Functional Analysis
Homework 4

Due Tuesday, 13 February 2018

Problem 1. A sequence (e_n) of elements from a normed space X is said to be a *Schauder basis* if for every $x \in X$, there is a unique sequence (c_n) of scalars such that $\sum_{i=1}^{n} c_n e_n \to x$. Show that if a normed space has a Schauder basis, then it is separable.

Problem 2. Let X be a vector space and Y subspace of X. The *quotient* X/Y is the collection of cosets $\{x + Y \mid x \in X\}$.

(a) Define an addition operation on X/Y by

$$(x_1 + Y) + (x_2 + Y) := (x_1 + x_2) + Y.$$

Prove that this addition on X/Y is well-defined.

(b) Define a scalar multiplication operation on X/Y in the natural way. That is,

$$c(x + Y) := (cx) + Y.$$

Prove that this scalar multiplication on X/Y is well-defined.

Problem 3. Suppose that X is a normed space and Y is subspace of X.

(a) Prove that the function $\|\cdot\|_0 : X/Y \to \mathbb{R}$ defined by

$$\|x + Y\|_0 := \inf_{y \in Y} \|x + y\|,$$

is a pseudo-norm on X/Y.

(b) Show that $\|\cdot\|_0$ is a norm on X/Y if Y is closed.

(c) Prove that if X is a Banach space and Y is closed, then X/Y is a Banach space.

Problem 4. Give examples of subspaces of ℓ^∞ and ℓ^2 which are *not* closed.

Problem 5. Consider the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbb{R}^n$. We already know that these are equivalent norms because $\mathbb{R}^n$ is finite-dimensional. Prove the more explicit assertion that for all $x \in \mathbb{R}^n$,

$$\frac{1}{\sqrt{n}} \|x\|_1 \leq \|x\|_2 \leq \|x\|_1 .$$

Problem 6. Suppose X is a compact metric space and $C \subseteq X$ is closed. Prove that C is compact.