Bang bang control of elliptic and parabolic PDEs

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Model problem

\[(P)\quad \min_{u \in U_{ad}, y \in Y_{ad}} J(y, u) = \frac{1}{2} \int_D |y - z|^2 + \frac{\alpha}{2} \|u\|_U^2 \]
subject to \(y = G(Bu)\).

\(y = G(Bu)\) iff \(Ay = Bu\) in \(D\), plus b.c. (plus i.c.)

Examples

- \(Ay := -\sum_{i,j=1}^d \partial_{x_j} \left( a_{ij} y_{x_i} \right) + \sum_{i=1}^d b_i y_{x_i} + cy\) uniformly elliptic operator, \(Y = H^1(\Omega)\)

- \(Ay := y_t - \sum_{i,j=1}^d \partial_{x_j} \left( a_{ij} y_{x_i} \right) + \sum_{i=1}^d b_i y_{x_i} + cy\) parabolic operator, \(Y = W(0, T)\).

Constraint sets:
\(U_{ad} = \{a \leq u \leq b\}\) and \(Y_{ad} = \{y \leq c\}\), or \(\{\nabla y \leq \delta\}\).

Bang bang control: \(\alpha = 0\).
Existence and uniqueness of solutions, optimality conditions

- For every $u$ we have a unique $y(u) = \mathcal{G}(Bu)$. So we may minimize the reduced functional
  \[ J(u) := J(\mathcal{G}(Bu), u) \]
  instead.

- Problem ($P$) admits a unique solution $u$ with corresponding state $y = \mathcal{G}(Bu)$.

- $\langle J'(u), v - u \rangle_{U^*, U} \geq 0$ for all
  \[ v \in F_{ad} := \{ v \in U_{ad} \mid \mathcal{G}(Bv) \in Y_{ad} \} \].
Necessary optimality conditions \((\alpha > 0)\), \(Y_{ad} = \{y \leq c\}\)

Constraint qualification (Slater condition)

\[
\exists \tilde{u} \in U_{ad} \text{ such that } \mathcal{G}(B\tilde{u}) < c \text{ in } \bar{D}
\]

Then there exist \(\mu \in \mathcal{M}(\bar{D})\) and some \(p\) such that with \(y\) and \(u\) there holds

\[
Ay = Bu \\
A^* p = y - z + \mu \\
u = P_{U_{ad}} \left( -\frac{1}{\alpha} RB^* p \right),
\]

\[\mu \geq 0, \ y \leq c \text{ in } D \text{ and } \int_{\bar{D}} (c - y) d\mu = 0,\]

Regularity

- elliptic case: \(p \in W^{1,s}(\Omega)\) for all \(s < d/(d - 1)\).
- parabolic case: \(p \in L^s(W^{1,\sigma})\) for all \(s, \sigma \in [1, 2]\) with \(2/s + d/\sigma > d + 1\).

Low regularity of adjoint states introduces difficulties in the numerical analysis.
Only state constraints ($\alpha > 0$), $U_{ad} \equiv U$

Optimality conditions:

\begin{align*}
Ay &= Bu \\
A^* p &= y - z + \mu \\
\frac{1}{\alpha} &= \frac{1}{\alpha} R B^* p,
\end{align*}

$\mu \geq 0$, $y \leq c$ in $D$ and $\int_D (c - y) d\mu = 0$,

- The state $y$ in general is smoother than the adjoint $p$.
- The optimal control and the adjoint state $p$ are coupled via an algebraic relation $\Rightarrow$ a discretization of $p$ induces a discretization of $u$.
- A discretization of $y$ ideally should deliver feasible discrete states.
Only control constraints \((\alpha > 0)\), \(Y_{ad} \equiv Y\)

Optimality conditions:

\[
Ay = Bu \\
A^* p = y - z \\
u = P_{Uad} \left( -\frac{1}{\alpha} RB^* p \right),
\]

