Remarks on essential codimension

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Abstract. We look for generalizations of the Brown–Douglas–Fillmore essential codimension result, leading to interesting local uniqueness theorems in KK theory. We also study the structure of Paschke dual algebras.

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1. Introduction

The notion of *essential codimension* was introduced by Brown–Douglas–Fillmore (BDF) in their groundbreaking paper [5] where they classified all essentially normal operators using Fredholm indices. Since then, this notion has had manifold applications (e.g., [1, 7]). This includes, among other things, an explanation for the mysterious integers appearing in Kadison's Pythagorean theorem ([16, 17, 18, 24]) as well as other Schur–Horn type results ([4, 14]).

Here is the BDF definition of essential codimension:

Definition 1.1. *(BDF) Let $P, Q \in \mathbb{B}(l_2)$ be projections such that $P - Q \in \mathcal{K}$. The* essential codimension *of Q in P is given by*

$$[P : Q] =_{df} \begin{cases} Tr(P) - Tr(Q) & \text{if } Tr(P) + Tr(Q) < \infty \\ Ind(V^*W) & \text{if } Tr(P) = Tr(Q) = \infty, \\ & \text{where } V^*V = W^*W = 1, \\ & WW^* = P, VV^* = Q. \end{cases}$$

In the above, "Ind" means Fredholm index.

It is not hard to show that, if $Q \leq P$, then essential codimension reduces to the usual codimension. Basic properties of essential codimension and their proofs can be found in [6]. We note that, given that $P - Q \in \mathcal{K}$, the essential codimension essentially measures "local differences".

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A fundamental result on essential codimension which was stated in [5] (a proof can be found in [6]) is the following:

Theorem 1.2. *Let $P, Q \in \mathbb{B}(l_2)$ be projections such that $P - Q \in \mathcal{K}$.*

Then there exists a unitary $U \in \mathbb{C}1 + \mathcal{K}$ such that $UPU^* = Q$ if and only if $[P : Q] = 0$.

The main goal of this paper is to find generalizations of this result. We are following the path first travelled on by [6], [21], and [22] (see also, [23] and [9]). Lee ([21]) observed that essential codimension is a basic example of KK^0, and thus the BDF essential codimension result (Theorem 1.2) is connected to powerful uniqueness theorems, and our goal is to work out some of the operator theoretic consequences.

In Section 2 we undertake a study of the Paschke dual algebra $\mathcal{A}^d_{\mathcal{B}}$ of $\mathcal{A}$ relative to $\mathcal{B}$ in the context of when $\mathcal{A}$ is a unital separable nuclear C*-algebra and $\mathcal{B}$ is a separable stable C*-algebra. In this setting we prove a number of results. We first establish that the Paschke dual algebra is K_1-injective (Theorem 2.5 and Theorem 2.9) under certain restrictions on the canonical ideal, which is essential for proving our theorems in Section 3. We note that the Paschke dual algebra is a unital properly infinite C*-algebra, and it is an open problem whether every properly infinite unital C*-algebra is K_1 injective[1]. We then prove that the Paschke dual algebra is dual in the sense that $\mathcal{A}$ and $\mathcal{A}^d_{\mathcal{B}}$ are each other's relative commutants in the corona algebra $\mathcal{C}(\mathcal{B})$, where $\mathcal{A}$ is identified with its image under the Busby map (Theorem 2.10). This generalizes a remark of Valette ([28]). The key technique throughout this section is the Elliott–Kucerovsky theory of absorbing extensions [10].

In Section 3 we prove a few theorems (Theorems 3.4 and 3.5) which can be considered as generalizations of BDF's Theorem 1.2 to the realm of KK-theory where the essential codimension is interpreted as an element of KK^0, and the unitary which is a compact perturbation of the identity is replaced by the notion of proper asymptotic unitary equivalence due to Dadarlat and Eilers [9]. In order to make this abstract notion of essential codimension more concrete, we simply take $\mathcal{A} = \mathbb{C}$ and, with a few modest hypotheses, arrive at a generalization of Theorem 1.2 that bears true resemblance to it (see Theorem 3.7).

In Section 4, we prove a technical lemma which is used in one of the main results in a previous section.

In a separate paper,[2] we study the connection between essential codimension and projection lifting.

2. The Paschke dual algebra

We briefly fix some notation and recall some preliminaries from extension theory. The reader is advised to refer to [2] for more details.

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[1] See, for example, [3]

[2] J. Loreaux and P. W. Ng, Remarks on essential codimension: Lifting projections. Preprint.

For a nonunital C*-algebra $\mathcal{B}$, $\mathcal{M}(\mathcal{B})$ and $\mathcal{C}(\mathcal{B}) =_{df} \mathcal{M}(\mathcal{B})/\mathcal{B}$ denote the multiplier and corona algebras of $\mathcal{B}$ respectively. Recall that, roughly speaking, $\mathcal{M}(\mathcal{B})$ is the "largest" unital C*-algebra containing $\mathcal{B}$ as an essential ideal. $\pi : \mathcal{M}(\mathcal{B}) \to \mathcal{C}(\mathcal{B})$ denotes the natural quotient map.

Recall that to each extension of C*-algebras

$$0 \to \mathcal{B} \to \mathcal{E} \to \mathcal{A} \to 0 \tag{2.1}$$

we can associate a *-homomorphism

$$\phi : \mathcal{A} \to \mathcal{C}(\mathcal{B})$$

called the *Busby invariant* of the extension. Moreover, to each *-homomorphism $\phi : \mathcal{A} \to \mathcal{C}(\mathcal{B})$, we can associate an extension as in (2.1) whose Busby invariant is ϕ. Two extensions have the same Busby invariant if and only if the extensions are strongly isomorphic in the terminology of Blackadar. (See [2] 15.1-15.4.) All properties of an extension that we are interested in are invariant under strong isomorphism, and, following the convention of extension theory, we identify an extension with its Busby invariant.

Given the considerations in the previous paragraph, henceforth, we freely move back and forth between the terminologies ****-homomorphism*** *and* ***extension*** *to refer to a *-homomorphism $\mathcal{A} \to \mathcal{C}(\mathcal{B})$, and freely identify such a *-homomorphism with an extension as in (2.1) in the natural (Busby invariant) way.*

If $\mathcal{A}$ as above is unital, then the extension (2.1) is called *unital* if the corresponding Busby invariant ϕ is a unital *-homomorphism, i.e., $\phi(1_{\mathcal{A}}) = 1_{\mathcal{C}(\mathcal{B})}$.

We recall that the extension (2.1) is essential if and only if the corresponding Busby invariant ϕ is injective. Also, the extension (2.1) splits if and only if its Busby invariant ϕ is *trivial*, i.e., there exists a *-homomorphism $\phi_0 : \mathcal{A} \to \mathcal{M}(\mathcal{B})$ such that $\phi = \pi \circ \phi_0$; in the case where $\mathcal{A}$ is unital and ϕ_0 is a unital *-homomorphism, ϕ is called a *strongly unital trivial extension.* (See [2] 15.2 and 15.5.)

We note that we will also use the generalized homomorphism picture of KK theory (see, for example, Theorem 3.4). In contrast to extension theory, in the generalized homomorphism picture of KK, the local aspects of relevant operators or *-homomorphisms are important. Hence, whenever we have a *-homomorphism $\phi_0 : \mathcal{A} \to \mathcal{M}(\mathcal{B})$, we will **not** identify it with the corresponding trivial extension $\pi \circ \phi_0$. *Nonetheless, whenever we say that a map ϕ_0 as before is a* ***trivial extension****, we mean that it is a *-homomorphism $\mathcal{A} \to \mathcal{M}(\mathcal{B})$. Whenever we call such ϕ_0 an* ***unital/absorbing/essential/etc.*** *trivial extension, we mean that it is a *-homomorphism such that the corresponding trivial extension $\pi \circ \phi_0$ is* ***unital/absorbing/essential/etc.****.*

Let $\phi, \psi : \mathcal{A} \to \mathcal{C}(\mathcal{B})$ be *-homomorphisms. We say that ϕ and ψ are *unitarily equivalent* and write

$$\phi \sim \psi \tag{2.2}$$

if there exists a unitary $u \in \mathcal{M}(\mathcal{B})$ such that

$$\pi(u)\phi(a)\pi(u)^* = \psi(a)$$

for all $a \in \mathcal{A}$.

Suppose that $\mathcal{B}$ is a stable C*-algebra, and let $\phi, \psi : \mathcal{A} \to \mathcal{C}(\mathcal{B})$ be two extensions. The *BDF sum* of ϕ and ψ is defined to be

$$(\phi \oplus \psi)(\cdot) =_{df} \pi(S)\phi(\cdot)\pi(S)^* + \pi(T)\psi(\cdot)\pi(T)^*,$$

where $S, T \in \mathcal{M}(\mathcal{B})$ are two isometries such that $SS^* + TT^* = 1$. The BDF sum $\oplus$ is well-defined up to unitary equivalence. Sometimes, we will also use "$\oplus$" to mean putting ϕ and ψ into the diagonals of a matrix, i.e., $\phi \oplus \psi = diag(\phi, \psi) : \mathcal{A} \to \mathbb{M}_2 \otimes \mathcal{C}(\mathcal{B})$. The context will make clear which convention that we are using. We note that here (and in some similar places) there is essentially no difference between the two definitions, since stability of $\mathcal{B}$ implies that $\mathbb{M}_2 \otimes \mathcal{M}(\mathcal{B}) \cong \mathcal{M}(\mathcal{B})$. Similar for the case where $\mathcal{M}(\mathcal{B})$ is replaced with $\mathcal{C}(\mathcal{B})$ or any unital C*-algebra with a unital copy of O_2 as a unital *-subalgebra (and the definition with isometries is the context dependent natural variation and is well-defined up to the natural unitary equivalence).

Similar remarks apply when the extensions ϕ and ψ in the previous paragraph are replaced with operators x, y in $\mathcal{M}(\mathcal{B})$ or $\mathcal{C}(\mathcal{B})$ or some other unital C*-algebra containing a unital copy of O_2. The context will make clear what type of convention we are using.

Definition 2.1. *Let $\mathcal{A}$ be a unital separable C*-algebra, let $\mathcal{B}$ be a separable stable C*-algebra, and let $\phi : \mathcal{A} \to \mathcal{M}(\mathcal{B})$ be a unital absorbing trivial extension, which always exists by Thomsen* [27]. *The* Paschke dual algebra of $\mathcal{A}$ relative to $\mathcal{B}$ *is defined to be $\mathcal{A}^d_{\mathcal{B}} =_{df} (\pi \circ \phi(\mathcal{A}))' \in \mathcal{C}(\mathcal{B})$. Sometimes, to emphasize the map ϕ, we will use the notation $\mathcal{D}_\phi =_{df} \mathcal{A}^d_{\mathcal{B}}$.*

We note that $\mathcal{A}^d_{\mathcal{B}}$ is, up to *-isomorphism, independent of ϕ. However, the map ϕ is quite important, and in many treatments of Paschke duality, one has "ϕ" in the notation. Hence, we also use the alternate notation "$\mathcal{D}_\phi$". There is also a definition for nonunital $\mathcal{A}$, but we focus on the unital case where the definition is simpler (essentially Paschke's and Valette's original definition). We so name the Paschke dual algebra because of Paschke duality, which asserts the existence of group isomorphisms $K_j(\mathcal{A}^d_{\mathcal{B}}) \cong KK^{j+1}(\mathcal{A}, \mathcal{B})$ for $j = 0, 1$. (See [13], [26], [27], [28].) We will show below (Theorem 2.10) that the Paschke dual algebra is also dual in another sense, thus generalizing a remark of Valette ([28]).

Paschke ([26]) focused on the case where $\mathcal{B} = \mathcal{K}$. However, many of his assertions and arguments remain true in general. Sometimes the modifications are straightforward and other times they are quite nontrivial.

The argument of the first result is very similar to that of [26] Lemma 1, but every occurrence of Voiculescu's noncommutative Weyl–von Neumann theorem ([29]) is replaced with the Elliott–Kucerovsky theory of absorbing extensions ([10]). We go through the proof for the convenience of the reader, expanding some details.

Lemma 2.2. *Let $\mathcal{A}$ be a unital separable nuclear C^*-algebra, and let $\mathcal{B}$ be a separable stable C^*-algebra.*

Then we have the following:

(a) *The unit of $\mathcal{A}^d_{\mathcal{B}}$ is properly infinite. In fact, $1 \oplus 0 \sim 1 \oplus 1$ in $\mathbb{M}_2 \otimes \mathcal{A}^d_{\mathcal{B}}$.*
(b) *The equivalence classes of full properly infinite projections in $\mathcal{A}^d_{\mathcal{B}}$ constitute all of $K_0(\mathcal{A}^d_{\mathcal{B}})$.*

Proof. (a): Let $\phi : \mathcal{A} \to \mathcal{M}(\mathcal{B})$ be a unital trivial absorbing extension. Hence, we may identify $\mathcal{A} = \pi \circ \phi(\mathcal{A}) \subset \mathcal{C}(\mathcal{B})$, and we may thus view $\mathcal{A}$ as a unital C*-subalgebra of $\mathcal{C}(\mathcal{B})$. And by [10], the inclusion map $\iota : \mathcal{A} \hookrightarrow \mathcal{C}(\mathcal{B})$ is a unital trivial absorbing extension. (For triviality, note that the map $\phi(\mathcal{A}) \to \mathcal{M}(\mathcal{B}) : \pi \circ \phi(a) \mapsto \phi(a)$ is a *-homomorphism, and note that we are identifying $\mathcal{A} = \pi \circ \phi(\mathcal{A})$.)

We may also identify $\mathcal{A}^d_{\mathcal{B}} = (\pi \circ \phi(\mathcal{A}))' \subseteq \mathcal{C}(\mathcal{B})$.

Since ι is trivial and absorbing

$$\iota \oplus \iota \sim \iota.$$

(In the above, we are using $\oplus$ to denote BDF sum.)

Therefore, there exists an isometry $\widetilde{v} \in \mathbb{M}_2 \otimes \mathcal{M}(\mathcal{B})$ such that

$$v(\iota \oplus \iota)v^* = \iota \oplus 0$$

where $v =_{df} \pi(\widetilde{v})$. (In the above, we use $\oplus$ to mean putting into the diagonals of a 2 by 2 matrix. Of course, this use is spiritually the same as the previous use.)

