Functional Analysis
Homework 4

Due Tuesday, 13 February 2018

Problem 1. A sequence (e_n) of elements from a normed space X is said to be a *Schauder basis* if for every $x \in X$, there is a unique sequence (c_n) of scalars such that $\sum_{i=1}^{n} c_n e_n \to x$. Show that if a normed space has a Schauder basis, then it is separable.[1]

Proof. Suppose X is a normed space with a Schauder basis (e_n). Note that the scalar field $\mathbb{F}$ of X has a countable dense set which we will call $\mathbb{G}$ (for $\mathbb{R}$ it is $\mathbb{Q}$, and for $\mathbb{C}$ it is $\mathbb{Q} + i\mathbb{Q}$). Let Y be the collection of all finite linear combinations of elements of the Schauder basis with coefficients in $\mathbb{G}$; that is,

$$Y := \bigcup_{n \in \mathbb{N}} Y_n, \quad \text{where } Y_n := \left\{ \sum_{i=1}^{n} c_i e_i \;\middle|\; c_i \in \mathbb{G} \right\} = \operatorname{span}_{\mathbb{G}}\{e_1, \ldots, e_n\}.$$

Note that Y_n is countable because there is a natural bijective[2] map $\mathbb{G}^n \to Y_n$ given by $(c_1, \ldots, c_n) \to \sum_{i=1}^{n} c_i e_i$. Since Y is a countable union of countable sets, it is countable.

Now we claim that Y is dense in X. To this end, take any $x \in X$. Since (e_n) is a Schauder basis, there exists a sequence of scalars (c_n) from $\mathbb{F}$ such that $\sum_{i=1}^{n} c_i e_i \to x$. Therefore, given $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that

$$\left\| \sum_{i=1}^{N} c_i e_i - x \right\| < \frac{\varepsilon}{2}.$$

Moreover, since $\mathbb{G}$ is dense in $\mathbb{F}$, for each $1 \leq i \leq N$, there is some $a_i \in \mathbb{G}$ such that $|a_i - c_i| \leq \frac{\varepsilon}{2N\|e_i\|}$. Therefore,

$$\begin{aligned}
\left\| \sum_{i=1}^{N} a_i e_i - x \right\| &\leq \left\| \sum_{i=1}^{N} (a_i - c_i) e_i \right\| + \left\| \sum_{i=1}^{N} c_i e_i - x \right\| \\
&\leq \sum_{i=1}^{N} \big(|a_i - c_i| \cdot \|e_i\| \big) + \left\| \sum_{i=1}^{N} c_i e_i - x \right\| \\
&< \sum_{i=1}^{N} \left(\frac{\varepsilon}{2N\|e_i\|} \|e_i\| \right) + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary and $x \in X$ was arbitrary, this proves that Y is dense in X. Because Y is countable and dense, X is separable. ∎

[1] A natural question is whether every separable Banach space has a Schauder basis. Even though all of the standard examples of separable Banach spaces *do* have a Schauder basis, unfortunately, this question has a negative answer in general. This was a long-standing open problem until Enflo constructed a counterexample.

[2] All we really need is that this map is surjective, but elements of a Schauder basis are linearly independent (can you prove that?), so it is injective too.

Problem 2. Let X be a vector space and Y subspace of X. The *quotient* X/Y is the collection of cosets $\{x + Y \mid x \in X\}$.

(a) Define an addition operation on X/Y by

$$(x_1 + Y) + (x_2 + Y) := (x_1 + x_2) + Y.$$

Prove that this addition on X/Y is well-defined.

(b) Define a scalar multiplication operation on X/Y in the natural way. That is,

$$c(x + Y) := (cx) + Y.$$

Prove that this scalar multiplication on X/Y is well-defined.

Proof. Let X be a vector space and Y a subspace of X.

(a) We first prove a basic fact: if $x \in X$ and $y \in Y$, then $(x + y) + Y = x + Y$. The containment $(x+y)+Y \subseteq x+Y$ is immediate because Y is a subspace and so is closed under addition. The reverse containment follows from noting that $-y \in Y$ and $x + Y = (x + y - y) + Y \subseteq (x + y) + Y$.