- The adjoint \(p\) in general is smoother than the state \(y\).
- The optimal control and the adjoint state \(p\) are coupled via an algebraic relation \(\Rightarrow\) a discretization of \(p\) induces a discretization of \(u\).
- Reduction to \(p\), say delivers

\[
AA^* p - BP_{Uad} \left( -\frac{1}{\alpha} RB^* p \right) = Az,
\]

which in the parabolic case is a bvp for \(p\) in space-time.
Control constraints, $\alpha = 0$

The function $u \in U_{ad}$ is a solution of the optimal control problem iff there exists an adjoint state $p$ such that $y = G(u)$, $p = G(y - z)$ and

$$(\alpha u + RB^* p, v - u) \geq 0 \text{ for all } v \in U_{ad}.$$ 

Recall: $u = P_{U_{ad}}\left(-\frac{1}{\alpha} RB^* p\right)$ for $\alpha > 0$, i.e.

$$u = \begin{cases} 
    a, & \alpha u + RB^* p > 0, \\
    -\frac{1}{\alpha} RB^* p, & \alpha u + RB^* p = 0, \\
    b, & \alpha u + RB^* p < 0,
\end{cases}$$

If $\alpha = 0$:

$$u \begin{cases} 
    = a, & RB^* p > 0, \\
    \in [a, b], & RB^* p = 0, \\
    = b, & RB^* p < 0.
\end{cases}$$

- Control is determined through the sign of $RB^* p$.
- Control in the generic case is either at upper or lower bound.
- Use homothopy $\alpha \to 0$ to compute bang bang control.
Tailored discretization

Discrete counterpart to (P):

\[(P_h) \min_{u_h \in U^h_{ad}, y_h \in Y^h_{ad}} J_h(y_h, u_h) \text{ s.t. } PDE_h(y_h) = B_h(u_h)\]

Questions:

- Appropriate choice of \(U^h_{ad}\) and Ansatz for \(u_h\)?
- Appropriate choice of \(Y^h_{ad}\) and Ansatz for \(y_h\)??

Aim: Capture as much structure as possible of \((P)\) on the discrete level.
Discretization – a variational concept\(^1\)

Discrete optimal control problem:

\[
(P_h) \quad \begin{cases}
\min_{u \in U_{ad}} J_h(u) := \frac{1}{2} \int_D (y_h - z)^2 + \frac{\alpha}{2} \|u\|^2_U \\
\text{subject to } y_h = G_h(Bu) \text{ and } y_h \leq l_h c.
\end{cases}
\]

This problem is still \(\infty\)-dimensional.

Here, \(y_h(u) = G_h(Bu)\) denotes e.g. the

- p.l. and continuous fe approximation to \(y(u)\) (elliptic case),
- \(dg(0)\) in time and p.l. and continuous fe in space approximation to \(y(u)\) (parabolic case), i.e.

\[a(y_h, v_h) = \langle Bu, v_h \rangle \text{ for all } v_h \in X_h.\]

The control is not discretized explicitely!

Discrete necessary optimality conditions ($\alpha > 0$)

Problem ($P_h$) admits a unique solution $u_h \in U_{ad}$. Furthermore, uniform convergence of discrete states implies discrete Slater condition

$$G_h(B\tilde{u}) < l_h b \text{ in } \bar{D} \text{ for } h \text{ small enough}$$

Then there exist $\mu_h \in \mathcal{M}(\bar{D})$ and some $p_h \in X_h$ such that with $y_h$ and $u_h$ there holds

$$a(y_h, v_h) = (Bu_h, v_h) \quad \forall v_h \in X_h,$$
$$a(w_h, p_h) = \int_D (y_h - z)w_h + \int_{\bar{D}} w_h d\mu_h \quad \forall w_h \in Y_h,$$
$$u_h = P_{U_{ad}}\left(-\frac{1}{\alpha}RB^*p_h\right)$$
$$\mu_h \geq 0, \ y_h \leq l_h c, \text{ and } \int_{\bar{D}} (l_h c - y_h) d\mu_h = 0.$$
Problem \((P_h)\) admits a unique solution \(u_h \in U_{ad}\). Furthermore, uniform convergence of discrete states implies discrete Slater condition

\[ \mathcal{G}_h(B\bar{u}) < l_h b \text{ in } \bar{D} \text{ for } h \text{ small enough} \]