In particular, we have that

$$v(x \oplus x)v^* = x \oplus 0$$

for all $x \in \mathcal{A}$. Hence, since $\mathcal{A}$ is unital,

$$v^*v = 1 \oplus 1 \text{ and } vv^* = 1 \oplus 0. \tag{2.3}$$

From the above, we have that for all $x \in \mathcal{A}$,

$$\begin{aligned} v(x \oplus x) &= (x \oplus 0)v \\ &= (x \oplus x)v \text{ (since } vv^* = 1 \oplus 0). \end{aligned}$$

Hence, $v \in \mathbb{M}_2 \otimes \mathcal{A}^d_{\mathcal{B}}$. From this and (2.3), the unit of $\mathcal{A}^d_{\mathcal{B}}$ is Murray–von Neumann equivalent to two copies of itself.

(b): This follows immediately from (a) and [8] Theorem 1.4. □

We note that it is an open problem whether every unital properly infinite C*-algebra is K_1 injective [3], and the Paschke dual algebra is an interesting and important case of this. We now move towards proving K_1 injectivity under additional hypotheses.

The next lemma ensures that under appropriate conditions, given any unitary u in the commutant of $\mathcal{A}$ (relative to some larger unital algebra), and given a unital trivial absorbing extension, the image of u in the Paschke dual of $\mathcal{A}$ lies in the connected component of the identity in the unitary group.

Lemma 2.3. *Let $\mathcal{C}$ be a unital C^*-algebra and $\mathcal{A} \subseteq \mathcal{C}$ a separable nuclear unital C^*-subalgebra. Say that $u \in \mathcal{A}'$ $(\subseteq \mathcal{C})$ is a unitary. Let $\mathcal{B}$ be a separable stable C^*-algebra. Let $\phi : C^*(\mathcal{A}, u) \to \mathcal{M}(\mathcal{B})$ be a unital trivial absorbing extension.*

Then there exists a norm-continuous path of unitaries $\{v_t\}_{t\in[0,1]}$ in $(\pi \circ \phi(\mathcal{A}))'$ $(\subseteq \mathcal{C}(\mathcal{B}))$ such that $v_0 = \pi \circ \phi(u)$ and $v_1 = 1$.

Proof. Since $\mathcal{B}$ is stable, we may work with $\mathcal{B} \otimes \mathcal{K}$ instead of $\mathcal{B}$.

By the universal property of the maximal tensor product, $C^*(\mathcal{A}, u)$ is a quotient of $\mathcal{A} \otimes_{\max} C(S^1)$, which is nuclear since $\mathcal{A}$ and $C(S^1)$ are nuclear. Hence, $C^*(\mathcal{A}, u)$ is a nuclear C*-algebra.

Since $C^*(\mathcal{A}, u)$ is separable, let $\{\sigma_n\}_{n=1}^\infty$ be a dense sequence in $\widehat{C^*(\mathcal{A}, u)}$ (the space of irreducible *-representations of $C^*(\mathcal{A}, u)$) such that every term in $\{\sigma_n\}$ reoccurs infinitely many times. Let $\sigma' : C^*(\mathcal{A}, u) \to \mathbb{B}(l_2)$ be the unital essential *-representation given by

$$\sigma' = \bigoplus_{n=1}^{\infty} \sigma_n.$$

Then by [19] Theorem 6 (see also [2] Theorem 15.12.4 and [10] Theorem 17), the map

$$\sigma : C^*(\mathcal{A}, u) \to \mathcal{M}(\mathcal{B} \otimes \mathcal{K}) : x \mapsto 1_{\mathcal{M}(\mathcal{B})} \otimes \sigma'(x)$$

is a unital trivial absorbing extension. Hence, since ϕ is also a unital trivial absorbing extension, there exists a unitary $w \in \mathcal{M}(\mathcal{B} \otimes \mathcal{K})$ such that

$$\phi(x) - w\sigma(x)w^* \in \mathcal{B} \otimes \mathcal{K}$$

for all $x \in C^*(\mathcal{A}, u)$.

Note that for all n, since σ_n is an irreducible *-representation of $C^*(\mathcal{A}, u)$, and since u commutes with every element of $C^*(\mathcal{A}, u)$, $\sigma_n(u) \in S^1$. So let $\theta_n \in [0, 2\pi)$ such that $\sigma_n(u) = e^{i\theta_n}1$.

Now for all $t \in [0, 1]$, let

$$v_t' =_{df} w(1_{\mathcal{M}(\mathcal{B})} \otimes \bigoplus_{n=1}^{\infty} e^{i(1-t)\theta_n} 1)w^*.$$

And let

$$v_t =_{df} \pi(v_t').$$

Then $\{v_t'\}_{t\in[0,1]}$ is a norm continuous path of unitaries in $w\sigma(\mathcal{A})'w^*$ $(\subseteq \mathcal{M}(\mathcal{B} \otimes \mathcal{K}))$, and so $\{v_t\}_{t\in[0,1]}$ is a norm continuous path of unitaries such that

$$v_0 = \pi \circ \phi(u), \;\; v_1 = 1$$

and $v_t \in (\pi \circ \phi(\mathcal{A}))'$ for all $t \in [0, 1]$. $\square$

Recall that for a unital C*-algebra $\mathcal{D}$, $U(\mathcal{D})$ denotes the unitary group of $\mathcal{D}$, and $U(\mathcal{D})_0$ denotes the elements of $U(\mathcal{D})$ that are in the connected component of the identity.

We first focus on the case where the canonical ideal is either $\mathcal{K}$ or simple purely infinite. It is well-known that this is exactly the case with "nicest" extension theory, since, among other things, a BDF–Voiculescu type absorption result holds. In fact, in this context, under a nuclearity hypothesis, Kasparov's KK^1 classifies all nonunital essential extensions.

The next result generalizes [26] Lemma 3(2).

Lemma 2.4. *Let $\mathcal{A}$ be a unital separable nuclear C^*-algebra, and $\mathcal{B}$ a separable stable simple C^*-algebra such that either $\mathcal{B} \cong \mathcal{K}$ or $\mathcal{B}$ is purely infinite.*

Then the map

$$U(\mathcal{A}^d_{\mathcal{B}})/U(\mathcal{A}^d_{\mathcal{B}})_0 \to U(\mathbb{M}_2 \otimes \mathcal{A}^d_{\mathcal{B}})/U(\mathbb{M}_2 \otimes \mathcal{A}^d_{\mathcal{B}})_0$$

given by

$$[u] \to [u \oplus 1]$$

is injective.

Proof. Let $\phi : \mathcal{A} \to \mathcal{M}(\mathcal{B})$ be a unital trivial absorbing extension. We may identify $\mathcal{A}^d_{\mathcal{B}} = (\pi \circ \phi(\mathcal{A}))'$.

Let $u \in \mathcal{A}^d_{\mathcal{B}}$ be a unitary such that

$$u \oplus 1 \sim_h 1 \oplus 1$$

in $\mathbb{M}_2 \otimes \mathcal{A}^d_{\mathcal{B}}$. (In the above, $\oplus$ means putting into the diagonals of a 2 by 2 matrix.)

Let $\sigma : C^*(\pi \circ \phi(\mathcal{A}), u) \to \mathcal{M}(\mathcal{B})$ be a unital trivial absorbing extension.

Since $\sigma|_{\pi \circ \phi(\mathcal{A})}$ is a unital trivial absorbing extension, conjugating σ by an appropriate unitary if necessary, we may assume that $\pi \circ \sigma(x) = x$ for all $x \in \pi \circ \phi(\mathcal{A})$. (After all, by [10], the map $\pi \circ \phi(\mathcal{A}) \to \mathcal{M}(\mathcal{B}) : \pi \circ \phi(a) \to \phi(a)$ is also a unital trivial absorbing extension.)

By Lemma 2.3, we have that

$$\pi \circ \sigma(u) \sim_h 1 \tag{2.4}$$

in $(\pi \circ \sigma(\pi \circ \phi(\mathcal{A}))' = (\pi \circ \phi(\mathcal{A}))' = \mathcal{A}^d_{\mathcal{B}}$.

Since either $\mathcal{B} \cong \mathcal{K}$ or $\mathcal{B}$ is simple purely infinite, it follows, by [10] Theorem 17, that the inclusion map $\iota : C^*(\pi \circ \phi(\mathcal{A}), u) \to \mathcal{C}(\mathcal{B})$ is a unital trivial absorbing extension. Hence,

$$\iota \oplus (\pi \circ \sigma) \sim \iota.$$

(In the above, $\oplus$ means BDF sum.)

Hence, there exists an isometry $W \in \mathbb{M}_2 \otimes \mathcal{M}(\mathcal{B})$ such that $W^*W = 1 \oplus 1 = 1_{\mathbb{M}_2 \otimes \mathcal{M}(\mathcal{B})}$, $WW^* = 1 \oplus 0$ and if $w =_{df} \pi(W)$, then

$$w(\iota \oplus (\pi \circ \sigma))w^* = \iota \oplus 0.$$

(In the above, $\oplus$ means putting into the diagonal of a matrix.)

As a consequence, we have that

$$w(u \oplus (\pi \circ \sigma(u)))w^* = u \oplus 0, \tag{2.5}$$

and

$$w(x \oplus x)w^* = x \oplus 0 \tag{2.6}$$

for all $x \in \pi \circ \sigma(\mathcal{A})$.

Note that by (2.6), for all $x \in \pi \circ \sigma(\mathcal{A})$,

$$\begin{aligned} w(x \oplus x) &= (x \oplus 0)w \\ &= (x \oplus x)w \text{ (since } ww^* = 1 \oplus 0 \text{)} \end{aligned}$$

Hence,

$$w \in \mathbb{M}_2 \otimes \mathcal{A}^d_{\mathcal{B}}.$$

Now by (2.4),

$$u \oplus (\pi \circ \sigma(u)) \sim_h u \oplus 1$$

in $\mathbb{M}_2 \otimes \mathcal{A}^d_{\mathcal{B}}$. Also, by the hypothesis on u,

$$u \oplus 1 \sim_h 1 \oplus 1$$

in $\mathbb{M}_2 \otimes \mathcal{A}^d_{\mathcal{B}}$. So

$$u \oplus (\pi \circ \sigma(u)) \sim_h 1 \oplus 1$$

in $\mathbb{M}_2 \otimes \mathcal{A}^d_{\mathcal{B}}$. Conjugating the continuous path of unitaries by w and applying (2.5), we have that

$$u \sim_h 1$$

in $\mathcal{A}^d_{\mathcal{B}}$. □

Theorem 2.5. *Let $\mathcal{A}$ be a unital separable nuclear C*-algebra and $\mathcal{B}$ a separable simple stable C*-algebra such that either $\mathcal{B} \cong \mathcal{K}$ or $\mathcal{B}$ is purely infinite.*

Then $\mathcal{A}^d_{\mathcal{B}}$ is K_1-injective. Moreover, for all $n \geq 1$, the map

$$U(\mathbb{M}_n \otimes \mathcal{A}^d_{\mathcal{B}})/U(\mathbb{M}_n \otimes \mathcal{A}^d_{\mathcal{B}})_0 \to U(\mathbb{M}_{2n} \otimes \mathcal{A}^d_{\mathcal{B}})/U(\mathbb{M}_{2n} \otimes \mathcal{A}^d_{\mathcal{B}})_0$$

given by

$$[u] \mapsto [u \oplus 1]$$

is injective.

Proof. By Lemma 2.2, we have that the unit of the Paschke algebra $\mathcal{A}^d_{\mathcal{B}}$ satisfies $1 \oplus 1 \sim 1$. Hence, for all n, $\mathcal{A}^d_{\mathcal{B}} \cong \mathbb{M}_n \otimes \mathcal{A}^d_{\mathcal{B}}$. Thus, the result follows from Lemma 2.4. □

We now move towards understanding K_1 injectivity of the Paschke dual algebra, when the canonical ideal is no longer elementary nor simple purely infinite. Outside of these small number of cases, our knowledge of extension theory is highly incomplete and the questions that arise are much more challenging.

Let $\mathcal{D}$ be a C*-algebra and $\mathcal{C} \subseteq \mathcal{D}$ a C*-subalgebra. We say that $\mathcal{C}$ is *strongly full* in $\mathcal{D}$ if every nonzero element of $\mathcal{C}$ is full in $\mathcal{D}$. For every nonzero $x \in \mathcal{D}$, we say that x is *strongly full* in $\mathcal{D}$ if $C^*(x)$ is a strongly full C*-subalgebra of $\mathcal{D}$.

Lemma 2.6. *Let $\mathcal{D}$ be a unital C*-algebra and $\mathcal{A} \subseteq \mathcal{D}$ a unital simple C*-subalgebra. Suppose that $u \in \mathcal{A}'$ is a strongly full unitary element of $\mathcal{D}$.*

Then $C^*(u, \mathcal{A})$ is strongly full in $\mathcal{D}$.

Proof. It suffices to prove that every nonzero positive element of $C^*(u, \mathcal{A})$ is full in $\mathcal{D}$.

Let $c \in C^*(u, \mathcal{A})$ be a nonzero positive element. Hence, there exists a continuous function $g : S^1 \to [0, 1]$, and an element $a \in \mathcal{A}_+$ such that $g(u)a \neq 0$ and $0 \leq g(u)a \leq c$.

Since $\mathcal{A}$ is unital and simple, let $x_1, x_2, ..., x_n \in \mathcal{A}$ be such that

$$\sum_{j=1}^{n} x_j a x_j^* = 1.$$

Hence,

$$\sum_{n=1}^{n} x_j g(u) a x_j^* = \sum_{n=1}^{n} g(u) x_j a x_j^* = g(u).$$

Since $g(u)$ is a full element of $\mathcal{D}$, it follows that $g(u)a$ is a full element of $\mathcal{D}$. Hence, c is a full element of $\mathcal{D}$. Since c was arbitrary, $C^*(u, \mathcal{A})$ is a strongly full C*-subalgebra of $\mathcal{D}$. □

Recall that a separable stable C*-algebra $\mathcal{B}$ is said to have the *corona factorization property* (CFP) if every norm-full projection in $\mathcal{M}(\mathcal{B})$ is Murray–von Neumann equivalent to $1_{\mathcal{M}(\mathcal{B})}$ ([20]).

