t), h(t)\big)\frac{\partial \psi}{\partial h} + \frac{\partial JW^*}{\partial h}\psi\big(h(t)\big) + \rho(t)\frac{\partial \psi}{\partial h} = 0 \quad (27)$$

Adjoint Equation

$$\dot{\rho}(t) = r\rho(t) + B\big(A(t)\big) - h(t) + \psi\big(h(t)\big)\Big(\rho(t) + JW^*\big(t, A(t), h(t)\big)\Big) \quad (28)$$

Transversality Condition

$$\rho(T) = 0 \quad (29)$$

Functional Forms

Benefits Function : B_1 - benefits parameter

$$B\big(A(t)\big) = B_1 A(t)$$

Hazard Function : b - background hazard of fire

v - hazard management effectiveness parameter

$$\psi\big(h(t)\big) = be^{-vh(t)}$$

Kill Function : k - fire severity parameter

k_1 - severity management effectiveness parameter

k_2 - severity suppression effectiveness parameter

$$K(h, x) = \frac{k}{(k_1 + h)(k_2 + x)}$$

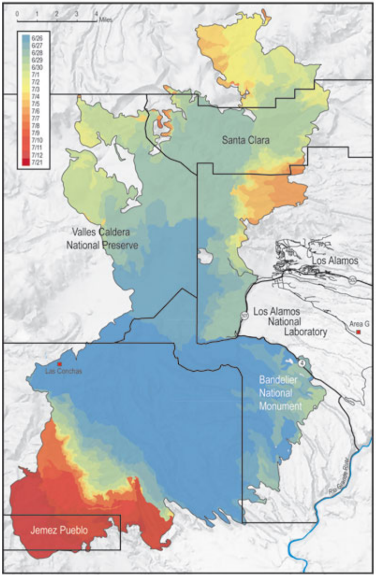
Nontimber Damage Function : c - cost parameter

$$D\big(K(h, x)\big) = cK(h, x)$$

Numerical Methods

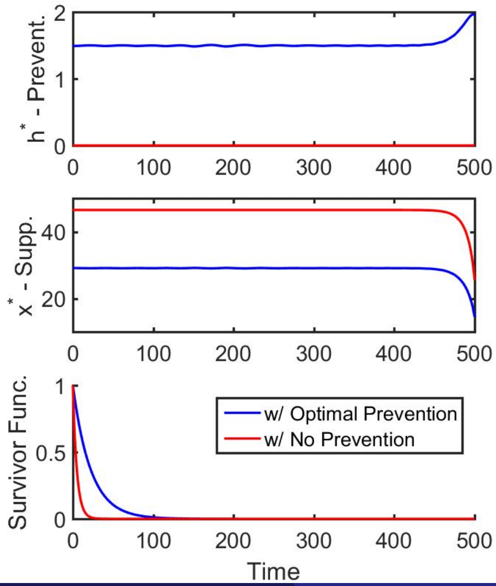
- Due to the complexity of ψ and JW^* an explicit closed form cannot be determined for h^* from the optimality condition.
- We numerically determine h^* by maximizing the conditional current-value Hamiltonian with respect to h at each time step.
- The MATLAB function `fminbnd` is used to optimize h. Since the control h is not bounded above, a large upper bound was used on `fminbnd`.

Results - 2011 Las Conchas Fire, NM



- When: June - August 2011
- Location: Santa Fe National Forest, Bandelier National Monument, Valles Caldera National Preserve
- Acres Burned: $> 150{,}000$
- Suppression Costs: $\approx \$40.9$ Million
- Structures Destroyed: 63 homes, 49 outbuildings
- Parameters chosen for this specific fire

Results - Las Conchas



No Prevention

Value of Forest: $J(0)$ =\$773M
Expected Time to Fire: 10 yrs

Optimal Prevention

Value of Forest: $J(h^*)$ =\$801M
Expected Time to Fire: 45 yrs

Results - Las Conchas

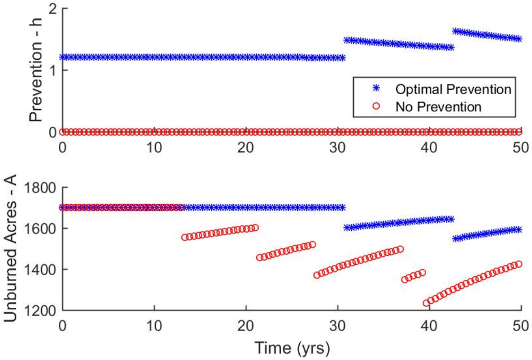
<u>Preliminary Results</u>

When the optimal prevention management spending rate is applied, we see:

- An increase in the expected value of the forest.
- An increase in the mean time of fire.

Fire Sequences

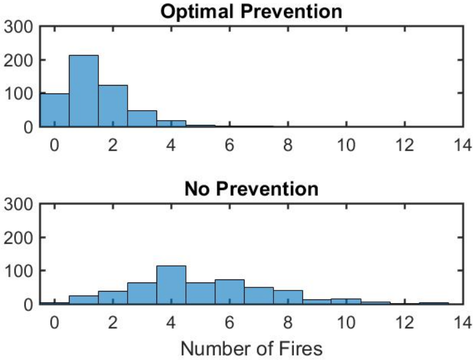
Using a different initial condition for $A(t)$, it is possible to consider sequences of fires by successively applying our optimal control problem.

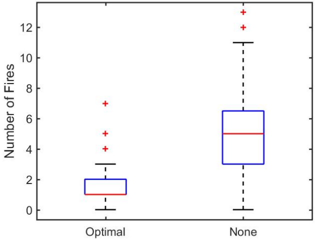


We can determine the time of fire by sampling from the cumulative distribution function for the time of fire random variable.

Fire Sequences

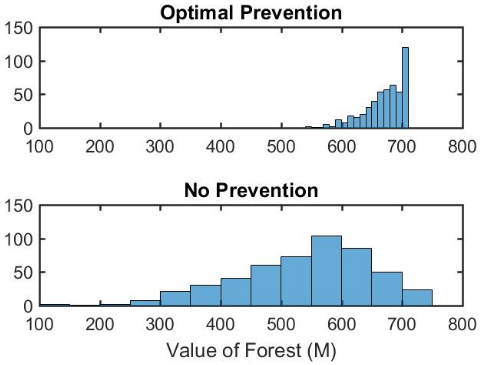
Results for 500 Simulations - Number of Fires

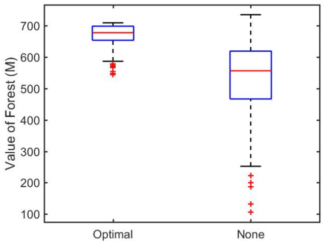




Number of Fires		
Prev.	Optimal	None
Mean	1.4	5.0
Median	1	5
Std.	1.1	2.4

- With optimal prevention, on average there are fewer severe fire events in a time period of 50 years.





Value of Forest $M		
Prev.	Optimal	None
Mean	671	536
Median	677	556
Std.	34.0	111.7

- Larger mean/median for optimal prevention case.
- Furthermore, std. for no prevention case is *over triple* the std. of optimal

Conclusions

From our work with fire sequences, we see that on average with prevention management:

- The overall value of a forest is increased by 25% and has less variation than when no prevention management efforts are made.
- There is a 72% reduction in the number of forest fires. Furthermore, the forest is at less risk for fire.
- There is an 82% reduction in suppression spending and a 55% reduction in management and suppression spending in total.

This work showcases a valuable tool which could guide forest managers and policymakers in their development of forest fire management plans.

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THANK YOU