Then there exist \(\mu_h \in \mathcal{M}(\bar{D})\) and some \(p_h \in X_h\) such that with \(y_h\) and \(u_h\) there holds

\[ a(y_h, v_h) = (Bu_h, v_h) \quad \forall v_h \in X_h, \]
\[ a(w_h, p_h) = \int_D (y_h - z) w_h + \int_{\bar{D}} w_h d\mu_h \quad \forall w_h \in Y_h, \]
\[ u_h = P_{U_{ad}} \left(-\frac{1}{\alpha} RB^* p_h\right) \]
\[ \mu_h \geq 0, \ y_h \leq l_h c, \text{ and } \int_{\bar{D}} (l_h c - y_h) d\mu_h = 0. \]
Discrete necessary optimality conditions ($\alpha > 0$)

Problem ($P_h$) admits a unique solution $u_h \in U_{ad}$. Furthermore, uniform convergence of discrete states implies discrete Slater condition

$$G_h(B\tilde{u}) < I_h b \text{ in } \tilde{D} \text{ for } h \text{ small enough}$$

Then there exist $\mu_h \in \mathcal{M}(\tilde{D})$ and some $p_h \in X_h$ such that with $y_h$ and $u_h$ there holds

$$a(y_h, v_h) = \langle Bu_h, v_h \rangle \quad \forall \ v_h \in X_h,$$

$$a(w_h, p_h) = \int_D (y_h - z) w_h + \int_{\tilde{D}} w_h d\mu_h \quad \forall \ w_h \in Y_h,$$

$$u_h = P_{U_{ad}} \left( -\frac{1}{\alpha} RB^* p_h \right)$$

$$\mu_h \geq 0, \ y_h \leq I_h c, \text{ and } \int_{\tilde{D}} (I_h c - y_h) d\mu_h = 0.$$

Post processing\(^2\) exploits the projection formula with a discrete adjoint state associated to a fully discrete control.

\( \alpha = 0 \): optimality conditions for discrete problem

The variational-discrete optimal control problems admits a solution \( u_h \in U_{ad} \), which is unique in the case \( \alpha > 0 \). The state \( y_h \) is unique (also in the case \( \alpha = 0 \)).

Let \( u_h \in U_{ad} \) be a solution of the optimal control problem. Then there exists a unique adjoint state \( p_h \) such that \( y_h = G_h(u_h), p_h = G_h(y_h - z) \) and

\[
(\alpha u_h + RB^* p_h, v - u_h) \geq 0 \text{ for all } v \in U_{ad}.
\]

Recall \( u_h = P_{U_{ad}} \left( -\frac{1}{\alpha} RB^* p_h \right) \) for \( \alpha > 0 \), i.e.

\[
u_h = \begin{cases} 
  a, & \alpha u_h + RB^* p_h > 0, \\
  -\frac{1}{\alpha} RB^* p_h, & \alpha u_h + RB^* p_h = 0, \\
  b, & \alpha u_h + RB^* p_h < 0.
\end{cases}
\]

If \( \alpha = 0 \):

\[
u_h \begin{cases} 
  = a, & RB^* p_h > 0, \\
  \in [a, b] & RB^* p_h = 0, \\
  = b, & RB^* p_h < 0.
\end{cases}
\]
Sketch of concept in 1d

\( \alpha > 0 \)

\( \frac{1}{2} \chi \nabla h \)

\( u_h = \mathcal{P}(\frac{1}{2} \chi \nabla h) \)

1. Finite element grid
2. Boundary of active set

\( \alpha = 0 \)