Many C*-algebras have the CFP. For example, all separable simple C*-algebras that are either purely infinite or have strict comparison of positive elements, including all simple C*-algebras classified in the Elliott program, have the CFP. In fact, it is quite difficult to construct a simple separable C*-algebra without CFP.

Recall also, that a map $\phi : \mathcal{A} \to \mathcal{C}$ between C*-algebras is said to be *norm full* or *full* if for every $a \in \mathcal{A} - \{0\}$, $\phi(a)$ is a full element of $\mathcal{C}$, i.e., $Ideal(\phi(a)) = \mathcal{C}$.

We say that a *-homomorphism $\phi : \mathcal{A} \to \mathcal{M}(\mathcal{B})$ *absorbs 0* if $\pi \circ \phi \oplus 0 \sim \pi \circ \phi$. (Here, $\oplus$ means BDF sum.)

In [20], the unital case of following result was proven. The nonunital version, where the extension ϕ absorbs zero, was not contained in [20] because of an error in [10]. The version where ϕ absorbs zero was first considered in [11].

Theorem 2.7. *Let $\mathcal{B}$ be a separable stable C*-algebra with the CFP, $\mathcal{A}$ a separable C*-algebra, and $\phi : \mathcal{A} \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ an essential extension such that either ϕ is unital or ϕ absorbs* 0.

Then ϕ is nuclearly absorbing if and only if ϕ is norm-full.

As a consequence, if, in addition, $\mathcal{A}$ is nuclear, then ϕ is absorbing if and only if ϕ is norm-full.

In the above, when $\phi(1) = 1$ and we say that ϕ is absorbing, we mean that ϕ is absorbing in the unital sense.

Let $\mathcal{B}$ be a separable stable simple C*-algebra with a nonzero projection $e \in \mathcal{B}$. We let $T_e(\mathcal{B})$ denote the set of all tracial states on $e\mathcal{B}e$. It is well known that $T_e(\mathcal{B})$, with the weak* topology, is a Choquet simplex. Moreover, it is also well known that $\mathcal{B} \cong e\mathcal{B}e \otimes \mathcal{K}$ and that every $\tau \in T_e(\mathcal{B})$ gives rise to a densely defined, norm lower semicontinuous trace $\tau \otimes \mathrm{tr}$ (which can take the value ∞) on $B_+ \cong (e\mathcal{B}e \otimes \mathcal{K})_+$. This, in turn, extends uniquely to a strictly lower semicontinuous trace $\bar{\tau}$ on $\mathcal{M}(B)_+$. If $e' \in \mathcal{B}$ is another nonzero projection, then $T_e(\mathcal{B})$ and $T_{e'}(\mathcal{B})$ are homeomorphic, and $T_e(\mathcal{B})$ has finitely many extreme points if and only if $T_{e'}(\mathcal{B})$ has finitely many extreme points. Our results will be independent of the choice of nonzero projection in $\mathcal{B}$, and hence, we will write $T(\mathcal{B})$ to mean $T_e(\mathcal{B})$ for some $e \in Proj(\mathcal{B}) - \{0\}$. Moreover, for a positive element $a \in \mathcal{M}(\mathcal{B})_+$ we let $\tau(a) =_{df} \bar{\tau}(a)$.

Recall that for all $a \in \mathcal{B}_+$ and for all $\tau \in T(\mathcal{B})$,

$$d_\tau(a) =_{df} \lim_{n\to\infty} \tau(a^{1/n}).$$

Recall that $\mathcal{B}$ is said to have *strict comparison* for positive elements if for all $a, b \in \mathcal{B}_+$,

$$d_\tau(a) < d_\tau(b) \text{ whenever } d_\tau(b) < \infty \ \forall \tau \in T(\mathcal{B}) \text{ only if } a \preceq b.$$

In the above, $a \preceq b$ means that there exists $\{x_k\}$ in $\mathcal{B}$ such that $x_k b x_k^* \to a$.

In the next proof, we use a key technical lemma, Lemma 4.7, whose proof we provide in the later Section 4.

Lemma 2.8. *Let $\mathcal{A}$ be a unital separable simple nuclear C*-algebra, and $\mathcal{B}$ a separable stable simple C*-algebra with a nonzero projection, strict comparison of positive elements and for which $T(\mathcal{B})$ has finitely many extreme points.*

*Suppose that there exists a *-embedding $\mathcal{A} \hookrightarrow \mathcal{B}$.*

Then the map

$$U(\mathcal{A}^d_{\mathcal{B}})/U(\mathcal{A}^d_{\mathcal{B}})_0 \to U(\mathbb{M}_2 \otimes A^d_{\mathcal{B}})/U(\mathbb{M}_2 \otimes \mathcal{A}^d_{\mathcal{B}})_0$$

given by

$$[u] \mapsto [u \oplus 1]$$

is injective.

Proof. By the hypotheses, there exist a sequence $\{p_n\}_{n=1}^\infty$ of pairwise orthogonal projections in $\mathcal{B}$, a sequence $\{\phi_n\}_{n=1}^\infty$ of *-embeddings from $\mathcal{A}$ to $\mathcal{B}$, and a sequence $\{v_{n,1}\}_{n=1}^\infty$ of partial isometries in $\mathcal{B}$ such that the following statements are true:

1. $p_m \sim p_n$ for all m, n. In fact, $v_{n,1}^* v_{n,1} = p_1$ and $v_{n,1} v_{n,1}^* = p_n$ for all n.
2. $\sum_{n=1}^\infty p_n = 1_{\mathcal{M}(\mathcal{B})}$, where the sum converges strictly.
3. $\phi_n(1) = p_n$ for all n.
4. $v_{n,1}\phi_1(x)v_{n,1}^* = \phi_n(x)$, for all $x \in \mathcal{A}$ and for all n.

Let $\phi : \mathcal{A} \to \mathcal{M}(\mathcal{B})$ be the unital *-homomorphism given by

$$\phi =_{df} \sum_{n=1}^{\infty} \phi_n.$$

Then by [23] (see also [10] Theorem 17), $\pi \circ \phi$ is a unital trivial absorbing extension. (In the literature, ϕ is often called the "Lin extension".)

We may identify $\mathcal{A}^d_{\mathcal{B}} = (\pi \circ \phi(\mathcal{A}))'$.

Let $u \in \mathcal{A}^d_{\mathcal{B}}$ be a unitary such that

$$u \oplus 1 \sim_h 1 \oplus 1$$

in $\mathbb{M}_2 \otimes \mathcal{A}^d_{\mathcal{B}}$.

By Lemma 4.7, there exists a unitary $v \in \mathcal{A}^d_{\mathcal{B}}$ such that

$$u \sim_h v$$

in $\mathcal{A}^d_{\mathcal{B}}$, and v is strongly full in $\mathcal{C}(\mathcal{B})$. Hence, we may assume that u is a strongly full element of $\mathcal{C}(\mathcal{B})$. Hence, by Lemma 2.6, $C^*(u, \pi \circ \phi(\mathcal{A}))$ is a strongly full unital C*-subalgebra of $\mathcal{C}(\mathcal{B})$.

Hence, by Theorem 2.7, the inclusion map

$$\iota : C^*(u, \pi \circ \phi(\mathcal{A})) \hookrightarrow \mathcal{C}(\mathcal{B})$$

is a unital absorbing extension.

The rest of the proof is exactly the same as that of Lemma 2.4. □

Theorem 2.9. *Let $\mathcal{A}$ be a unital separable simple nuclear C*-algebra, and $\mathcal{B}$ a separable stable simple C*-algebra with a nonzero projection, strict comparison of positive elements, and for which $T(\mathcal{B})$ has finitely many extreme points.*

*Suppose also that there exists a *-embedding $\mathcal{A} \hookrightarrow \mathcal{B}$.*

Then $\mathcal{A}^d_{\mathcal{B}}$ is K_1-injective. Moreover, for all $n \geq 1$, the map

$$U(\mathbb{M}_n \otimes \mathcal{A}^d_{\mathcal{B}})/U(\mathbb{M}_n \otimes \mathcal{A}^d_{\mathcal{B}})_0 \to U(\mathbb{M}_{2n} \otimes \mathcal{A}^d_{\mathcal{B}})/U(\mathbb{M}_{2n} \otimes \mathcal{A}^d_{\mathcal{B}})_0$$

given by

$$[u] \mapsto [u \oplus 1]$$

is injective.

Proof. The proof is exactly the same as that of Theorem 2.5, except that Lemma 2.4 is replaced with Lemma 2.8. □

We fix a terminology that will only be used in the next theorem. Let $\mathcal{A}$ be a unital separable nuclear C*-algebra, and let $\mathcal{B}$ be a separable stable C*-algebra. Let $\phi : \mathcal{A} \to \mathcal{M}(\mathcal{B})$ be a unital trivial absorbing extension. Recall that we can identify $\mathcal{A}^d_{\mathcal{B}} = (\pi \circ \phi(\mathcal{A}))'$ ($\subseteq \mathcal{C}(\mathcal{B})$). Since $\pi \circ \phi$ is injective, we may identify $\mathcal{A}$ with $\pi \circ \phi(\mathcal{A})$. When $\mathcal{A}$ and $\mathcal{A}^d_{\mathcal{B}}$ sit in $\mathcal{C}(\mathcal{B})$ in the above manner, we say that $\mathcal{A}$ and $\mathcal{A}^d_{\mathcal{B}}$ are in *standard position* in $\mathcal{C}(\mathcal{B})$.

Theorem 2.10. *Let $\mathcal{A}$ be a separable simple unital nuclear C*-algebra, and let $\mathcal{B}$ be a separable stable simple C*-algebra. Suppose that $\mathcal{A}$ and $\mathcal{A}^d_{\mathcal{B}}$ are in standard position in $\mathcal{C}(\mathcal{B})$.*

Then

$$\mathcal{A}' = \mathcal{A}^d_{\mathcal{B}} \text{ and } (\mathcal{A}^d_{\mathcal{B}})' = \mathcal{A}.$$

Proof. The first equality follows trivially from the definition of $\mathcal{A}^d_{\mathcal{B}}$.

The proof of the second equality is exactly the same as that of [25] Theorem 1. We note that, in our context, the inclusion map $\iota : \mathcal{A} \to \mathcal{C}(\mathcal{B})$ is a unital trivial absorbing extension. Hence, the hypothesis, that $[\iota] \in \mathcal{T}$ (notation as in [25] Theorem 1) in [25] Theorem 1 is satisfied. Also, since ι is absorbing, the hypothesis that $\mathcal{B}$ satisfies the CFP in [25] Theorem 1 is unnecessary. □

Thus, the Paschke dual algebra is "dual" in still another sense.

3. Essential codimension

In what follows, we will let KK denote the generalized homomorphism picture of KK theory (see, for example, [15] Chapter 4).

In [21], Lee observed that the BDF notion of essential codimension (Definition 1.1) is a special case of an element of KK^0. He thus gave the following definition:

Definition 3.1. *Let $\mathcal{B}$ be a separable stable C*-algebra, and let $P, Q \in \mathcal{M}(\mathcal{B})$ be projections such that $P - Q \in \mathcal{B}$.*

*Let $\phi, \psi : \mathbb{C} \to \mathcal{M}(\mathcal{B})$ be *-homomorphisms for which $\phi(1) = P$ and $\psi(1) = Q$.*

The essential codimension *of Q in P is given by*

$$[P : Q] =_{df} [\phi, \psi] \in KK(\mathbb{C}, \mathcal{B}) \cong K_0(\mathcal{B}).$$

Here, $[\phi, \psi]$ is the class of the generalized homomorphism (ϕ, ψ) in $KK(\mathbb{C}, \mathcal{B})$.

It is not hard to see (e.g., [22] Remark 2.2) that in the case where $\mathcal{B} = \mathcal{K}$, Definition 3.1 coincides with the original BDF essential codimension (Definition 1.1). Thus, KK^0 concerns the local aspects of operator theory, as opposed to KK^1 which deals with the asymptotic aspects (e.g., classifying essentially normal operators up to unitary equivalence modulo the compacts).

Towards generalizing the BDF essential codimension result (Theorem 1.2), we recall the notion of proper asymptotic unitary equivalence (see [9]).

Definition 3.2. *Let $\mathcal{A}$, $\mathcal{B}$ be C*-algebras, with $\mathcal{B}$ nonunital. Let $\phi, \psi : \mathcal{A} \to \mathcal{M}(\mathcal{B})$ be two *-homomorphisms.*

1. *ϕ and ψ are said to be* asymptotically unitarily equivalent *($\phi \sim_{asymp} \psi$) if there exists a (norm-) continuous path $\{u_t\}_{t \in [0,\infty)}$ of unitaries in $\mathcal{M}(\mathcal{B})$ such that for all $a \in \mathcal{A}$,*

i. *$\phi(a) - u_t\psi(a)u_t^* \in \mathcal{B}$, for all t, and*
ii. *$\|\phi(a) - u_t\psi(a)u_t^*\| \to 0$ as $t \to \infty$.*

2. *ϕ and ψ are said to be* properly asymptotically unitarily equivalent *($\phi \approxeq \psi$) if ϕ and ψ are asymptotically unitarily equivalent where the path of unitaries satisfy that $u_t \in \mathbb{C}1 + \mathcal{B}$ for all t.*

We note that proper asymptotic unitary equivalence is a *local notion* in the sense that if $\phi \approxeq \psi$, then $\phi(a) - \psi(a) \in \mathcal{B}$ for all $a \in \mathcal{A}$, or equivalently, $\pi \circ \phi = \pi \circ \psi$; and the path of unitaries is in $\mathbb{C}1 + \mathcal{B}$. This is in fitting with the BDF essential codimension theorem.

