Functional Analysis
Homework 5

Due Thursday, 29 February 2018

Problem 1. Consider the space $CS[0,1]$ of all $\mathbb{F}$-valued functions that are continuous except on a (possibly empty) finite set of points which varies from function to function, and for which all the left- and right-hand limits exist. Equivalently, this statement says that there is a partition of $[0,1]$ into a finite set of subintervals such that the function is uniformly continuous on each subinterval.

$$CS[0,1] = \{f : [0,1] \to \mathbb{F} \mid \exists t_0 < \cdots < t_n, f \text{ uniformly continuous on each } (t_i, t_{i+1})\}$$

You do not have to prove that these two definitions are equivalent, you may them interchangeably in your proof.

Let $F_0[0,1]$ denote the vector subspace of $\mathbb{F}$-valued functions which are zero everywhere except on a finite set which varies from function to function. Note that the Riemann integral is well-defined for any function in $CS[0,1]$ or in $F_0[0,1]$, and moreover if $f \in F_0[0,1]$, then $\int_0^1 f(x)\,dx = 0$, and similarly for $|f|$.

The focus of this problem is the vector space $X := CS[0,1]/F_0[0,1]$. Informally, X is the space of functions which are (uniformly) continuous on a finite partition of $[0,1]$ into subintervals, and for which the function values at the endpoints are undefined. This informal description is for intuition only, you cannot use it as a definition. Another informal way to think about it is that X consists of the sums of continuous functions with step functions, which you will prove in some sense below. In what follows, remember that X consists of equivalence classes (in particular, cosets of the form $f + F_0[0,1]$) of functions; any function in a given equivalence class is called a *representative* for that equivalence class.

(a) Explain why is doesn't make sense to talk about the value of an element of X at a point.

(b) Prove that any element of X has a unique representative function $f \in CS[0,1]$ which is the sum of a continuous function and a step function. (Note: technically, the values of the step function at each point of discontinuity are not uniquely determined, but the constant values it takes on each interval are determined.)

(c) Prove that the usual "L^p norm" actually defines a norm on X. More precisely, show that

$$\|f + F_0[0,1]\|_p := \left(\int_0^1 |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

is a norm on X.

(d) Prove that the natural map $C[0,1] \to X$ (that's not a typo, I really meant continuous functions) defined by $f \mapsto f + F_0[0,1]$ is a linear isometry when $C[0,1]$ is equipped with the L^p norm.

Note that this problem shows we may view X as an intermediate space between $C[0,1]$ and its completion $L^p[0,1]$, all equipped with the L^p norm. By the way, you probably won't see the notation $CS[0,1]$ anywhere outside this homework problem since I made it up, but the C is for *continuous* and the S is for *step*, because of the second part of this problem..

Proof. (a) The representatives of $f + F_0[0,1] \in X$ do not share (collectively) the same value at any point in $[0,1]$. Indeed, let $h_{x,r} \in F_0[0,1]$ be the function which takes the nonzero real-value r when its argument is x, and is 0 otherwise. Then $f + h_{x,r}$ is a representative of $f + F_0[0,1]$, and yet $(f + h_{x,r})(x) - f(x) = r \neq 0$.

(b) Let $f \in CS[0,1]$. Then there are $0 = t_0 < \cdots < t_n = 1$ such that f is uniformly continuous on each subinterval (t_i, t_{i+1}) where $0 \leq i < n$. Then for $1 \leq i < n$, define $c_i := \lim_{x\to t_i^+} f(x) - \lim_{x\to t_i^-} f(x)$ and the limits are guaranteed to exists by the definition of $CS[0,1]$. For convenience set $c_0 := 0$. Now define a step function h on $[0,1]$ by

$$h(x) = \begin{cases} \sum_{i=0}^{k} c_i & \text{if } x \in (t_k, t_{k+1}] \\ 0 & \text{if } x = 0. \end{cases}$$

Then we claim that the left- and right-hand limits of $f - h$ agree at every point of $[0,1]$ (the endpoints don't matter). Indeed, if $x \in (t_k, t_{k+1})$, then $f - h$ is continuous at x, and so $\lim_{y\to x}(f-h)(y) = (f-h)(x)$. If $x = t_k$, then

$$\lim_{y\to t_k^-}(f-h)(y) = \lim_{y\to t_k^-} f(y) - \lim_{y\to t_k^-} h(y) = \lim_{y\to t_k^-} f(y) - \sum_{i=0}^{k-1} c_i,$$

and similarly,

$$\lim_{y\to t_k^+}(f-h)(y) = \lim_{y\to t_k^+} f(y) - \lim_{y\to t_k^+} h(y) = \lim_{y\to t_k^+} f(y) - \sum_{i=0}^{k} c_i.$$

