Functional Analysis
Homework 5

Due Thursday, 29 February 2018

Problem 1. Consider the space $CS[0,1]$ of all $\mathbb{F}$-valued functions that are continuous except on a (possibly empty) finite set of points which varies from function to function, and for which all the left- and right-hand limits exist. Equivalently, this statement says that there is a partition of $[0,1]$ into a finite set of subintervals such that the function is uniformly continuous on each subinterval.

$$CS[0,1] = \{f : [0,1] \to \mathbb{F} \mid \exists t_0 < \cdots < t_n, f \text{ uniformly continuous on each } (t_i, t_{i+1})\}$$

You do not have to prove that these two definitions are equivalent, you may them interchangeably in your proof.

Let $F_0[0,1]$ denote the vector subspace of $\mathbb{F}$-valued functions which are zero everywhere except on a finite set which varies from function to function. Note that the Riemann integral is well-defined for any function in $CS[0,1]$ or in $F_0[0,1]$, and moreover if $f \in F_0[0,1]$, then $\int_0^1 f(x)\,dx = 0$, and similarly for $|f|$.

The focus of this problem is the vector space $X := CS[0,1]/F_0[0,1]$. Informally, X is the space of functions which are (uniformly) continuous on a finite partition of $[0,1]$ into subintervals, and for which the function values at the endpoints are undefined. This informal description is for intuition only, you cannot use it as a definition. Another informal way to think about it is that X consists of the sums of continuous functions with step functions, which you will prove in some sense below. In what follows, remember that X consists of equivalence classes (in particular, cosets of the form $f + F_0[0,1]$) of functions; any function in a given equivalence class is called a *representative* for that equivalence class.

(a) Explain why is doesn't make sense to talk about the value of an element of X at a point.

(b) Prove that any element of X has a unique representative function $f \in CS[0,1]$ which is the sum of a continuous function and a step function. (Note: technically, the values of the step function at each point of discontinuity are not uniquely determined, but the constant values it takes on each interval are determined.)

(c) Prove that the usual "L^p norm" actually defines a norm on X. More precisely, show that

$$\|f + F_0[0,1]\|_p := \left(\int_0^1 |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

is a norm on X.

(d) Prove that the natural map $C[0,1] \to X$ (that's not a typo, I really meant continuous functions) defined by $f \mapsto f + F_0[0,1]$ is a linear isometry when $C[0,1]$ is equipped with the L^p norm.

Note that this problem shows we may view X as an intermediate space between $C[0,1]$ and its completion $L^p[0,1]$, all equipped with the L^p norm. By the way, you probably won't see the notation $CS[0,1]$ anywhere outside this homework problem since I made it up, but the C is for *continuous* and the S is for *step*, because of the second part of this problem..

Problem 2. Consider the maps $T_1, \dots, T_5$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by

$$\begin{aligned}
(x, y) &\mapsto (x, 0)\\
(x, y) &\mapsto (y, -x)\\
(x, y) &\mapsto (y, x)\\
(x, y) &\mapsto (cx, cy)
\end{aligned}$$

where $c, d, x, y \in \mathbb{R}$. Show that each T_k is linear, and provide the geometric interpretation of each linear map.

Problem 3. Suppose that $T : X \to Y$ is a linear operator between normed spaces and let $Z = \ker T$. Consider the map $T_0 : X/Z \to Y$ defined by

$$T_0(x + \ker T) = Tx.$$

(a) Prove that T_0 is well-defined and linear.

(b) Prove that T_0 is bounded if T is bounded.