Now, suppose $x_1, x_1', x_2 \in X$ with $x_1 + Y = x_1' + Y$. Then there is some $y_1 \in Y$ such that $x_1 + y_1 = x_1'$. Hence

$$(x_1' + Y) + (x_2 + Y) = (x_1' + x_2) + Y = (x_1 + y_1 + x_2) + Y = (x_1 + x_2) + Y = (x_1 + Y) + (x_2 + Y).$$

Therefore the addition operation is independent of the choice of representative in the first coordinate, and by the commutativity of addition in X, this addition operation is commutative and so it is also independent of the choice of representative in the second coordinate. Therefore addition is well-defined on X/Y.

(b) First notice that for any $x \in X$, $0(x + Y) = (0x) + Y = 0 + Y = Y$ which is independent of the choice of representative for the coset $x + Y$. So we may suppose $c \neq 0$.

Let $x, x' \in X$ with $x + Y = x' + Y$. Then there is some $y \in Y$ such that $x + y = x'$. Moreover,

$$c(x' + Y) = (cx') + Y = (cx + cy) + Y = (cx) + Y = c(x + Y),$$

where the penultimate equality follows from the fact that $cy \in Y$ since Y is a subspace and the fact we proved at the beginning of the previous part of the problem. Therefore, scalar multiplication is independent of the choice of representative, and so is well-defined.

Note that once we have shown these operations are well-defined, all the standard vector space properties for X/Y (existence of zero, additive inverses, distributitvity of scalar multiplication over addition, etc.) follow directly from the same properties of X. Thus X/Y is a vector space. ■

Problem 3. Suppose that X is a normed space and Y is subspace of X.

(a) Prove that the function $\|\cdot\|_0 : X/Y \to \mathbb{R}$ defined by

$$\|x + Y\|_0 := \inf_{y \in Y} \|x + y\|,$$

is a pseudo-norm on X/Y.

(b) Show that $\|\cdot\|_0$ is a norm on X/Y if Y is closed.

(c) Prove that if X is a Banach space and Y is closed, then X/Y is a Banach space.

Proof. Suppose that X is a normed space and Y is a subspace of X.

(a) To show that $\|\cdot\|_0$ is a pseudo-norm, we need to show that it is nonnegative, homogeneous and subadditive. Nonnegativity is immediate because it is an infimum of nonnegative quantities. Moreover, it is homogeneous because

$$\|0(x+Y)\|_0 = \|Y\|_0 = \inf_{y\in Y} \|y\| \le \|0\| = 0,$$

and if $c \neq 0$, then $y \in Y$ if and only if $cy \in Y$, and hence

$$\|c(x+Y)\|_0 = \inf_{y\in Y} \|cx+y\| = \inf_{cy\in Y} \|cx+cy\| = \inf_{cy\in Y} |c|\, \|x+y\| = |c| \inf_{y\in Y} \|x+y\| = |c|\, \|x+Y\|_0 .$$

To see that $\|\cdot\|_0$ is subadditive, let $y, y' \in Y$ and $x, x' \in X$. Then $y+y' \in Y$, and hence

$$\|(x+x')+Y\|_0 = \inf_{z\in Y} \|(x+x')+z\| \le \|(x+x')+(y+y')\| \le \|x+y\| + \|x'+y'\| .$$

Taking the infimum over $y \in Y$ we obtain

$$\|(x+x')+Y\|_0 \le \inf_{y\in Y} \left(\|x+y\| + \|x'+y'\| \right) = \left(\inf_{y\in Y} \|x+y\| \right) + \|x'+y'\|$$

Then taking the infimum over $y' \in Y$, we obtain

$$\|(x+x')+Y\|_0 \le \inf_{y\in Y} \|x+y\| + \inf_{y'\in Y} \|x'+y'\| = \|x+Y\|_0 + \|x'+Y\|_0 \qquad \blacksquare$$

(b) Recall that from a previous homework, we showed that if (X,d) is a metric space, $x \in X$ and $Y \subseteq X$, then $x \in \overline{Y}$ if and only if $D(x,Y) := \inf_{y\in Y} d(x,y) = 0$. Note that in our problem, Y is a subspace of X, and so $y \in Y$ if and only if $-y \in Y$, and therefore

$$\|x+Y\|_0 = \inf_{y\in Y} \|x+y\| = \inf_{y\in Y} \|x-(-y)\| = \inf_{y\in Y} \|x-y\| = D(x,Y).$$

Therefore, $\|x+Y\|_0 = 0$ if and only if $x \in \overline{Y}$.