In [9], the following generalization of Theorem 1.2 was given: Let $\mathcal{A}, \mathcal{B}$ be separable C*-algebras with $\mathcal{B}$ stable, and let $\phi, \psi : \mathcal{A} \to \mathcal{M}(\mathcal{B})$ be *-homomorphisms such that $\phi(a) - \psi(a) \in \mathcal{B}$, for all $a \in \mathcal{A}$. Then $[\phi, \psi] = 0$ in $KK(\mathcal{A}, \mathcal{B})$ if and only if there exists a *-homomorphism $\sigma : \mathcal{A} \to \mathcal{M}(\mathcal{B})$ such that $\phi \oplus \sigma \approxeq \psi \oplus \sigma$. This is a generalization of Theorem 1.2 because when $\mathcal{A} = \mathbb{C}$ and $\mathcal{B} = \mathcal{K}$, then $P =_{df} \phi(1), Q =_{df} \psi(1)$ are projections in $\mathbb{B}(l_2)$ whose difference is compact, and $[P : Q] = [\phi, \psi] = 0$ (it makes sense to call these things equal because $KK(\mathbb{C}, \mathcal{K}) \simeq K_0(\mathcal{K}) \simeq \mathbb{Z}$). Moreover, in this case, the proper asymptotic unitary equivalence $\phi \approxeq \psi$ can be replaced by actual unitary equivalence and the unitary can be chosen of the form $U \in 1 + \mathcal{K}$.

We note that [9] was inspired by and extensively used ideas from the earlier stable uniqueness paper [23]. We also note that results of the above type can be used to produce (unbounded) stable uniqueness theorems. This idea is essentially due to Lin ([23]).

Before we introduce and prove our generalization of Theorem 1.2, we need a small lemma concerning asymptotic unitary equivalence.

Recall that a trivial extension ϕ is said to *absorb the zero extension* if $\pi \circ \phi \oplus 0 \sim \pi \circ \phi$.

Lemma 3.3. *Let $\mathcal{A}, \mathcal{B}$ be separable C^*-algebras with $\mathcal{A}$ unital and nuclear, and $\mathcal{B}$ stable. Let $\phi, \psi : \mathcal{A} \to \mathcal{M}(\mathcal{B})$ be unital absorbing trivial extensions such that $\phi(a) - \psi(a) \in \mathcal{B}$ for all $a \in \mathcal{A}$. Then $\phi \sim_{asymp} \psi$.*

Proof. Note that $\phi \oplus 0, \psi \oplus 0 : \mathcal{A} \to \mathbb{M}_2 \otimes \mathcal{M}(\mathcal{B})$ are *nonunital* absorbing trivial extensions for which $(\phi \oplus 0)(a) - (\psi \oplus 0)(a) \in \mathbb{M}_2 \otimes \mathcal{B}$ for all $a \in \mathcal{A}$. By [21] Theorem 2.5, $\phi \oplus 0 \sim_{asymp} \psi \oplus 0$, so that there is a norm continuous path of unitaries $\{u_t\}_{t \in [0,\infty)}$ in $\mathbb{M}_2 \otimes \mathcal{M}(\mathcal{B})$ such that for all $t \in [0, \infty)$ and all $a \in \mathcal{A}$,

$$u_t(\phi(a) \oplus 0)u_t^* - (\psi(a) \oplus 0) \in \mathbb{M}_2 \otimes \mathcal{B},$$

and as $t \to \infty$,

$$\|u_t(\phi(a) \oplus 0)u_t^* - (\psi(a) \oplus 0)\| \to 0.$$

In particular, for $P =_{df} 1 \oplus 0 = \phi(1) \oplus 0 = \psi(1) \oplus 0$, we find that $u_t P - P u_t \to 0$ as $t \to \infty$, and also $u_t P - P u_t \in \mathbb{M}_2 \otimes \mathcal{B}$. Therefore $P^\perp u_t P, P u_t P^\perp \to 0$ as $t \to \infty$ and $P^\perp u_t P, P u_t P^\perp \in \mathbb{M}_2 \otimes \mathcal{B}$ as well. Consequently $\pi(u_t) = \pi(P u_t P \oplus P^\perp u_t P^\perp)$ is a block diagonal unitary in $\mathbb{M}_2 \otimes \mathcal{C}(\mathcal{B})$.

Since u_t is unitary, $P = P u_t^* P u_t P + P u_t^* P^\perp u_t P \to P u_t^* P u_t P$, and similarly, $P = P u_t P u_t^* P + P u_t P^\perp u_t^* P \to P u_t P u_t^* P$. So for all large enough

t, defining $v_t \oplus 0 := P u_t P$, we find v_t is invertible in $\mathcal{M}(\mathcal{B})$ and $\pi(v_t)$ is unitary in $\mathcal{C}(\mathcal{B})$. Moreover, we see that

$$(v_t \oplus 0)(\phi(a) \oplus 0)(v_t^* \oplus 0) - u_t(\phi(a) \oplus 0)u_t^* = P^\perp u_t P(\phi(a) \oplus 0)u_t^* \\ + P u_t P(\phi(a) \oplus 0) P u_t^* P^\perp$$

lies in $\mathbb{M}_2 \otimes \mathcal{B}$ and this converges in norm to zero as $t \to \infty$. Combining this with the asymptotic unitary equivalence between $\phi \oplus 0$ and $\psi \oplus 0$, we obtain that $v_t \phi(a) v_t^* - \psi(a) \in \mathcal{B}$ and converges in norm to zero as $t \to \infty$.

Define $u_t' =_{df} v_t |v_t|^{-1} \in \mathcal{M}(\mathcal{B})$ to be the unitary in the polar decomposition of v_t. Then $\pi(u_t') = \pi(v_t|v_t|^{-1}) = \pi(v_t)$, so $u_t' - v_t \in \mathcal{B}$, and since $|v_t|^2 \to 1$, we also have $u_t' - v_t \to 0$ as $t \to \infty$. Therefore, by omitting an initial segment to ensure v_t is invertible, the path of unitaries $\{u_t'\}_{t \in [0,\infty)}$ in $\mathcal{M}(\mathcal{B})$ implements the asymptotic unitary equivalence between ϕ, ψ. □

We now introduce and prove our generalization of Theorem 1.2. The proof essentially follows that of [21] Theorem 2.11 which follows that of [9] Theorem 3.12. As noted above, [9] used extensively the ideas of [23]. In fact, the argument is essentially that of [23]: A proper asymptotic unitary equivalence induces a continuous path of automorphisms on $\phi(\mathcal{A}) + \mathcal{B}$. Then, following [23], we prove innerness of the automorphisms. We sketch the proof for the convenience of the reader.

Recall that KK denotes the generalized homomorphism picture of KK theory (e.g., see [15] Chapter 4). In the next proof, we will let KK_{Higson} denote Higson's definition of KK theory (e.g., see [12] Section 2).

Theorem 3.4. *Let $\mathcal{A}, \mathcal{B}$ be separable C^*-algebras with $\mathcal{A}$ nuclear and $\mathcal{B}$ stable and simple purely infinite. Let $\phi, \psi : \mathcal{A} \to \mathcal{M}(\mathcal{B})$ be essential trivial extensions such that $\phi(a) - \psi(a) \in \mathcal{B}$ for all $a \in \mathcal{A}$.*

Suppose also that either both ϕ and ψ are unital, or both ϕ and ψ absorb the zero extension.

Then $[\phi, \psi] = 0$ in $KK(\mathcal{A}, \mathcal{B})$ if and only if $\phi \approxeq \psi$.

Proof. The "if" direction follows directly from Lemma 3.3 of [9].

We now prove the "only if" direction. Note that both ϕ and ψ are absorbing trivial extensions[3] by [10] Theorem 17 for the unital case, and [11] for the nonunital case.

Let $\widetilde{\mathcal{A}}$ denote the unitization of $\mathcal{A}$ if $\mathcal{A}$ is nonunital, and $\mathcal{A} \oplus \mathbb{C}$ if $\mathcal{A}$ is unital. By [10], if $\phi : \mathcal{A} \to \mathcal{M}(\mathcal{B})$ is an absorbing trivial extension, then the map $\widetilde{\phi} : \widetilde{\mathcal{A}} \to \mathcal{M}(\mathcal{B})$ given by $\widetilde{\phi}|_{\mathcal{A}} = \phi$ and $\widetilde{\phi}(1) = 1$ is a unital absorbing trivial extension. Moreover, $(\widetilde{\phi}, \widetilde{\psi})$ is a generalized homomorphism. Additionally, $[\widetilde{\phi}, \widetilde{\psi}] = 0$ because a homotopy of generalized homomorphisms (ϕ_s, ψ_s) between (ϕ, ψ) and $(0, 0)$ lifts to a homotopy $(\widetilde{\phi_s}, \widetilde{\psi_s})$, and hence $[\widetilde{\phi}, \widetilde{\psi}] = [\tilde{0}, \tilde{0}] = 0$. Thus, we may assume that $\mathcal{A}$ is unital and ϕ and ψ are unital absorbing trivial extensions.

[3] Of course, when both are unital, we mean that they are unitally absorbing.

As in the previous section, we may identify the Paschke dual algebra $\mathcal{A}^d_{\mathcal{B}} = (\pi \circ \phi(\mathcal{A}))' \in \mathcal{C}(\mathcal{B})$.

By 3.3, $\phi \sim_{asymp} \psi$. I.e., there exists a norm continuous path $\{u_t\}_{t \in [0,\infty)}$ of unitaries in $\mathcal{M}(\mathcal{B})$ such that

$$u_t \phi(a) u_t^* - \psi(a) \in \mathcal{B}$$

for all t and for all $a \in \mathcal{A}$, and

$$\| u_t \phi(a) u_t^* - \psi(a) \| \to 0$$

as $t \to \infty$, for all $a \in \mathcal{A}$.

It is trivial to see that this implies that

$$[\phi, u_0 \phi u_0^*] = [\phi, \psi] = 0,$$

and that $\pi(u_t) \in (\pi \circ \phi(\mathcal{A}))' = \mathcal{A}^d_{\mathcal{B}}$ for all t.

It is well-known that we have a group isomorphism

$$KK(\mathcal{A}, \mathcal{B}) \to KK_{Higson}(\mathcal{A}, \mathcal{B}) : [\phi, \psi] \to [\phi, \psi, 1].$$

Hence, $[\phi, u_0 \phi u_0^*, 1] = 0$ in $KK_{Higson}(\mathcal{A}, \mathcal{B})$. Hence, by [12] Lemma 2.3, $[\phi, \phi, u_0^*] = 0$ in $KK_{Higson}(\mathcal{A}, \mathcal{B})$.

By Thomsen's Paschke duality theorem ([27] Theorem 3.2), there is a group isomorphism $K_1(\mathcal{A}^d_{\mathcal{B}}) \to KK_{Higson}(\mathcal{A}, \mathcal{B})$ which sends $[\pi(u_0)]$ to $[\phi, \phi, u_0^*]$. Hence, $[\pi(u_0)] = 0$ in $K_1(\mathcal{A}^d_{\mathcal{B}})$. Hence, by Theorem 2.5, $\pi(u_0) \sim_h 1$ in $\mathcal{A}^d_{\mathcal{B}} = (\pi \circ \phi(\mathcal{A}))'$. Hence, there exists a unitary $v \in \mathbb{C}1 + \mathcal{B}$ such that $v^* u_0 \sim_h 1$ in $\pi^{-1}(\mathcal{A}^d_{\mathcal{B}})$.

Hence, modifying an initial segment of $\{v^* u_t\}_{t \in [0,\infty)}$ if necessary, we may assume that $\{v^* u_t\}_{t \in [0,\infty)}$ is a norm continuous path of unitaries in $\pi^{-1}(\mathcal{A}^{\mathcal{B}}_d)$ such that $v^* u_0 = 1$.

Now for all $t \in [0, \infty)$, let $\alpha_t \in Aut(\phi(\mathcal{A}) + \mathcal{B})$ be given by $\alpha_t(x) =_{df} v^* u_t x u_t^* v$ for all $x \in \phi(\mathcal{A}) + \mathcal{B}$. Thus, $\{\alpha_t\}_{t \in [0,\infty}$ is a norm continuous path of automorphisms of $\phi(\mathcal{A}) + \mathcal{B}$ such that $\alpha_0 = id$. Hence, by [9] Proposition 2.15 (see also [23] Theorem 3.2 and 3.4), there exist a continuous path $\{v_t\}_{t \in [0,\infty)}$ of unitaries in $\phi(\mathcal{A}) + \mathcal{B}$ such that $v_0 = 1$ and $\| v_t x v_t^* - v^* u_t x u_t^* v \| \to 0$ as $t \to \infty$ for all $x \in \phi(\mathcal{A}) + \mathcal{B}$. Thus, $\| v v_t x v_t^* v^* - u_t x u_t^* \| \to 0$ as $t \to \infty$ for all $x \in \phi(\mathcal{A}) + \mathcal{B}$.

We now proceed as in the last part of the proof of [9] Proposition 3.6 Step 1 (see also the proof of [23] Theorem 3.4). For all $t \in [0, \infty)$, let $a_t \in \mathcal{A}$ and $b_t \in \mathcal{B}$ such that $v v_t = \phi(a_t) + b_t$. Since $\pi \circ \phi$ is injective, we have that for all t, a_t is a unitary in $\mathcal{A}$, and hence, $\phi(a_t)$ is a unitary in $\phi(\mathcal{A}) + \mathcal{B}$. Note also that since $\pi \circ \phi = \pi \circ \psi$ and both maps are injective, $\| a_t a a_t^* - a \| \to 0$ as $t \to \infty$ for all $a \in \mathcal{A}$. For all t, let $w_t =_{df} v v_t \phi(a_t)^* \in 1 + \mathcal{B}$. Then $\{w_t\}_{t \in [0,1)}$

is a norm continuous path of unitaries in $1+\mathcal{B}$, and for all $a \in \mathcal{A}$,

$$\begin{aligned}\|w_t\phi(a)w_t^* - \psi(a)\| &\leq \|w_t\phi(a)w_t^* - vv_t\phi(a)v_t^*v^*\| \\ &\quad + \|vv_t\phi(a)v_t^*v^* - u_t\phi(a)u_t^*\| \\ &\quad + \|u_t\phi(a)u_t^* - \psi(a)\| \\ &= \|vv_t\phi(a_taa_t^* - a)v_t^*v^*\| \\ &\quad + \|vv_t\phi(a)v_t^*v^* - u_t\phi(a)u_t^*\| \\ &\quad + \|u_t\phi(a)u_t^* - \psi(a)\| \\ &\to 0. \end{aligned}$$

□

We have another generalization of the BDF essential codimension theorem:

Theorem 3.5. *Let $\mathcal{A}$ be a unital separable simple nuclear C^*-algebra, and $\mathcal{B}$ a separable simple stable C^*-algebra with a nonzero projection, strict comparison of positive elements and for which $T(\mathcal{B})$ has finitely many extreme points.*

*Suppose that there exists a *-embedding $\mathcal{A} \hookrightarrow \mathcal{B}$.*

Let $\phi, \psi : \mathcal{A} \to \mathcal{M}(\mathcal{B})$ be unital trivial extensions such that $\phi(a)-\psi(a) \in \mathcal{B}$ for all $a \in \mathcal{A}$.

