Topology
Homework 2

Due Thursday, 7 February 2019

Problem 1. Show that there is an ambient isotopy of $\mathbb{R}^2$ which deforms any quadrilateral into any triangle. (Hint: it might be helpful to do this in three steps: deform any quadrilateral into a standard square, deform a standard square into an equilateral triangle, then deform any triangle into an equilateral triangle. The one for triangles is an example in the book.).

Problem 2. Let $\mathbb{S}^2$ be the standard 2-sphere in $\mathbb{R}^3$. Let $A = \mathbb{S}^2 \cup \{(0,0,z) \in \mathbb{R}^3 \mid 1 \le z \le 2\}$ be the sphere with a whisker, and let $B = \mathbb{S}^2 \cup \{(0,0,z) \in \mathbb{R}^3 \mid 0 \le z \le 1\}$.

(a) In what sense are A and B the same? Give a proof of your claim.

(b) In what sense are A and B different? Give a proof of your claim.

(c) Are A, B isotopic relative to $\mathbb{R}^4$ (considered as embedded in $\mathbb{R}^4$ with the 4th coordinate zero)? Give a sketch of your argument.

Problem 3. Let $A_0 = \{(0,y) \in \mathbb{R}^2 \mid 0 \le y \le 1\}$, and for $n \in \mathbb{N}$, let $A_n = \left\{\left(\frac{1}{n}, y\right) \,\middle|\, 0 \le y \le 1\right\}$. Determine the path components of $A_0 \cup A_1 \cup A_2 \cup \cdots$. You should prove that your path components are path connected, and that no points in different path components are path connected.

Problem 4. Find the path components of $\mathbb{R}^{n+1} \setminus \mathbb{S}^n$. Be sure to prove that these really are the path components. (Hint: it will help to consider an appropriate continuous function relating to the definition of $\mathbb{S}^n$. Also, beware, the case $n = 0$ is slightly different than the rest.)

Problem 5. Show that the interval $[0,1]$ is not homeomorphic to the square region $[0,1] \times [0,1]$.

Problem 6. Write the upper-case letters of the alphabet using line segments and arcs of circles (note: you need to actually write them out on your homework so I can see how you write the letters).

(a) Place the letters into groups in which all members are homeomorphic to each other.

(b) Are there any letter which are homeomorphic but not isotopic relative to $\mathbb{R}^2$?

Problem 7. The figure above shows a knotted arc spanning the region between two concentric spheres. Explain how to construct an ambient isotopy of the region between two spheres so that the endpoints of the arc do not move during the isotopy and so that the arc is deformed to a straight line segment between the spheres. Draw a series of pictures to accompany your explanation.

Problem 8. Recall the following two metrics d_2 and d_∞ defined on $\mathbb{R}^2$. For $x, y \in \mathbb{R}^2$, we have

$$d_2(x,y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2},$$

and

$$d_\infty = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

If you recall from class, we described how balls in the d_2 metric look like disks, and balls in the d_∞ metric look like squares. Prove that the metrics d_2 and d_∞ are equivalent metrics.