If Y is closed, then $\|x+Y\|_0 = 0$ if and only if $x \in Y$ if and only if $x+Y = 0+Y$, which is the zero vector in X/Y. Therefore, if Y is a closed subspace of X, the $\|\cdot\|_0$ is a norm on X/Y.

Similarly, if Y is not closed, then there is some $x \in \overline{Y}\setminus Y$, and then $x+Y \neq 0+_Y$, and yet $\|x+Y\|_0 = 0$. Therefore, this pseudo-norm is a norm *if and only if* Y is closed.

(c) Suppose that X is a Banach space and Y is closed. Take any sequence of cosets (x_n+Y) which is Cauchy in X/Y. Then for each $k \in \mathbb{N}$, there is some n_k such that $\|(x_n - x_m)+Y\|_0 < 2^{-k}$ if $n, m \ge n_k$. Moreover, we can ensure that $n_{k+1} > n_k$. Then we choose $y_k \in Y$ inductively as follows. Let $y_1 = 0$, and then choose $y_{k+1} \in Y$ so that $\left\|(x_{n_{k+1}} - x_{n_k}) + y_{k+1}\right\| < 2^{-k}$. Therefore, the sequence defined by $z_1 = x_{n_1}$ and $z_{k+1} = x_{n_{k+1}} - x_{n_k} + y_{k+1}$ is absolutely summable. Since X is a Banach space, the series $\sum_{k=1}^{\infty} z_k$ converges to some element $x \in X$. Moreover, the partial sums are

$$\sum_{k=1}^{m} z_k = x_{n_m} + \sum_{k=1}^{m} y_k .$$

Therefore, the subsequence $(x_{n_k}+Y)$ converges to $x+Y$. Indeed,

$$\|(x_{n_k} - x)+Y\|_0 \le \left\| x_{n_k} - x + \sum_{i=1}^{k} y_i \right\| = \|z_k - x\| \to 0 \text{ as } k \to \infty.$$

Since (x_n+Y) is Cauchy and a subsequence converges to $x+Y$, the entire sequence converges to $x+Y$ as well.

Problem 4. Give examples of subspaces of ℓ^∞ and ℓ^2 which are *not* closed.

Proof. Let c_{00} denote the collection of sequences with *finite support* (i.e., for each $(x_n) \in c_{00}$, there is some $N \in \mathbb{N}$ such that if $n \geq N$, then $x_n = 0$). Note that the sum of sequences with finite support still has finite support, and similarly for scalar multiples. Therefore, c_{00} is a vector subspace, both of ℓ^∞ and ℓ^2.

I claim that the closure of c_{00} in the ℓ^∞ norm is c_0. Indeed, take any sequence $x = (x_n) \in c_0$. Then, for $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that $|x_n| < \varepsilon$ whenever $n > N$. Now consider the sequence $y \in c_{00}$ defined by $y = (x_1, \ldots, x_N, 0, 0, \ldots)$. Then

$$\|x - y\|_\infty = \sup_{n \in \mathbb{N}} |x_n - y_n| = \sup_{n > N} |x_n| \leq \varepsilon.$$

Therefore c_{00} is dense (in ℓ^∞ norm) in c_0. Moreover, c_0 is closed in ℓ^∞ norm[3]. Therefore, c_0 is the closure of c_{00} in ℓ^∞ norm (and obviously $c_{00} \neq c_0$ since $({}^1\!/\!{}_n) \in c_0 \setminus c_{00}$).