Then $[\phi, \psi] = 0$ in $KK(\mathcal{A}, \mathcal{B})$ if and only if $\phi \approxeq \psi$.

Proof. Note that since $\mathcal{A}$ is simple, ϕ and ψ are both norm full trivial extensions. Hence, since $\mathcal{B}$ has the CFP, it follows, by Theorem 2.7, that ϕ and ψ are both unitally absorbing trivial extensions.

The rest of the proof is exactly the same as that of Theorem 3.4, except that Theorem 2.5 is replaced with Theorem 2.9. □

We note once more, that, as in Theorem 1.2, Theorems 3.4 and 3.5 are essentially about local phenomena.

Towards more concrete generalizations, we first need a technical result.

Lemma 3.6. *If $\mathcal{B}$ is a nonunital C^*-algebra and $P, Q \in \mathcal{M}(\mathcal{B})$ are projections with $P - Q \in \mathcal{B}$ and $\|P-Q\| < 1$, then there exists a unitary $U \in 1+\mathcal{B}$ such that $P = UQU^*$.*

Moreover, we can choose U as above so that $\|U-1\| \leq 4\|P-Q\|$.

Proof. Brief sketch of standard argument: $Z = PQ + (1-P)(1-Q)$ satisfies $Z - 1 = (1-2P)(P-Q)$, and thus $\|Z-1\| < 1$. Hence, Z is invertible and if $U = Z|Z|^{-1}$ is the unitary in the polar decomposition of Z, then $UQU^* = P$. Moreover, since $P - Q \in \mathcal{B}$, $Z \in 1+\mathcal{B}$ and hence, $U \in 1+\mathcal{B}$.

Also,

$$\begin{aligned}\|Z^*Z - 1\| &\leq \|Z^*Z - Z\| + \|Z-1\| \\ &\leq \|Z^*-1\|\|Z\| + \|Z-1\| \\ &\leq 3\|Z-1\| = 3\|P-Q\|.\end{aligned}$$

So $\||Z| - 1\| \leq \||Z|^2 - 1\| \leq 3\|P - Q\|$. So

$$\|U - 1\| \leq \|U - U|Z|\| + \|Z - 1\| = \|1 - |Z|\| + \|P - Q\| \leq 4\|P - Q\|. \quad \square$$

We now move towards a more concrete generalization of the BDF essential codimension theorem. We will be using the notion of generalized essential codimension in Definition 3.1.

Theorem 3.7. *Let $\mathcal{B}$ be a separable stable simple purely infinite C*-algebra, and $P, Q \in \mathcal{M}(\mathcal{B})$ projections such that $P, Q, 1-P, 1-Q \notin \mathcal{B}$, and $P-Q \in \mathcal{B}$.*

Then $[P : Q] = 0$ in $K_0(\mathcal{B})$ if and only if there exists a unitary $U \in 1+\mathcal{B}$ such that $UPU^* = Q$.

Proof. Since $\mathcal{B}$ is simple purely infinite, it follows, from the hypotheses, that $P \sim 1-P \sim Q \sim 1-Q \sim 1$. Let $\phi, \psi : \mathbb{C} \to \mathcal{M}(\mathcal{B})$ be *-homomorphisms such that $\phi(1) = P$ and $\psi(1) = Q$. Then ϕ and ψ are absorbing trivial extensions. (And both absorb the zero extension.)

The "if" direction then follows immediately from Theorem 3.4. (See also [22] Lemma 2.4.)

We now prove the "only if" direction. We have that $[\phi, \psi] = [P : Q] = 0$. Hence, by Theorem 3.4, there exists a norm continuous path $\{u_t\}_{t \in [0,1]}$ of unitaries in $\mathbb{C}1 + \mathcal{B}$ such that $\|u_t P u_t^* - Q\| \to 0$ as $t \to \infty$.

Choose $s \in [0, \infty)$ such that $\|u_s P u_s^* - Q\| < 1$. We may assume that $u_s \in 1 + \mathcal{B}$. Then, by Lemma 3.6, there exists a unitary $V \in 1 + \mathcal{B}$ such that $V u_s P u_s^* V^* = Q$. Take $U =_{df} V u_s$. $\square$

We note that there is a mistake in [21] Theorem 2.14. It is *not true* that if $\mathcal{B}$ is a separable simple stable purely infinite C*-algebra for which $\mathcal{M}(\mathcal{B})$ has real rank zero, and if $P, Q \in \mathcal{M}(\mathcal{B})$ are projections with $P - Q \in \mathcal{B}$, $P \notin \mathcal{B}$, for which $[P, Q] = 0$ in $K_0(\mathcal{B})$ then there exists a unitary $U \in 1 + \mathcal{B}$ such that $UPU^* = Q$. Here is a counterexample: Take $\mathcal{B} = O_2 \otimes \mathcal{K}$ and let $r \in O_2 \otimes \mathcal{K}$ be a nonzero projection. Note that $[r] = 0$ in $K_0(O_2)$. Let $P =_{df} 1_{\mathcal{M}(O_2 \otimes \mathcal{K})}$ and $Q = P - r$. Then $P - Q = r \in O_2 \otimes \mathcal{K}$, $P \notin O_2 \otimes \mathcal{K}$, and $[P : Q] = 0$ in $K_0(O_2 \otimes \mathcal{K})$. But it is *not true* that P is unitarily equivalent to Q.

The mistake in the argument of [21] Theorem 2.14 is essentially a mistake about absorbing extensions. If $\phi : \mathcal{A} \to \mathcal{M}(\mathcal{B})$ is an absorbing trivial extension then $\phi \oplus 0 \sim \phi$, i.e., ϕ must absorb the 0 extension, and thus $ran(\phi)^\perp$ must be big. (Of course, this must be separated from the unital case where $\phi(1) = 1$ and ϕ is *unitally absorbing* – meaning absorbing all *strongly unital* trivial extensions.)

Finally, we note that in a separate paper, where we also investigate the relationship between essential codimension and projection lifting, we will look more extensively at concrete generalizations of the BDF essential codimension result, as in the above.

4. Technical lemma

For $\delta > 0$, let $f_\delta : [0, \infty) \to [0, 1]$ be the unique continuous function for which

$$f_\delta(t) = \begin{cases} 1 & t \in [\delta, \infty) \\ 0 & t = 0 \\ \text{linear on} & [0, \delta]. \end{cases}$$

If $\mathcal{C}$ is a unital C*-algebra and $p \in \mathcal{C}$ is a projection, we follow standard convention by letting $p^\perp =_{df} 1 - p$.

In what follows, for elements a, b in a C*-algebra, we use $a \approx_\epsilon b$ to denote $\|a - b\| < \epsilon$.

Lemma 4.1. *Let $\mathcal{B}$ be a separable stable C*-algebra with an approximate unit $\{e_n\}$ consisting of increasing projections. (We define $e_0 =_{df} 0$.)*

Suppose that $A, A', A'' \in \mathcal{C}(\mathcal{B})_+$ are contractive elements and $\delta > 0$ such that

$$AA' = A'$$

and

$$A'' \in her((A' - \delta)_+).$$

Let $A_0 \in \mathcal{M}(\mathcal{B})$ be any contractive lift of A, and let $\epsilon > 0$ be given.

Then for every $M \geq 0$, there exists an $A_0'' \in e_M^\perp \mathcal{M}(\mathcal{B}) e_M^\perp$ which is a contractive positive lift of A'' such that for all $l \geq 1$,

$$A_0 (A_0'')^{1/l} \approx_\epsilon (A_0'')^{1/l} \approx_\epsilon (A_0'')^{1/l} A_0.$$

Proof. Choose $\delta_1 > 0$ such that for any contractive operators B, C, with $C \geq 0$, if

$$BC \approx_{\delta_1} C \approx_{\delta_1} CB$$

then

$$Bf_\delta(C) \approx_\epsilon f_\delta(C) \approx_\epsilon f_\delta(C)B.$$

(Sketch of argument for choosing δ_1: By the Weierstrass approximation theorem, find a polynomial $p(t)$, with $p(0) = 0$, such that $|f_\delta(t) - p(t)| < \epsilon/2$ for all $t \in [0, 1]$. Now use the concrete structure of $p(t)$ to determine δ_1.)

Let $A_0' \in e_M^\perp \mathcal{M}(\mathcal{B}) e_M^\perp$ be any contractive positive lift of A'. Note that we can restrict to the corner $e_M^\perp$ because the image of $\pi(e_M^\perp) = 1$ since $e_M \in \mathcal{M}(\mathcal{B})$. Because $AA' = A'$ (and since they are positive, we also have $A' = A'A$) it follows that $AA'^{1/2} = A'^{1/2}$ and so also $A'^{1/2}A = A'^{1/2}$. Therefore, there exist $c, c' \in \mathcal{B}$ for which

$$A_0 {A_0'}^{1/2} = {A_0'}^{1/2} + c,$$

and

$${A_0'}^{1/2} A_0 = {A_0'}^{1/2} + c'.$$

Since $\{e_n\}$ is an approximate identity for $\mathcal{B}$, we can choose $N \geq 1$ so that

$$ce_N^\perp \approx_{\delta_1} 0 \approx_{\delta_1} e_N^\perp c'.$$

Then, combining the above displays yields

$$A_0 {A'_0}^{1/2} e_N^{\perp} \approx_{\delta_1} {A'_0}^{1/2} e_N^{\perp},$$

and

$$e_N^{\perp} {A'_0}^{1/2} A_0 \approx_{\delta_1} e_N^{\perp} {A'_0}^{1/2}.$$

Hence, if we define

$$D =_{df} {A'_0}^{1/2} e_N^{\perp} {A'_0}^{1/2}$$

then

$$A_0 D \approx_{\delta_1} D \approx_{\delta_1} D A_0.$$

Hence, by the definition of δ_1,

$$A_0 f_\delta(D) \approx_\epsilon f_\delta(D) \approx_\epsilon f_\delta(D) A_0. \tag{4.1}$$

Note that $\pi(D) = A'$, which follows since $\pi(e_N^{\perp}) = 1$. Because the algebra $\pi(\overline{(D-\delta)_+ \mathcal{M}(\mathcal{B})(D-\delta)_+}) = her((A'-\delta)_+)$, we can find a contractive positive lift $A''_0 \in \overline{(D-\delta)_+ \mathcal{M}(\mathcal{B})(D-\delta)_+}$ of A''. Note that $A''_0 \in e_M^{\perp} \mathcal{M}(\mathcal{B}) e_M^{\perp}$ since A'_0, and consequently, D and $(D-\delta)_+$ are.

We remark that $f_\delta(D)(D-\delta)_+ = (D-\delta)_+$, and for all $l \geq 1$, ${A''_0}^{1/l}$ is a contraction. Combining these facts with (4.1) we obtain

$$A_0 (A''_0)^{1/l} = A_0 f_\delta(D)(A''_0)^{1/l} \approx_\epsilon f_\delta(D)(A''_0)^{1/l} = (A''_0)^{1/l}.$$

Similarly,

$$(A''_0)^{1/l} A_0 \approx_\epsilon (A''_0)^{1/l}.$$

□

We now fix some notation which will be used for the rest of this section.

Let $\mathcal{B}$ be a separable simple stable C*-algebra with a nonzero projection. Let $\{p_k\}_{k=1}^{\infty}$ be a sequence of pairwise orthogonal projections of $\mathcal{B}$ such that

$$\sum_{k=1}^{\infty} p_k = 1_{\mathcal{M}(\mathcal{B})}.$$

where the series converges strictly.

For all $m \leq n$, let

$$p_{m,n} =_{df} \sum_{k=m}^{n} p_n$$

and let

$$e_n =_{df} \sum_{k=1}^{n} p_n.$$

(Hence, $\{e_n\}$ is an approximate unit for $\mathcal{B}$.)

Let $U \in \mathcal{C}(\mathcal{B})$ be a unitary and let $V \in \mathcal{M}(\mathcal{B})$ be a partial isometry such that

$$\pi(V) = U.$$

In fact, in the computations that follow, that V is a partial isometry is not really needed. All that is needed, for the rest of this paper, is that V is a contractive operator which lifts U.

Also, we let $\overline{B(0,1)}$ denote the closed unit ball of the complex plane, i.e., $\overline{B(0,1)} =_{df} \{\alpha \in \mathbb{C} : |\alpha| \leq 1\}$.

Recall next that for a C*-algebra $\mathcal{C}$, for a $\tau \in T(\mathcal{C})$ and for any $a \in \mathcal{C}_+$,

$$d_\tau(a) =_{df} \lim_{n \to \infty} \tau(a^{1/n}).$$

Recall also that a *Laurent polynomial* on the punctured closed disk $\overline{B(0,1)} - \{0\}$ is a continuous function $h : \overline{B(0,1)} - \{0\} \to \mathbb{C}$ which has the form

$$h(\lambda) = \sum_{n=0}^{N} \beta_n \lambda^n + \sum_{m=1}^{M} \gamma_m \overline{\lambda}^m$$

for all $\lambda \in \overline{B(0,1)} - \{0\}$. Here, $M, N \geq 1$ are integers and $\beta_n, \gamma_m \in \mathbb{C}$.

In what follows, we will use that the algebra of Laurent polynomials, when restricted to the circle S^1, is uniformly dense in $C(S^1)$.

Next, if $\mathcal{C}$ is a unital C*-algebra and h is a Laurent polynomial as above, then for all contractive $x \in \mathcal{C}$, we define

$$h(x) =_{df} \sum_{n=0}^{N} \beta_n x^n + \sum_{m=1}^{M} \gamma_m (x^*)^m.$$

This is well-defined by the uniqueness of Laurent series expansion. Note that when x is a unitary, this is consistent with the continuous functional calculus.

