Topology
Homework 1

Due Tuesday, 22 January 2019

Problem 1. Consider the three binary relations on $\mathbb{Z}$ defined by $a \leq b$, $a \sim b$ if and only if $ab \neq 0$, and $a \approx b$ if and only if $|a - b| < 1$.

(a) Determine which of these relations is reflexive and symmetric, but not transitive.

(b) Determine which of these relations is reflexive and transitive, but not symmetric.

(c) Determine which of these relations is symmetric and transitive, but not reflexive.

In each case, you should justify why the relation you chose does not have the stated property by providing a counterexample.

Problem 2. Suppose $\sim$ is an equivalence relation on a set X. For any $x \in X$, let $[x] := \{y \in X \mid y \sim x\}$ be the *equivalence class* represented by x.

(a) Show that equivalence classes are *pairwise disjoint*, i.e., if $[x] \cap [y] \neq \emptyset$, then $[x] = [y]$.

(b) Show that the union of the equivalence classes is X, i.e., $\bigcup_{x \in X} [x] = X$.

(c) Suppose that for each element α in some index set A that X_α is a nonempty subset of a set X. Suppose these sets are pairwise disjoint and their union is X. Define an equivalence relation on X so that X_α are the equivalence classes.

Problem 3. We saw in class how to define rational numbers as equivalence classes of ordered pairs of integers (with the second one nonzero). You will do a similar construction of integers from natural numbers in this problem. Consider the relation on $\mathbb{N}^2$ defined by $(a, b) \sim (b, d)$ if and only if $a + d = b + c$.

(a) Show that this is an equivalence relation. Be sure you give a proof that does not use the concept of negative numbers (hint: cancellation properties are different from negative numbers).

(b) Provide a bijection *phi* from the equivalence classes of this equivalence relation to the integers.

(c) Define an addition operation on the equivalence classes which corresponds to addition of the associated integers. That is, your addition should satisfy $\varphi(x + y) = \varphi(x) + phi(y)$ for any equivalence classes x, y; note that the addition symbol on the left means addition of equivalence classes, and the addition symbol on the right means addition of integers.

(d) Prove that the addition operation you defined in the previous problem is well-defined; in particular, that it does not depend on the choice of representative from your equivalence classes.

(e) Which equivalence classes corresponds to zero?

(f) What is the "negative" of an equivalence class?

Problem 4. Two sequences $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ are *eventually equal* if and only if there is a natural number N such that if $n \geq N$, then $a_n = b_n$. Show that this defines an equivalence relation on the set of sequences of real numbers.

Problem 5. Let $\mathbb{R}$ denote the set of all real numbers. A subset X of $\mathbb{R}$ has *measure zero* if and only if for every $\epsilon > 0$ there is a sequence of intervals (a_n, b_n) such that

$$X \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n) \quad \text{and} \quad \sum_{n=1}^{\infty} (b_n - a_n) < \epsilon.$$

Informally, a set has measure zero if it can be covered by intervals of arbitrarily small total length. Two functions $f, g : \mathbb{R} \to \mathbb{R}$ are *equal almost everywhere* if and only if the set $\{x \in \mathbb{R} \mid f(x) \neq g(x)\}$ has measure zero.

(a) Show that the union of two sets of measure zero has measure zero.

(b) Show that the relation of being equal almost everywhere is an equivalence relation on the set of functions $f : \mathbb{R} \to \mathbb{R}$.

NB: For the remaining problems, your solutions should be correct and convincing, but you don't need "rigorous proofs" in the usual sense. Here you need to provide the requested items along with some sort of reasoning about why they satisfy the desired properties.

Problem 6. Give both a geometric and an analytic description of a bijection from the Cartesian product $\mathcal{S}^1 \times [0, 1]$ to the annulus $\{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 4\}$. (Note: your descriptions should be describing the same bijection, although you do not need to prove this fact. The purpose of this problem is to see if you understand how to turn "proof by picture" into "proof by careful reasoning".)

Problem 7. Give a geometric description of a bijection between the lines in $\mathbb{R}^2$ through the origin and a subset of the unit circle.

Problem 8. Find a subset of $\mathbb{R}^2$ and a bijection from the subset to the congruence classes of rectangles. Which points correspond to squares?

Problem 9. Find a subset of $\mathbb{R}^2$ and a bijection from the subset to the similarity classes of triangles. Which points correspond to isosceles triangles? Which point corresponds to the class of equilateral triangles?

Problem 10. Define an equivalence relation on the closed interval $[0, 1]$ so that there is a simple bijection between the set of equivalence classes and the circle $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, and then describe this bijection.