1. Switching points
Variational versus conventional discretization ($\alpha > 0$)
Variational discretization in the bang–bang case $\alpha = 0$

\[ u(x) = -\text{sign}(p(x)), \quad p(x) = \frac{1}{128} \sin(8\pi x_1) \sin(8\pi x_2) \]
Parabolic bang bang with $\alpha$-homothopy

$$p(t, x) = -\frac{T}{2\pi a} \sin \left(\frac{2\pi at}{T}\right) \sin 2\pi x_1 \sin 2\pi x_2, \ a = 2, \ a_1 = 0.2, \ b_1 = 0.4 \text{ bounds.}$$

$$u(t) = \begin{cases} a_1, & B^* p > 0, \\ b_1, & B^* p < 0, \end{cases}$$

where $B^* p(t) = \int_{\Omega} \sin 2\pi x_1 \sin 2\pi x_2 p(t, x) \, dx$.

Here, $\alpha = h^2 = k^{4/3}$ with $h = 2^{-l}$. Furthermore, with this coupling$^3$

$$\|u - u_{k, h, \alpha}\|_{L^1} \leq ch^2.$$

Define

\[ G_h(u) = u - P_{U_{ad}} \left( -\frac{1}{\alpha} p_h(y_h(u)) \right). \]

The optimality condition reads \( G_h(u) = 0 \) and motivates the fix–point iteration

\[ u^+ = P_{U_{ad}} \left( -\frac{1}{\alpha} p_h(y_h(u)) \right), \quad u = u^+. \]

1. Is this algorithm numerically implementable?

Yes, whenever for given \( u \) it is possible to numerically evaluate the expression

\[ P_{U_{ad}} \left( -\frac{1}{\alpha} p_h(y_h(u)) \right) \]

in the \( i-th \) iteration, with an numerical overhead which is independent of the iteration counter of the algorithm.
Semi–smooth Newton algorithm for $\alpha > 0$

2. Does the fix–point algorithm converge?

Yes, if $\alpha > \|RB^* S^*_h S_h B\|_{\mathcal{L}(U)}$, since $P_{U_{ad}}$ is non–expansive.

Condition too restrictive for our purpose → semi–smooth Newton method applied to $G_h(u) = 0$:

- $u$ given, solve until convergence

$$G'_h(u) u^+ = -G_h(u) + G'_h(u) u, \quad u = u^+.$$ 

1. This algorithm is implementable whenever the fix–point iteration is, since

$$-G_h(u) + G'_h(u) u = -P_{U_{ad}} \left( -\frac{1}{\alpha} p_h(u) \right) - \frac{1}{\alpha} P'_{U_{ad}} \left( -\frac{1}{\alpha} p_h(u) \right) S^*_h S_h u.$$ 

2. For every $\alpha > 0$ this algorithm is locally fast convergent$^4$.

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Error analysis, sketch for elliptic case\textsuperscript{5}

It is well known that
\[
\|y - y_h\| + \alpha \|u - u_h\| \sim \|y - y_h(u)\| + \|p - p_h(y(u))\|
\]

So one expects estimates for $y - y_h$ also in the case $\alpha = 0$.

Estimates for $\|u - u_h\|$?

Estimate for the states $(S := \{x \in \Omega \mid p(x) \neq 0\} \subset \bar{\Omega})$
\[
\|y - y_h\| \leq C \left( h^2 + (b - a) \|p - R_h p\|_{L^1(\Omega \setminus S)} + \|p - R_h p\|_{L^\infty} \|u - u_h\|_{L^1(S)} \right),
\]
\[
\|p - p_h\|_{L^\infty} \leq C \|y - y_h\| + \|p - R_h p\|_{L^\infty},
\]

follow from

- $0 \leq (p - p_h, u_h - u) = (R_h p - p_h, u_h - u) + (p - R_h p, u_h - u) \equiv I + II$.
- $I \leq \frac{1}{2} \|y - y_h\|^2 + \frac{1}{2} \|y - R_h y\|^2$
- $II = \int_{\Omega \setminus S} (p - R_h p)(u_h - u) + \int_S (p - R_h p)(u_h - u)$.