Finally, for a real-valued function f, we let $\mathrm{osupp}(f) := f^{-1}(\mathbb{R} \setminus \{0\})$ denote the *open support* of f. Of course, $\overline{\mathrm{osupp}(f)} = \mathrm{supp}(f)$.

Lemma 4.2. *Let h be a Laurent polynomial and let $X \in \mathcal{M}(\mathcal{B})$ be contractive.*

(a) *For every $\epsilon > 0$ and $L \geq 1$, there exists an $L_1 \geq 1$ where for all sequences $\{\alpha_j\}$ and $\{\alpha'_j\}$ in $\overline{B(0,1)}$ for which*

$$\alpha_j = \alpha'_j$$

for all $j \geq L$, we have that

$$h\left(\sum_{j=1}^{\infty} \alpha_j p_j X\right) e_{L_1}^{\perp} \approx_\epsilon h\left(\sum_{j=1}^{\infty} \alpha'_j p_j X\right) e_{L_1}^{\perp}.$$

(b) *For every $\epsilon > 0$ and $y \in \mathcal{B}$, there exists an $M \geq 1$ where for all sequences $\{\alpha_j\}$ and $\{\alpha'_j\}$ in $\overline{B(0,1)}$ for which*

$$\alpha_j = \alpha'_j$$

for all $j \leq M$, we have that

$$h\left(\sum_{j=1}^{\infty} \alpha_j p_j X\right) y \approx_\epsilon h\left(\sum_{j=1}^{\infty} \alpha'_j p_j X\right) y.$$

Proof. Let us suppose that $h(\lambda)$ has the form

$$h(\lambda) = \lambda^n$$

where

$$n \geq 1.$$

The proofs for the cases $\overline{\lambda}^n$, constants and linear combinations are similar or easier. In fact, for our arguments, the distinctions between X and X^*, and left and right, are not important. The required small changes are easy to see.

Let us first prove (a).

We prove that for every $\epsilon > 0$, for all $1 \leq k \leq n$, there exists $M_k \geq 1$ so that for every sequence $\{\alpha_j\}$ in $\overline{B(0,1)}$,

$$\left(\sum_{j=1}^{\infty} \alpha_j p_j X\right)^k e_{M_k}^{\perp} \approx_{\epsilon} \left(\sum_{j=L+1}^{\infty} \alpha_j p_j X\right)^k e_{M_k}^{\perp}$$

The proof is by induction on k.

Basis step $k = 1$. Since

$$e_L X \in \mathcal{B},$$

for every $\epsilon > 0$, we can find $M_1 \geq 1$ so that

$$e_L X e_{M_1}^{\perp} \approx_{\epsilon} 0.$$

It follows that for every sequence $\{\alpha_j\}$ in $\overline{B(0,1)}$,

$$\sum_{j=1}^{\infty} \alpha_j p_j X e_{M_1}^{\perp} \approx_{\epsilon} \sum_{j=L+1}^{\infty} \alpha_j p_j X e_{M_1}^{\perp}.$$

Induction step.

Let $\epsilon > 0$ be given.

Say that $k+1 \leq n$ and we have found $M_k \geq 1$ so that for every sequence $\{\alpha_j\}$ in $\overline{B(0,1)}$,

$$\left(\sum_{j=1}^{\infty} \alpha_j p_j X\right)^k e_{M_k}^{\perp} \approx_{\frac{\epsilon}{3}} \left(\sum_{j=L+1}^{\infty} \alpha_j p_j X\right)^k e_{M_k}^{\perp}.$$

Let

$$M_k' =_{df} \max\{M_k, L\}.$$

Since

$$e_{M_k'} X \in \mathcal{B},$$

we can find $M_{k+1} \geq 1$ so that

$$e_{M_k'} X e_{M_{k+1}}^{\perp} \approx_{\frac{\epsilon}{3}} 0.$$

It follows that for every sequence $\{\alpha_j\}$ in $\overline{B(0,1)}$,

$$\left(\sum_{j=1}^{\infty} \alpha_j p_j X\right)^{k+1} e_{M_{k+1}}^{\perp} \approx_{\epsilon} \left(\sum_{j=L+1}^{\infty} \alpha_j p_j X\right)^{k+1} e_{M_{k+1}}^{\perp}$$

as required.

This completes the induction and the proof of (a).

Let us now prove (b).

For simplicity, let us also assume that y is a contraction.

Let

$$N =_{df} 2^n + 1.$$

We construct a finite sequence of positive integers $m_1, m_2, ..., m_n$ such that for all $1 \leq k \leq n$,

$$c_k X ... c_2 X c_1 X y \approx_{\frac{\epsilon}{2N}} 0$$

if for all j, c_j is a contractive operator where

$$c_j \in e_{m_j} \mathcal{B} e_{m_j} \cup e_{m_j}^{\perp} \mathcal{B} e_{m_j}^{\perp}$$

and if there exists an l for which

$$c_l \in e_{m_l}^{\perp} \mathcal{B} e_{m_l}^{\perp}.$$

The construction is by induction on k.

Basis step $k = 1$. Since

$$Xy \in \mathcal{B},$$

we can find $m_1 \geq 1$ so that

$$e_{m_1}^{\perp} X y \approx_{\frac{\epsilon}{2N}} 0.$$

Therefore, for every contractive $c_1 \in e_{m_1}^{\perp} \mathcal{B} e_{m_1}^{\perp}$,

$$c_1 X y \approx_{\frac{\epsilon}{2N}} 0.$$

Induction step. Say that $m_1, ..., m_k$ have been chosen, and $k+1 \leq n$. We now choose m_{k+1}.

Since

$$X e_{m_k} \in \mathcal{B},$$

we can find $m_{k+1} \geq 1$ so that

$$e_{m_{k+1}}^{\perp} X e_{m_k} \approx_{\frac{\epsilon}{2N}} 0.$$

Hence, if $c_j \in e_{m_j} \mathcal{B} e_{m_j}$ is contractive for all $1 \leq j \leq k$, and if $c_{k+1} \in e_{m_{k+1}}^{\perp} \mathcal{B} e_{m_{k+1}}^{\perp}$ is contractive, then

$$c_{k+1} X c_k X ... c_2 X c_1 X y \approx_{\frac{\epsilon}{2N}} 0.$$

Now by the induction hypothesis,

$$d_k X d_{k-1} X ... d_2 X d_1 X y \approx_{\frac{\epsilon}{2N}} 0,$$

whenever for all j,

$$d_j \in e_{m_j}\mathcal{B}e_{m_j} \cup e_{m_j}^{\perp}\mathcal{B}e_{m_j}^{\perp}$$

is contractive and if there exists an l for which

$$d_l \in e_{m_l}^{\perp}\mathcal{B}e_{m_l}^{\perp}.$$

Hence, with d_j as above, if $c_{k+1} \in e_{m_{k+1}}\mathcal{B}e_{m_{k+1}} \cup e_{m_{k+1}}^{\perp}\mathcal{B}e_{m_{k+1}}^{\perp}$ is contractive, then

$$c_{k+1}Xd_kXd_{k-1}X...d_2Xd_1Xy \approx_{\frac{\epsilon}{2N}} 0.$$

This completes the inductive construction.

Now let

$$M =_{df} \max\{m_1, m_2, ..., m_n\} + 2.$$

Say that $\{\alpha_j\}$, $\{\alpha'_j\}$ are two sequences in $\overline{B(0,1)}$ such that

$$\alpha_j = \alpha'_j$$

for all $j \leq M$.

Now recall that we are assuming that $h(\lambda)$ has the form $h(\lambda) = \lambda^n$. Therefore,

$$\begin{aligned} & h\left(\sum_{j=1}^{\infty}\alpha_j p_j X\right) y \\ = \quad & \left(\sum_{j=1}^{\infty}\alpha_j p_j X\right)^n y \\ \approx_{\frac{2^n\epsilon}{2N}} \quad & \prod_{l=1}^{n}\left(\sum_{j=1}^{m_l}\alpha_j p_j X\right) y \text{ (by the definition of the } m_j \text{)} \\ = \quad & \prod_{l=1}^{n}\left(\sum_{j=1}^{m_l}\alpha'_j p_j X\right) y \text{ (by the definition of } M) \\ \approx_{\frac{2^n\epsilon}{2N}} \quad & \left(\sum_{j=1}^{\infty}\alpha'_j p_j X\right)^n y \text{ (by the definition of the } m_j \text{)} \\ = \quad & h\left(\sum_{j=1}^{\infty}\alpha'_j p_j X\right) y. \end{aligned}$$

Since $N =_{df} 2^n + 1$, we are done. $\square$

Lemma 4.3. *Let h be a Laurent polynomial and $X \in \mathcal{M}(\mathcal{B})$ be contractive.*

Then for every $\epsilon > 0$ and $K \geq 1$, there exists an $L \geq 1$ such that for every sequence $\{\alpha_j\}$ in $\overline{B(0,1)}$,

$$\left\| e_K h\left(\sum_{j=1}^{\infty} \alpha_j p_j X\right) e_L^{\perp} \right\|, \left\| e_L^{\perp} h\left(\sum_{j=1}^{\infty} \alpha_j p_j X\right) e_K \right\| < \epsilon.$$

Proof. The proof is a variation on that of Lemma 4.2 (b). We sketch the argument here, referring to parts of the proof of Lemma 4.2 (b).

For simplicity, we will prove that for every $K \geq 1$, there exists an $L \geq 1$ such that for every sequence $\{\alpha_j\}$ in $\overline{B(0,1)}$,

$$\left\| e_L^{\perp} h\left(\sum_{j=1}^{\infty} \alpha_j p_j X\right) e_K \right\| < \epsilon,$$

leaving the other statement to the reader.

As in the proof of Lemma 4.2 (b), let us assume, for simplicity, that $h(\lambda)$ has the form

$$h(\lambda) = \lambda^n$$

where

$$n \geq 1.$$

As before, the arguments for the other cases are similar or easier.

In the proof of Lemma 4.2 (b), replace y with e_K.

Let $m_1, m_2, ..., m_n$ be the positive integers obtained from the inductive construction in the proof of Lemma 4.2 (b), and let

$$N =_{df} 2^n + 1.$$

Hence, from the proof of Lemma 4.2 (b), we have that

$$c_n X ... c_2 X c_1 X e_K \approx_{\frac{\epsilon}{2N}} 0$$

if for all j, c_j is a contractive operator where

$$c_j \in e_{m_j} \mathcal{B} e_{m_j} \cup e_{m_j}^{\perp} \mathcal{B} e_{m_j}^{\perp}$$

and if there exists an l for which

$$c_l \in e_{m_l}^{\perp} \mathcal{B} e_{m_l}^{\perp}.$$

Now let

$$L =_{df} m_n.$$

Hence, if $\{\alpha_j\}$ is a sequence in $\overline{B(0,1)}$, then we have that

$$e_L^{\perp} h\left(\sum_{j=1}^{\infty} \alpha_j p_j X\right) e_K = \left(\sum_{j=L+1}^{\infty} \alpha_j p_j X\right) \left(\sum_{j=1}^{\infty} \alpha_j p_j X\right)^{n-1} e_K \approx_{\frac{2^n \epsilon}{2N}} 0.$$

Since $N = 2^n + 1$, we are done. ∎

Lemma 4.4. *Let $h_1, h_2, h_3 : S^1 \to [0,1]$ be continuous functions and let $\delta_1 > 0$ be such that*

$$h_1 h_2 = h_2$$

and

$$\overline{\mathrm{osupp}(h_3)} \subset \mathrm{osupp}((h_2 - \delta_1)_+).$$

Let $\delta_2 > 0$ and $\widehat{h}$ be a Laurent polynomial such that

$$|\widehat{h}(\lambda) - h_1(\lambda)| < \frac{\delta_2}{10}$$

for all $\lambda \in S^1$.

Then for every $L, L' \geq 1$, there exist $L_1 > L'$, there exist contractive $A \in e_{L_1}^{\perp}\mathcal{M}(\mathcal{B})_+ e_{L_1}^{\perp}$ which is a lift of $h_3(U)$ such that for every contractive $a \in (\overline{A\mathcal{B}A})_+$ there exist $M > L$, $M_1 > L_1$ and $x \in p_{L_1+1,M_1}\mathcal{B}p_{L_1+1,M_1}$ for which

$$x\widehat{h}\left(\sum_{j=1}^{\infty} \alpha_j p_j V\right) x^* \approx_{\delta_2} a$$

for every sequence $\{\alpha_j\}$ in $\overline{B(0,1)}$ (closed unit ball of the complex plane) such that $\alpha_j = 1$ for all $L \leq j \leq M$.

Proof. Let $A_0 \in \mathcal{M}(\mathcal{B})_+$ be a contractive lift of $h_1(U)$.

Since U is unitary and because of the conditions on $\hat{h}$, we know $\hat{h}(U) \approx_{\frac{\delta_2}{10}} h_1(U)$. Moreover, since $\hat{h}$ is a Laurent polynomial, $\hat{h}(U) = \hat{h}(\pi(V)) = \pi(\hat{h}(V))$ and also $h_1(U) = \pi(A_0)$. Using these facts and by Lemma 4.2 (a), we can choose $L_1 > L'$ so that

$$e_{L_1}^{\perp}\widehat{h}(V)e_{L_1}^{\perp} \approx_{\frac{\delta_2}{10}} e_{L_1}^{\perp}A_0e_{L_1}^{\perp} \tag{4.2}$$

and

$$\widehat{h}\left(\sum_{j=1}^{\infty} \alpha_j p_j V\right) e_{L_1}^{\perp} \approx_{\frac{\delta_2}{10}} \widehat{h}\left(\sum_{j=1}^{\infty} \alpha'_j p_j V\right) e_{L_1}^{\perp} \tag{4.3}$$

for all sequences $\{\alpha_j\}$ and $\{\alpha'_j\}$ in $\overline{B(0,1)}$ such that $\alpha_j = \alpha'_j$ for all $j \geq L$.