Therefore,

$$\begin{aligned}
\lim_{y\to t_k^+}(f-h)(y) - \lim_{y\to t_k^-}(f-h)(y) &= \left(\lim_{y\to t_k^+} f(y) - \sum_{i=0}^{k} c_i\right) - \left(\lim_{y\to t_k^-} f(y) - \sum_{i=0}^{k-1} c_i\right) \\
&= \left(\lim_{y\to t_k^+} f(y) - \lim_{y\to t_k^-} f(y)\right) - \left(\sum_{i=0}^{k} c_i - \sum_{i=0}^{k-1} c_i\right) \\
&= \left(\lim_{y\to t_k^+} f(y) - \lim_{y\to t_k^-} f(y)\right) - c_k \\
&= 0.
\end{aligned}$$

Finally, define a function g by $g(x) := (f-h)(x) - \lim_{y\to x}(f-h)(y)$. By the above analysis, $g(x) = 0$ unless $x = t_k$ for some $0 \leq k \leq n$, and therefore $g \in F_0[0,1]$. Moreover, $k := f - h - g \in C[0,1]$ by construction (because $\lim_{y\to x} g(y) = 0$ for all $x \in [0,1]$). Thus $f = k + h + g$.

Now suppose that $f' \in CS[0,1]$ such that $f + F_0[0,1] = f' + F_0[0,1]$. We can perform the operation above to obtain $f' = k' + h' + g'$. Then $(k + h - k' - h') + (g - g') = f - f' \in F_0[0,1]$. Since $g - g' \in F_0[0,1]$. this implies $k + h - k' - h'$ is zero everywhere except the finitely many points t_k and t'_k. However, this means that $k - k' = h' - h$ away from these finitely many points. By the left-hand side is continuous and the right-hand side is a step function, and the only continuous step functions are the constant functions. Thus k and k' differ by a constant c, and h and h' differ by a constant function $-c$ (at least, aside from these finitely many points). So, technically, the continuous function and step functions are not unique, but only unique up to a constant factor. However, if we canonicalize in some way (like requiring the step functions to be zero at 0 and right-continuous), then this uniquely identifies the continuous function and the step function.

(c) Let $g \in F_0[0,1]$ be arbitrary. Then it is a basic fact that $\int_0^1 |g(x)|\, dx = 0$. Indeed, let n be the number of points at which $g(x) \neq 0$ and let $M = \max x \in [0,1]\, |g(x)|$. Then if $0 = s_0 < \cdots < s_m = 1$ is any partition of width δ, then the upper Riemann sum is bounded above by $\delta n M$, and the lower Riemann sum is bounded below by 0, thus as $\delta \to 0$, we find $\int_0^1 |g(x)|\, dx = 0$.

Let $t_0 < \cdots < t_n$ be the finitely many points at which $g(x) \neq 0$. Let $f \in CS[0,1]$. Then define $h(t_k) = |f(t_k) + g(t_k)|^p - |f(t_k)|^p$ and zero otherwise, so $h \in F_0[0,1]$. This yields $|f(x) + g(x)|^p = |f(x)|^p + h(x)$ for all $x \in [0,1]$. Therefore,

$$\int_0^1 |f(x) + g(x)|^p\, dx = \int_0^1 |f(x)|^p + h(x)\, dx = \int_0^1 |f(x)|^p\, dx + \int_0^1 h(x)\, dx = \int_0^1 |f(x)|^p\, dx.$$

Therefore $\|f + F_0[0,1]\|_p^p = \|(f+g) + F_0[0,1]\|_p^p$. Since $g \in F_0[0,1]$ was arbitrary, this is independent of the choice of representative.

Nonnegativity for this norm is trivial. to see that it is definitely, suppose that $f \in CS[0,1] \setminus F_0[0,1]$. Then there is some $x \in [0,1]$ such that f is continuous at x and $f(x) \neq 0$. Then there is some $\delta > 0$ for continuity at x corresponding to $\epsilon = \frac{|f(x)|}{2}$. Finally, for $y \in (x-\delta, x+\delta)$, we know that $abs f(y) \geq \frac{|f(x)|}{2}$.

$$\begin{aligned}
\|f + F_0[0,1]\|_p &= \left(\int_0^1 |f(y)|^p\, dy \right)^{1/p} \\
&\geq \left(\int_{x-\delta}^{x+\delta} |f(y)|^p\, dy \right)^{1/p} \\
&\geq \left(\int_{x-\delta}^{x+\delta} \frac{|f(x)|^p}{2^p}\, dy \right)^{1/p} \\
&= \left(\frac{\delta\, |f(x)|^p}{2^{p-1}} \right)^{1/p}.
\end{aligned}$$

which is strictly greater than zero since $f(x) \neq 0$.