Error estimates

If the adjoint state $p$ in the solution with some $\beta \in (0, 1]$ satisfies the structural assumption

$$\exists C > 0 \forall \epsilon > 0 : \mathcal{L}(\{x \in \overline{\Omega}; |p(x)| \leq \epsilon\}) \leq C \epsilon^\beta,$$

one for $h$ small enough gets a unique variational discrete control $u_h$, which together with the discrete state $y_h$ and the discrete adjoint $p_h$ satisfies

$$\|y - y_h\| + \|p - p_h\|_{L^\infty} \leq C \left( h^2 + \|p - R_h p\|_{L^\infty}^{\frac{1}{2-\beta}} \right),$$

$$\|u - u_h\|_{L^1} \leq C \left( h^{2\beta} + \|p - R_h p\|_{L^\infty}^{\frac{\beta}{2-\beta}} \right).$$
Sketch of proof for $\beta = 1$

\[ \|u - u_h\|_{L^1}, \|y - y_h\|, \|p - p_h\|_{L^\infty} \leq C \left\{ h^2 + \|p - R_h p\|_{L^\infty} \right\} \]

Sketch of proof:

- $0 \leq (p - p_h, u_h - u) = (R_h p - p_h, u_h - u) + (p - R_h p, u_h - u) \equiv I + II$.
- $I \leq -\frac{1}{2} \|y - y_h\|^2 + \frac{1}{2} \|y - R_h y\|^2$
- $II = \int_S (p - R_h p)(u_h - u)$. Combine now
  - $\|u - u_h\|_{L^1} \leq (b - a) \mathcal{L}(\{p > 0, p_h \leq 0\} \cup \{p < 0, p_h \geq 0\})$
  - $\{p > 0, p_h \leq 0\} \cup \{p < 0, p_h \geq 0\} \subseteq \{|p(x)| \leq \|p - p_h\|_{\infty}\} \Rightarrow \mathcal{L}(\{|p(x)| \leq \|p - p_h\|_{\infty}\}) \leq C \|p - p_h\|_{\infty}$
  - $\|u - u_h\|_{L^1} \leq C \|p - p_h\|_{\infty}$
  - $\|p - p_h\|_{\infty} \leq \|p - R_h p\|_{\infty} + \|R_h p - p_h\|_{\infty} $
  - $\|R_h p - p_h\|_{\infty} \leq C \|y - y_h\|$

To estimate $II$. 
Numerical example with 2 switching points, fix-point iteration

\[ u(x) = -\text{sign}(-\sin(m\pi x)), \quad y(x) = \sin(m\pi x), \quad \text{and} \quad p(x) = -\sin(m\pi x), \quad m=2. \]

Experimental order of convergence:

- \( \|u - u_h\|_{L^1} : 3.00077834 \)
- Function values 1.99966106
- \( \|p - p_h\|_{L^\infty} : 1.99979367 \)
- \( \|y - y_h\|_{L^\infty} : 1.9997965 \)
- \( \|p - p_h\|_{L^2} : 1.99945711 \)
1. If the desired state $z$ is reachable with a feasible control, then
   \[
   \|y - y_h\| + \|p - p_h\|_{L^\infty} \leq C h^2.
   \]

2. If $p \in C^1(\bar{\Omega})$ satisfies
   \[
   \min_{x \in K} |\nabla p(x)| > 0, \quad \text{where } K = \{ x \in \bar{\Omega} \mid p(x) = 0 \}.
   \]
   Then, the structural assumption is satisfied with $\beta = 1$.

3. If $p \in W^{2,\infty}(\Omega)$ and satisfies the structural assumption, then
   \[
   \|y - y_h\| + \|p - p_h\|_{L^\infty} + \|u - u_h\|_{L^1} \leq C h^2 \log h^{\gamma(d)}.
   \]
Further reading


Thank you very much for your attention!