By Lemma 4.1 (instantiated with $A, A', A'', \delta, \epsilon, M$ chosen to be $h_1(U), h_2(U), h_3(U), \delta_1, \frac{\delta_2}{10}, L_1$), there exists $A \in e_{L_1}^{\perp}\mathcal{M}(\mathcal{B})e_{L_1}^{\perp}$ which is a contractive positive lift of $h_3(U)$ for which

$$e_{L_1}^{\perp}A_0e_{L_1}^{\perp}A^{1/l} \approx_{\frac{\delta_2}{10}} A^{1/l}$$

for all $l \geq 1$.

Hence, if we let $a \in \overline{A\mathcal{B}A}$ be an arbitrary contractive positive element, then because the previous display holds for all $l \geq 1$,

$$e_{L_1}^{\perp}A_0e_{L_1}^{\perp}a^{1/2} \approx_{\frac{\delta_2}{10}} a^{1/2}.$$

Chaining this with (4.2) yields

$$e_{L_1}^{\perp}\widehat{h}(V)e_{L_1}^{\perp}a^{1/2} \approx_{\frac{\delta_2}{5}} a^{1/2}.$$

Therefore, if we let $y =_{df} a^{1/2}$ then

$$ye_{L_1}^{\perp}\widehat{h}(V)e_{L_1}^{\perp}y^* \approx_{\frac{2\delta_2}{5}} a.$$

Since $y \in e_{L_1}^{\perp}\mathcal{B}e_{L_1}^{\perp}$,

$$e_{L_1}^{\perp}ye_{L_1}^{\perp}\widehat{h}(V)e_{L_1}^{\perp}y^*e_{L_1}^{\perp} \approx_{\frac{2\delta_2}{5}} a.$$

Since $y \in \mathcal{B}$, we can choose $M_1 > L_1$ such that if we define

$$x =_{df} p_{L_1+1,M_1}yp_{L_1+1,M_1}$$

then

$$x\widehat{h}(V)x^* \approx_{\frac{2\delta_2}{5}} a.$$

Hence, by (4.3),

$$x\widehat{h}\left(\sum_{j=1}^{\infty}\alpha_j p_j V\right)x^* \approx_{\frac{\delta_2}{2}} a$$

for every sequence $\{\alpha_j\}$ in $\overline{B(0,1)}$ for which $\alpha_j = 1$ for all $L \leq j$.

Hence, by Lemma 4.2 (b), we can choose $M > L$ such that

$$x\widehat{h}\left(\sum_{j=1}^{\infty}\alpha_j p_j V\right)x^* \approx_{\delta_2} a$$

for every sequence $\{\alpha_j\}$ in $\overline{B(0,1)}$ for which $\alpha_j = 1$ for all $L \leq j \leq M$. $\square$

Lemma 4.5. *Suppose that, in addition, $\mathcal{B}$ has strict comparison of positive elements and $T(\mathcal{B})$ has finitely many extreme points.*

Let $p \in \mathcal{B}$ be a nonzero projection and let $\epsilon > 0$ be given.

Let $h_1, h_2, h_3 : S^1 \to [0,1]$ be continuous functions, $\delta_1 > 0$ and $\lambda_1, ..., \lambda_m \in S^1$ such that

$$h_1h_2 = h_2$$

$$\overline{\mathrm{osupp}(h_3)} \subset \mathrm{osupp}((h_2 - \delta_1)_+)$$

and the function

$$\lambda \mapsto \sum_{j=1}^{m} h_3(\lambda_j\lambda)$$

is a full element in $C(S^1)$.

There exists $\delta_2 > 0$ such that if $\widehat{h}$ is a Laurent polynomial for which

$$|\widehat{h}(\lambda) - h_1(\lambda)| < \frac{\delta_2}{10}$$

for all $\lambda \in S^1$ then the following holds:

For every $L, L' \geq 1$, there exist $L < L_1 < L_2 < ... < L_m$, $L' < M < M'$, and contractive $x \in p\mathcal{B}p_{M+1,M'}$ such that

$$x\widehat{h}\left(\sum_{j=1}^{\infty} \alpha_j p_j V\right) x^* \approx_{\epsilon} p$$

where $\{\alpha_j\}$ is any sequence in $\overline{B(0,1)}$ such that $\alpha_j = \lambda_k$ for all $L_{k-1} < j \leq L_k$ and all $1 \leq k \leq m$. (Here, $L_0 =_{df} L$.)

Proof. Let $\mathcal{F}$ be the finitely many extreme points of $T(\mathcal{B})$.

Let $L, L' \geq 1$ be arbitrary.

Let $\epsilon > 0$ and let $\delta_2 > 0$ be any constant for which $\delta_2 < \frac{\epsilon}{m^2}$.

We construct a elements L_j, A_j, b_j, ϵ_j, M, M'_j, M''_j, M'''_j and x_j for $1 \leq j \leq m$. The construction is by induction on j.

Basis step: $j = 1$.

Let $\widehat{h}_1$ be the Laurent polynomial given by

$$\widehat{h}_1(\lambda) =_{df} \widehat{h}(\lambda_1 \lambda)$$

for all $\lambda \in S^1$.

Note that

$$\widehat{h}_1(\lambda) \approx_{\frac{\delta_2}{10}} h_1(\lambda_1 \lambda)$$

for all $\lambda \in S^1$. Also, it is clear that the remaining hypotheses of Lemma 4.4 are satisfied for $h_1(\lambda_1 \cdot)$, $h_2(\lambda_1 \cdot)$, $h_3(\lambda_1 \cdot)$ and $\widehat{h}_1(\cdot)$. (We have merely shifted everything by a factor λ_1.)

By Lemma 4.4, choose $M > L'$ and contractive positive $A_1 \in e_M^{\perp}\mathcal{M}(\mathcal{B})e_M^{\perp}$ such that

$$\pi(A_1) = h_3(\lambda_1 U).$$

We let

$$M'_1 =_{df} M.$$

Let $\mathcal{F}_1 =_{df} \{\tau \in \mathcal{F} : \tau(A_1) = \infty\}$.

Let $a_1 \in \overline{A_1 \mathcal{B} A_1}$ be a strictly positive element. Then

$$\tau(a_1) = d_{\tau}(a_1) = \infty$$

for all $\tau \in \mathcal{F}_1$.

Since d_{τ} is norm lower semicontinuous on $\mathcal{B}_+$, choose $\epsilon_1 > 0$ so that

$$d_{\tau}((a_1 - 2\epsilon_1)_+) > \tau(p)$$

for all $\tau \in \mathcal{F}_1$.

By Lemma 4.4, choose $L_1 > L$, $M''_1 > M'_1 = M$ and a contractive element $x_1 \in p_{M+1,M''_1}\mathcal{B}p_{M+1,M''_1}$ so that

$$x_1\widehat{h}_1\left(\sum_{j=1}^{\infty} \alpha_j p_j V\right) x_1^* \approx_{\delta_2} f_{\epsilon_1}(a_1)$$

for every sequence $\{\alpha_j\}$ in $\overline{B(0,1)}$ for which $\alpha_j = 1$ for all $L < j \leq L_1$.

Hence,

$$x_1 \widehat{h}\left(\sum_{j=1}^{\infty} \alpha_j p_j V\right) x_1^* \approx_{\delta_2} f_{\epsilon_1}(a_1)$$

for every sequence $\{\alpha_j\}$ in $\overline{B(0,1)}$ for which $\alpha_j = \lambda_1$ for all $L < j \leq L_1$.

For all $l \geq M+1$, let

$$b_{1,l} =_{df} p_{M+1,l} a_1 p_{M+1,l}.$$

Then

$$(b_{1,l} - 2\epsilon_1)_+ \to (a_1 - 2\epsilon_1)_+$$

in norm, as $l \to \infty$. Hence, since d_τ is norm lower semicontinuous,

$$\liminf_{l\to\infty} d_\tau((b_{1,l} - 2\epsilon_1)_+) \geq d_\tau((a_1 - 2\epsilon_1)_+),$$

for all $\tau \in T(\mathcal{B})$.

Hence, let $M_1''' > M_1''$ be a number that is big enough so that if we define

$$b_1 =_{df} b_{1,M_1'''}$$

then

$$d_\tau((b_1 - 2\epsilon_1)_+) > \tau(p)$$

for all $\tau \in \mathcal{F}_1$ and

$$x_1 \widehat{h}\left(\sum_{j=1}^{\infty} \alpha_j p_j V\right) x_1^* \approx_{\delta_2} f_{\epsilon_1}(b_1)$$

for every sequence $\{\alpha_j\}$ in $\overline{B(0,1)}$ for which $\alpha_j = \lambda_1$ for all $L < j \leq L_1$.

Induction step. Suppose that L_k, A_k, b_k, ϵ_k, M_k', M_k'' M_k''', ϵ_k and x_k have been chosen. We now construct the constants with k replaced with $k+1$.

By Lemma 4.3, choose $N > M_k'''$ big enough so that

$$\left\|e_{M_k'''} \widehat{h}(\sum_{j=1}^{\infty} \alpha_j p_j V) e_N^\perp\right\|, \left\|e_N^\perp \widehat{h}(\sum_{j=1}^{\infty} \alpha_j p_j V) e_{M_k'''}\right\| < \delta_2, \tag{4.4}$$

for every sequence $\{\alpha_j\}$ in $\overline{B(0,1)}$.

Let $\widehat{h}_{k+1}$ be the Laurent polynomial given by

$$\widehat{h}_{k+1}(\lambda) =_{df} \widehat{h}(\lambda_{k+1}\lambda)$$

for all $\lambda \in S^1$.

Once more, it is not hard to see that the hypotheses of Lemma 4.4 are satisfied by $h_1(\lambda_{k+1}\cdot)$, $h_2(\lambda_{k+1}\cdot)$, $h_3(\lambda_{k+1}\cdot)$, and $\widehat{h}_{k+1}(\cdot)$, with error estimate $\frac{\delta_2}{10}$. (Again, we have merely shifted everything by a factor λ_{k+1}.)

By Lemma 4.4, choose $M_{k+1}' > N$ and a contractive positive element $A_{k+1} \in e_{M_{k+1}'}^\perp \mathcal{M}(\mathcal{B}) e_{M_{k+1}'}^\perp$ such that

$$\pi(A_{k+1}) = h_3(\lambda_{k+1}U).$$

Let $\mathcal{F}_{k+1} =_{df} \{\tau \in \mathcal{F} : \tau(A_{k+1}) = \infty\}$. Let $a_{k+1} \in \overline{A_{k+1}\mathcal{B}A_{k+1}}$ be a strictly positive element. Since d_τ is norm lower semicontinuous, choose $\epsilon_{k+1} > 0$ so that

$$d_\tau((a_{k+1} - 2\epsilon_{k+1})_+) > \tau(p)$$

for all $\tau \in \mathcal{F}_{k+1}$.

By Lemma 4.4, choose $L_{k+1} > L_k$, $M''_{k+1} > M'_{k+1}$ and contractive $x_{k+1} \in p_{M'_{k+1}+1, M''_{k+1}} \mathcal{B} p_{M'_{k+1}+1, M''_{k+1}}$ so that

$$x_{k+1}\widehat{h}_{k+1}\left(\sum_{j=1}^{\infty} \alpha_j p_j V\right) x^*_{k+1} \approx_{\delta_2} f_{\epsilon_{k+1}}(a_{k+1})$$

for every sequence $\{\alpha_j\}$ in $\overline{B(0,1)}$ for which $\alpha_j = 1$ for all $L_k < j \leq L_{k+1}$.

Hence,

$$x_{k+1}\widehat{h}\left(\sum_{j=1}^{\infty} \alpha_j p_j V\right) x^*_{k+1} \approx_{\delta_2} f_{\epsilon_{k+1}}(a_{k+1})$$

for every sequence $\{\alpha_j\}$ in $\overline{B(0,1)}$ for which $\alpha_j = \lambda_{k+1}$ for all $L_k < j \leq L_{k+1}$.

Find $M'''_{k+1} > M''_{k+1}$ big enough so that if we define

$$b_{k+1} =_{df} p_{M'_{k+1}, M'''_{k+1}} a_{k+1} p_{M'_{k+1}, M'''_{k+1}}$$

then, as in the basis step, since d_τ is norm lower semicontinuous

$$d_\tau((b_{k+1} - 2\epsilon_{k+1})_+) > \tau(p) \tag{4.5}$$

for all $\tau \in \mathcal{F}_{k+1}$ and

$$x_{k+1}\widehat{h}\left(\sum_{j=1}^{\infty} \alpha_j p_j V\right) x^*_{k+1} \approx_{\delta_2} f_{\epsilon_{k+1}}(b_{k+1}) \tag{4.6}$$

for every sequence $\{\alpha_j\}$ in $\overline{B(0,1)}$ for which $\alpha_j = \lambda_{k+1}$ for all $L_k < j \leq L_{k+1}$.

Finally, note that for all $l \leq k$,

$$M'_l < M''_l < M'''_l < N < M'_{k+1} < M''_{k+1} < M'''_{k+1}.$$

Hence, since for all $l \leq k+1$,

$$x_l \in p_{M'_l+1, M''_l} \mathcal{B} p_{M'_l+1, M''_l},$$

it follows, from (4.4), that for all $l \leq k$, for every sequence $\{\alpha_j\}$ in $\overline{B(0,1)}$,

$$\left\| x_l \widehat{h}\left(\sum_{j=1}^{\infty} \alpha_j p_j V\right) x^*_{k+1} \right\|, \left\| x_{k+1} \widehat{h}\left(\sum_{j=1}^{\infty} \alpha_j p_j V\right) x^*_l \right\| < \delta_2. \tag{4.7}$$

This completes the inductive construction.

Define

$$M' =_{df} M'''_m.$$

Now let $x_0 \in p_{M+1,M'}\mathcal{B}p_{M+1,M'}$ be the contractive element defined by

$$x_0 =_{df} \sum_{j=1}^{m} x_j.$$

Let $\{\alpha_j\}$ be any sequence in $\overline{B(0,1)}$ such that $\alpha_j = \lambda_k$ for all $L_{k-1} < j \leq L_k$ and for all $1 \leq k \leq m$. (Here $L_0 =_{df} L$.)