Homeogeneity is relatively simple:

$$\begin{aligned}
\|c(f + F_0[0,1])\|_p &= \|(cf) + F_0[0,1]\|_p \\
&= \left(\int_0^1 |(cf)(x)|^p\, dx \right)^{1/p} \\
&= \left(|c|^p \int_0^1 |f(x)|^p\, dx \right)^{1/p} \\
&= |c| \left(\int_0^1 |f(x)|^p\, dx \right)^{1/p} \\
&= |c|\, \|f + F_0[0,1]\|_p
\end{aligned}$$

Finally, subadditivity follows immediately from Minkowski's inequality for integrals.

(d) Since $C[0,1]$ is a subspace of $CS[0,1]$, we just use the map $f \mapsto f + F_0[0,1]$. This map is certainly linear because $(cf) + F_0[0,1] = c(f + F_0[0,1])$ and $(f + F_0[0,1]) + (g + F_0[0,1]) = (f+g) + F_0[0,1]$ which we proved on a previous homework. Moreover, the previous item shows that this map is an isometry since $\|f + F_0[0,1]\|_p = \|f\|_p$.

■

Problem 2. Consider the maps $T_1, \dots, T_4$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by

$$\begin{aligned}
(x, y) &\mapsto (x, 0) \\
(x, y) &\mapsto (y, -x) \\
(x, y) &\mapsto (y, x) \\
(x, y) &\mapsto (cx, cy)
\end{aligned}$$

where $c, d, x, y \in \mathbb{R}$. Show that each T_k is linear, and provide the geometric interpretation of each linear map.

Proof. These maps T_i are linear because

$$\begin{aligned}
&T_1(k(x,y)+(v,w)) = T_1(kx+v, ky+w) = (kx+v, 0) = k(x,0)+(v,0) = kT_1(x,y)+T_1(v,w) \\
&T_2(k(x,y)+(v,w)) = T_2(kx+v, ky+w) = (ky+w, -kx-v) = k(y,-x)+(w,-v) = kT_2(x,y)+T_2(v,w) \\
&T_3(k(x,y)+(v,w)) = T_3(kx+v, ky+w) = (ky+w, kx+v) = k(y,x)+(w,v) = kT_3(x,y)+T_3(v,w) \\
&T_4(k(x,y)+(v,w)) = T_4(c(kx+v), c(ky+w)) = (kcx+cv, kcy+cw) = k(cx,cy)+(cv,cw) = kT_4(x,y)+T_4(v,w).
\end{aligned}$$

T_1 is the projection onto the horizontal coordinate. T_2 is counterclockwise rotation through an angle of $\frac{\pi}{2}$. T_3 is a reflection about the line $y = x$. T_4 is a dilation (scaling) by a factor of c. ■

Problem 3. Suppose that $T : X \to Y$ is a linear operator between normed spaces and let $Z = \ker T$. Consider the map $T_0 : X/Z \to Y$ defined by

$$T_0(x + \ker T) = Tx.$$

(a) Prove that T_0 is well-defined and linear.

(b) Prove that T_0 is bounded if T is bounded.

Proof. (a) We begin by showing T_0 is well-defined. For this, simply notice that if $x + \ker T = y + \ker T$, then $x - y \in \ker T$. Therefore

$$T_0(x + \ker T) = Tx = Tx + T(y - x) = T(x + y - x) = Ty = T_0(y + \ker T).$$

So the function value is independent of the choice of representative and so the function is well-defined. It is also linear because

$$T_0\big(c(x+\ker T)+(y+\ker T)\big) = T_0\big((cx+y)+\ker T\big) = T(cx+y) = cTx+Ty = cT_0(x+\ker T)+T_0(y+\ker T).$$

(b) Now suppose that T is bounded. Consider $x + \ker T$ and any $y \in \ker T$. Then we have

$$\|T_0(x + \ker T)\| = \|Tx\| = \|Tx + Ty\| = \|T(x + y)\| \le \|T\| \, \|x + y\| .$$

Taking the infimum over $y \in \ker T$, we find

$$\|T_0(x + \ker T)\| \le \|T\| \inf_{y \in \ker T} \|x + y\| = \|T\| \, \|x + \ker T\| .$$

Since $x + \ker T$ was arbitrary, this proves T_0 is bounded (and even that $\|T_0\| \le \|T\|$.

■