Topology
Homework 2

Due Thursday, 7 February 2019

Problem 1. (a) Show that a function $f : X \to Y$ is a surjection if and only if there is a function $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$.

(b) Show that a function $f : X \to Y$ with nonempty domain X is an injection if and only if there is a function $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$.

(c) Show that $f : X \to Y$ is a bijection if and only if f has an inverse function from Y to X.

(d) Consider a function $f : X \to Y$. Suppose $g : Y \to X$ is an inverse of the function f, and suppose $h : Y \to X$ is also an inverse of the function f. Prove that $g = h$.

Problem 2. Consider the reciprocal function $r : (0, 1) \to \mathbb{R}$ defined by $r(x) = \frac{1}{x}$.

(a) Where does the greatest stretching take place?

(b) Show that there is no upper bound for the stretching factor, i.e., that r is not Lipschitz. That is, for any $K > 0$, show that there are points $x, y \in (0, 1)$ with $|r(x) - r(y)| > K\,|x - y|$.

(c) Show that r is continuous (if you wish, you can cite a theorem, but if you are not sure exactly how to do it "manually", then you should figure that out; it is a good exercise).

Problem 3. The definition of continuity depends on the metric chosen for your space. When you choose to measure differently, you might get different notions of continuity. For example, consider the *discrete metric*, which can be defined on an arbitrary set X by

$$d_{disc}(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

In what follows, suppose that X has the discrete metric d_{disc}, and that Y is any metric space with metric d.

(a) Prove that d_{disc} is actually a metric.

(b) Prove that any function $f : X \to Y$ is continuous.

(c) Prove that if $f : \mathbb{R} \to X$ is continuous, then f is constant (here $\mathbb{R}$ is given the usual notion of distance).

Problem 4. (a) Show that any open interval (a, b) is homeomorphic to the open interval $(0, 1)$.

(b) Show that any open ray (a, ∞) or $(-\infty, b)$ is homeomorphic to the open interval $(0, 1)$.

(c) Show that the real line $\mathbb{R}$ is homeomorphic to the open interval $(0, 1)$.

(d) Conclude that any two open intervals of $\mathbb{R}$ are homeomorphic.

Problem 5. The pasting lemma will be useful here.

(a) Give a formula for a function $f : [-2, 2] \to [-2, 2]$ that stretches the interval $[-2, -1]$ linearly (i.e., constant stretching factor, can be implemented with a linear function) onto $[-2, 1]$ and shrinks the interval $[-1, 2]$ linearly onto $[1, 2]$.

(b) Show that f is a homeomorphism of $[-2, 2]$ onto itself.

(c) Show that f extends to a homeomorphism $F : \mathbb{R} \to \mathbb{R}$ by using the identity function outside the interval $(-2, 2)$.

Problem 6. Consider the picture below where $R = R_- \cup R_0 \cup R_+$.

(a) Give a formula for a function $f : R \to R$ that moves points horizontally mapping R_- onto $R_- \cup R_0$ and $R_0 \cup R_+$ onto R_+.

(b) Show that f is a homeomorphism of R onto itself.