I also claim that c_{00} is dense (in ℓ^2 norm) in ℓ^2 itself. Indeed, let $x \in \ell^2$ and let $\varepsilon > 0$. Then since

$$\sum_{n=1}^{\infty} |x_n|^2 = \|x\|_2^2 < \infty,$$

there is some $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} |x_n|^2 < \varepsilon^2$. Let $y = (x_1, \ldots, x_N, 0, 0, \ldots) \in c_{00}$. Then

$$\|x - y\|_2^2 = \sum_{n=1}^{\infty} |x_n - y_n|^2 = \sum_{n=N+1}^{\infty} |x_n|^2 < \varepsilon^2,$$

and hence $\|x - y\|_2 < \varepsilon$. Therefore c_{00} is dense in ℓ^2 (and obviously $c_{00} \neq \ell^2$ since $({}^1\!/\!{}_n) \in \ell^2 \setminus c_{00}$). ■

Problem 5. Consider the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbb{R}^n$. We already know that these are equivalent norms because $\mathbb{R}^n$ is finite-dimensional. Prove the more explicit assertion that for all $x \in \mathbb{R}^n$,

$$\frac{1}{\sqrt{n}} \|x\|_1 \leq \|x\|_2 \leq \|x\|_1 .$$

Proof. Take any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Notice that we can apply Hölder's inequality[4] to obtain

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| = \sum_{i=1}^{n} |1 \cdot x_i| \leq \left(\sum_{i=1}^{n} |1|^2 \right)^{1/2} \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} = \sqrt{n} \|x\|_2 ,$$

which yields the first inequality.

The second inequality follows by induction from a straightforward calculus fact: for $t \geq 0$, $\sqrt{1+t^2} \leq 1+t$. The calculus fact follows from taking derivatives. Consider the functions $f(t) = \sqrt{1+t^2}$ and $g(t) = 1+t$. Then $f(0) = 1 = g(0)$ and $f'(t) = \frac{t}{\sqrt{1+t^2}} \leq \frac{t}{t} = 1 = g'(t)$, and hence $f(t) \leq g(t)$ for $t \geq 0$.

We will now prove by induction that $\|x\|_2 \leq \|x\|_1$. Notice that for $x \in \mathbb{R}$, it is trivially true that $|x|_2 \leq |x|_1$. So suppose that $n \in \mathbb{N}$ and for $x \in \mathbb{R}^n$, $\|x\|_2 \leq \|x\|_1$. Consider $y = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. If $x_{n+1} = 0$, the result follows immediately by induction because in this case

[3] An explicit proof of this fact appears at the end of the homework assignment.

[4] Actually, in the special case $p = q = 2$, Hölder's inequality is often referred to as the *Cauchy–Schwarz inequality*.

$\|y\|_2 = \|x\|_2 \leq \|x\|_1 = \|y\|_1$. so assume $x_{n+1} \neq 0$. In this case,

$$\begin{aligned}
\|y\|_2 &= \left(\sum_{i=1}^{n+1} |x_i|^2\right)^{1/2} \\
&= |x_{n+1}| \left(1 + \sum_{i=1}^{n} \left|\frac{x_i}{x_{n+1}}\right|^2\right)^{1/2} \\
&\leq |x_{n+1}| \left(1 + \left(\sum_{i=1}^{n} \left|\frac{x_i}{x_{n+1}}\right|^2\right)^{1/2}\right) \\
&= |x_{n+1}| + \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \\
&= |x_{n+1}| + \|x\|_2 \\
&\leq |x_{n+1}| + \|x\|_1 = \|y\|_1 .
\end{aligned}$$

Above, the first inequality follows from our calculus fact, and the second inequality is an application of the inductive hypothesis. By induction, we have proven the result. ∎

Remark. It is helpful to notice that each of the these inequalities is *sharp*[5]. Indeed, if $x = (1, \ldots, 1)$, then $\|x\|_1 = n$ and $\|x\|_2 = \sqrt{n}$, making the first inequality sharp. Similarly, if $x = (1, 0, \ldots, 0)$, then $\|x\|_1 = 1 = \|x\|_2$ making the second inequality sharp.