Then

$$\begin{aligned} & x_0 \widehat{h}(\sum_{j=1}^{\infty} \alpha_j p_j V) x_0^* \\ \approx_{(m^2-m)\delta_2} \quad & \sum_{k=1}^{m} x_k \widehat{h}(\sum_{j=1}^{\infty} \alpha_j p_j V) x_k^* \text{ (by (4.7))} \\ \approx_{m\delta_2} \quad & \sum_{k=1}^{m} f_{\epsilon_k}(b_k) \text{ (by (4.6)).} \end{aligned}$$

Note that by hypothesis, the map

$$\lambda \mapsto \sum_{j=1}^{m} h_3(\lambda_j \lambda)$$

is full in $C(S^1)$. Hence, $\sum_{j=1}^{n} h_3(\lambda_j U)$ is full in $\mathcal{C}(\mathcal{B})$. Hence, since $\pi(A_j) = h_3(\lambda_j U)$ for all j,

$$\tau\left(\sum_{j=1}^{m} A_j\right) = \infty$$

for all $\tau \in T(\mathcal{B})$. Hence, by definition of the $\mathcal{F}_j$s (in the inductive construction), we must have that

$$\bigcup_{j=1}^{m} \mathcal{F}_j = \partial_{ext} T(\mathcal{B}).$$

Hence, from (4.5), it follows that for all $\tau \in T(\mathcal{B})$,

$$d_\tau\left(\sum_{k=1}^{m} (b_k - 2\epsilon_k)_+\right) > \tau(p)$$

Hence, since $\mathcal{B}$ has strict comparison for positive elements, there exists a projection $q \in \overline{(\sum_{k=1}^{m}(b_k - 2\epsilon_k)_+)\mathcal{B}(\sum_{k=1}^{m}(b_k - 2\epsilon_k)_+)}$ such that

$$q \sim p.$$

Hence,

$$q x_0 \widehat{h}\left(\sum_{j=1}^{\infty} \alpha_j p_j V\right) x_0^* q \approx_{m^2\delta_2} q$$

for every sequence $\{\alpha_j\}$ in $\overline{B(0,1)}$ for which $\alpha_j = \lambda_k$ for every $L_{k-1} < j \leq L_k$ and all $1 \leq k \leq m$.

Let $v \in \mathcal{B}$ be a partial isometry such that

$$v^* v = q \text{ and } vv^* = p.$$

Let

$$x =_{df} vqx_0.$$

Then

$$x\widehat{h}\left(\sum_{j=1}^{\infty} \alpha_j p_j V\right) x^* \approx_{m^2 \delta_2} p$$

for every sequence $\{\alpha_j\}$ in $\overline{B(0,1)}$ for which $\alpha_j = \lambda_k$ for every $L_{k-1} < j \leq L_k$ and all $1 \leq k \leq m$. Since $\delta_2 < \frac{\epsilon}{m^2}$ we are done. □

Let $\mathcal{D}$ be a unital C*-algebra. Recall that a nonzero element $x \in \mathcal{D}$ is said to be *strongly full* in $\mathcal{D}$ if every nonzero element of $C^*(x)$ is a full element of $\mathcal{D}$.

Lemma 4.6. *Say that, in addition, $\mathcal{B}$ has strict comparison for positive elements and $T(\mathcal{B})$ has finitely many extreme points. Then there exists a sequence $\{\alpha_j\}$ in S^1 such that the unitary $\pi(\sum_{j=1}^{\infty} \alpha_j p_j)U$ is a strongly full element of $\mathcal{C}(\mathcal{B})$.*

Proof. For every $k \geq 1$, let $h_{k,1,j}, h_{k,2,j}, h_{k,3,j} : S^1 \to [0,1]$ be continuous functions, $\lambda_{k,j} \in S^1$ (for $1 \leq j \leq k$), and $\delta_k > 0$ be such that

$$\sum_{j=1}^{k} h_{k,3,j}$$

is a full element of $C(S^1)$,

$$\overline{\text{osupp}(h_{k,3,j})} \subset \text{osupp}((h_{k,2,j} - \delta_k)_+),$$

$$h_{k,1,j} h_{k,2,j} = h_{k,2,j},$$

$$h_{k,3,j}(\lambda) = h_{k,3,1}(\lambda_{k,j}\lambda)$$

for all $1 \leq j \leq k$, and

$$\max_{1 \leq j \leq k} \text{diam}(\text{osupp}(h_{k,1,j})) \to 0$$

as $k \to \infty$.

Let $\{h_l\}_{l=1}^{\infty}$ be a sequence of continuous functions from S^1 to $[0,1]$ such that for all l, there exists k, j such that $h_l = h_{k,1,j}$ and for all k, j, $h_{k,1,j}$ occurs infinitely many times as a term in the sequence $\{h_l\}_{l=1}^{\infty}$.

Let $\{r_n\}$ be a sequence of pairwise orthogonal projections in $\mathcal{B}$ such $r_m \sim r_n$ for all m, n, and

$$\sum_{n=1}^{\infty} r_n = 1_{\mathcal{M}(\mathcal{B})}$$

where the sum converges strictly.

Let $\{\epsilon_k\}$ be a decreasing sequence in $(0,1)$ and $\{\epsilon_{k,l}\}$ a (decreasing in $k+l$) biinfinite sequence in $(0,1)$ such that

$$\sum_{k=1}^{\infty} \epsilon_k < \infty$$

and

$$\sum_{1 \leq k,l < \infty} \epsilon_{k,l} < \infty.$$

Note that for every $\gamma > 0$, $Y \in \mathcal{M}(\mathcal{B})$ and $y \in \mathcal{B}$, there exists $N \geq 1$ such that

$$\|yYe_N^{\perp}\| < \gamma.$$

Also, by Lemma 4.2, for every $\gamma > 0$, Laurent polynomial $\widehat{h}$ and $L \geq 1$, there exists $N \geq 1$ so that

$$\widehat{h}\left(\sum_{j=1}^{\infty} \beta_j p_n V\right) e_N^{\perp} \approx_{\gamma} \widehat{h}\left(\sum_{j=1}^{\infty} \beta'_j p_n V\right) e_N^{\perp}$$

for all sequences $\{\beta_j\}$ and $\{\beta'_j\}$ in $\overline{B(0,1)}$ for which $\beta_j = \beta'_j$ for all $j \geq L$.

By using the above two principles and by repeatedly applying Lemma 4.5 (first to h_1; then to h_2; then to h_3; and so forth), we can find a sequence $\{x_k\}$ of pairwise orthogonal contractive elements of $\mathcal{B}$, a sequence $\{\alpha_k\}$ in S^1, and a sequence $\{\widehat{h}_k\}$ of Laurent polynomials such that the following statements hold:

1. $\sum_{k=1}^{\infty} x_k$ converges strictly in $\mathcal{M}(\mathcal{B})$.
2. For all $k \geq 1$,
$$\max_{\lambda \in S^1} |h_k(\lambda) - \widehat{h}_k(\lambda)| < \epsilon_k.$$
3. For all $k \geq 1$, there exists a subsequence $\{x_{j_l}\}$ of $\{x_j\}$ such that
   (a) $\left\| x_{j_l} \widehat{h}_k \left(\sum_{n=1}^{\infty} \alpha_n p_n V\right) x_{j_s}^* \right\| < \epsilon_{l,s}$ for all $l \neq s$, and
   (b) $x_{j_l} \widehat{h}_k \left(\sum_{n=1}^{\infty} \alpha_n p_n V\right) x_{j_l}^* \approx_{\epsilon_{j_l}} r_{j_l}$ for all l.

We denote the above statements by "(*)".

(Sketch of argument on how to choose the subsequence in (*) (3) above: Firstly, from the construction of the sequence, we already have part (3)(b). Next, note that, from Lemma 4.5, there is a sequence of pairwise orthogonal projections $\{s_j\}$ in $\mathcal{B}$ such that $\sum_{j=1}^{\infty} s_j$ converges strictly in $\mathcal{M}(\mathcal{B})$ and $x_j = r_j x_j s_j$ for all j. Now fix a k. The subsequence is constructed in two steps (a subsequence of a subsequence). Step 1: Let $\{j_i\}$ be a subsequence of the positive integers for which $h_{j_i} = h_k$ for all i. Step 2: Extract the subsequence of $\{j_i\}$ by observing that for all δ, for all $Y \in \mathcal{M}(\mathcal{B})$, for all i_1, there exists i_2 such that for all $i \geq i_2$, $\|x_{j_{i_1}} Y x_{j_i}^*\| < \delta$.)

Let $m \geq 1$ be given. We will now show that $h_m(\pi(\sum_{n=1}^{\infty} \alpha_n p_n)U)$ is full in $\mathcal{C}(\mathcal{B})$. Let $\epsilon > 0$. Since each term of the sequence $\{h_l\}_{l=1}^{\infty}$ is repeated infinitely many times there is some k for which $\epsilon_k < \frac{\epsilon}{2}$ and $h_k = h_m$.

Choose a subsequence $\{x_{j_l}\}$ of $\{x_j\}$ as in (3) of (*), corresponding to $\hat{h}_k$. Let $A \in \mathcal{M}(\mathcal{B})_+$ be a contractive element so that

$$\pi(A) = h_k(\pi(\sum_{n=1}^{\infty} \alpha_n p_n)U).$$

We can choose $L \geq 1$ great enough so that if we define

$$X =_{df} \sum_{l=L}^{\infty} x_{j_l}$$

then

$$X\widehat{h}_k \left(\sum_{n=1}^{\infty} \alpha_n p_n V \right) X^* \approx_{\epsilon_k} XAX^*.$$

Increasing L if necessary, we may assume that

$$\sum_{l \geq L} \epsilon_l + \sum_{m,n \geq L} \epsilon_{m,n} < \frac{\epsilon}{2}.$$

Consider the projection $R =_{df} \sum_{l \geq L} r_{j_l} \in \mathcal{M}(\mathcal{B})$ and note that $R \sim 1_{\mathcal{M}(\mathcal{B})}$ since $\sum_{n=1}^{\infty} r_n \sim 1$, and because all the projections r_n are equivalent. From (3) of (*),

$$X\widehat{h}_k \left(\sum_{n=1}^{\infty} \alpha_n p_n V \right) X^* \approx_{\delta} R$$

where

$$\delta =_{df} \sum_{l=L}^{\infty} \epsilon_{j_l} + \sum_{L \leq l,s < \infty} \epsilon_{l,s}.$$

Therefore,

$$\|XAX^* - R\| < \delta + \epsilon_k < \epsilon.$$

Since $R \sim 1_{\mathcal{M}(\mathcal{B})}$, there is some partial isometry W implementing the equivalence so that $WW^* = 1_{\mathcal{M}(\mathcal{B})}$ and $W^*W = R$. Then

$$\|WXAX^*W^* - 1_{\mathcal{M}(\mathcal{B})}\| \leq \|XAX^* - R\| < \epsilon.$$

Applying π, we obtain

$$\|\pi(WX)\pi(A)\pi(X^*W^*) - 1_{\mathcal{C}(\mathcal{B})}\| < \epsilon$$

Therefore, $\pi(A) = h_k(\pi(\sum_{n=1}^{\infty} \alpha_n p_n)U) = h_m(\pi(\sum_{n=1}^{\infty} \alpha_n p_n)U)$ is full in $\mathcal{C}(\mathcal{B})$.

Since $m \geq 1$ was arbitrary, and by the definition of the sequence $\{h_k\}$, we claim that $\pi(\sum_{n=1}^{\infty} \alpha_n p_n)U$ is a strongly full element of $\mathcal{C}(\mathcal{B})$.

To see this, note that every nonnegative continuous function $f \in C(S^1)$ has some h_l which is in the ideal generated by f. Indeed, there is some arc of positive width η centered at $s \in S^1$ on which f is greater than some $\zeta > 0$. Since $\max_{1 \leq j \leq k} \mathrm{diam}(\mathrm{osupp}(h_{k,1,j})) \to 0$, there is some k such that the maximum of these diameters is less than $\frac{\eta}{3}$. Moreover, since $\sum_{j=1}^{k} h_{k,3,j}$ is full in $C(S^1)$, there is some $1 \leq j \leq k$ such that $h_{k,1,j}(s) \neq 0$. Then, because $\mathrm{diam}(\mathrm{osupp}(h_{k,1,j})) < \frac{\eta}{3}$, the support of $h_{k,1,j}$ is entirely contained within the arc on which $f \geq \zeta > 0$. Therefore $h_{k,1,j}$ is in the ideal generated by f. Finally, by the definition of $\{h_l\}_{l=1}^{\infty}$, there is some l for which $h_l = h_{k,1,j}$ (in fact, there are infinitely many such l). $\square$

Recall that if $\mathcal{D}$ is a unital C*-algebra and $x \in \mathcal{C}$, then x is *strongly full* in $\mathcal{D}$ if every nonzero element of $C^*(x)$ is full in $\mathcal{D}$.

Our technical lemma follows immediately from Lemma 4.6.

Lemma 4.7. *Say, in addition, that $\mathcal{B}$ has strict comparison for positive elements and $T(\mathcal{B})$ has finitely many extreme points.*

Suppose also that $\mathcal{A}$ is a unital C-algebra and for all n,*

$$\phi_n : \mathcal{A} \to p_n \mathcal{B} p_n$$

*is a unital *-homomorphism.*

Let

$$\phi : \mathcal{A} \to \mathcal{M}(\mathcal{B})$$

*be the unital *-homomorphism given by*

$$\phi =_{df} \sum_{n=1}^{\infty} \phi_n$$

where the sum converges in the pointwise-strict topology.

Then for every unitary $W \in (\pi \circ \phi(\mathcal{A}))' \subseteq \mathcal{C}(\mathcal{B})$, there exists a unitary $W' \in (\pi \circ \phi(\mathcal{A}))'$ which is strongly full in $\mathcal{C}(\mathcal{B})$ such that

$$W \sim_h W'$$

in $\pi \circ \phi(\mathcal{A})'$.

Proof. This follows immediately from Lemma 4.6. Note that, by the definition of ϕ, any diagonal unitary of the form $\pi\left(\sum_{j=1}^{\infty} \alpha_j p_j\right)$ is a unitary in $(\pi \circ \phi(\mathcal{A}))'$ which is connected, via a norm continuous path of similar diagonal unitaries, to 1 in $(\pi \circ \phi(\mathcal{A}))'$. $\square$

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