As a technical note, it can be shown that these are essentially the only two ways to make these inequalities sharp (i.e., a sequence where all the entries are equal in absolute value, and a sequence with only one nonzero value). In fact, one way to prove this technical note also yields *another* proof of this problem: Lagrange multipliers. The idea is this: consider the function $\|\cdot\|_1 : \mathbb{R}^n \to \mathbb{R}$ subject to the constraint $\|x\|_2^2 = 1$ (i.e., we are trying to find the maximum and minimum of the function $\|\cdot\|_1$ on the unit sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\}$.).

Now, we can't immediately apply Lagrange multipliers because although the constraint function $\|x\|_2^2 = 1$ is continuously differentiable, the function $\|\cdot\|_1$ is only differentiable in regions where the coordinates don't have sign changes. So, we actually have two constraints: $\|x\|_2^2 = 1$ and $x_i > 0$ for all $1 \leq i \leq n$. Applying Lagrange multipliers to this setting yields the maximum value of $\|x\|_1 = \sqrt{n}$ which occurs at $x = (1, \ldots, 1)$. Since there are no other critical points, we know that $\|x\|_1 < \sqrt{n}$ for all other vectors x subject to the constraints, and by continuity, we know that there are no maxima on the boundary region where at least one of the coordinates is zero.

Moreover, we know the minimum of $\|\cdot\|_1$ must occur on one of these boundary regions (where one of the coordinates is zero). Then, because each of these boundary regions looks exactly like the region we started with except in a smaller dimension, we can apply induction to conclude that the minimum of $\|\cdot\|_1$ must occur when all the coordinates are zero except one. In this case, the only option is that one of the coordinates is 1 and the rest are zero, which yields a minimum value of $\|x\|_1 = 1$ occurring at $x = (1, 0, \ldots, 0)$.

Problem 6. Suppose X is a compact metric space and $C \subseteq X$ is closed. Prove that C is compact.

Proof. Suppose X is a compact metric space and $C \subseteq X$ is closed. Let $\{V_\alpha \mid \alpha \in I\}$ be any open cover of C. Since C is closed, $X \setminus C$ is open. Notice that $\{X \setminus C\} \cup \{V_\alpha \mid \alpha \in I\}$ is an open cover of X. Indeed,

$$(X \setminus C) \cup \bigcup_{\alpha \in I} V_\alpha \supseteq (X \setminus C) \cup C = X.$$

[5] An inequality is said to be sharp if there is some choice of the variables which actually yields equality. In this sense, there is no "extra room" between the two quantities in general.

Therefore, since X is compact, this cover has a finite subcover $X \setminus C, V_1, \ldots, V_n$ (note: $X \setminus C$ is not *necessarily* part of any finite subcover, but it never hurts to throw it in). As such,

$$(X \setminus C) \cup \bigcup_{i=1}^{n} V_i = X,$$

and therefore $\bigcup_{i=1}^{n} V_i \supseteq C$. Hence $V_1, \ldots, V_n$ is a finite subcover (of our original open cover) of C, and thus C is compact. ■

Lemma. The vector space c_0 of sequences converging to zero is closed in the ℓ^∞ norm.

Proof. Let $x^k = (x_n^k) \in c_0$ be a sequence of sequences converging in ℓ^∞ norm to some $x \in \ell^\infty$. We must show that $x \in c_0$. Let $\varepsilon > 0$. Since $x^k \to x$ in ℓ^∞, there is some $N \in \mathbb{N}$ such that if $k \geq N$, then $\left\|x^k - x\right\|_\infty < \varepsilon$. Moreover, since $x^N \in c_0$, there is some $M \in \mathbb{N}$ such that if $n \geq M$, then $\left|x_n^N\right| < \varepsilon$. Therefore, for such $n \geq M$,

$$|x_n| \leq \left|x_n - x_n^N\right| + \left|x_n^N\right| < \left\|x - x^N\right\|_\infty + \varepsilon < 2\varepsilon.$$

Therefore $x \in c_0$. ■