Functional Analysis
Homework 3

Due Tuesday, 6 February 2018

Problem 1. Show that $C[0,1]$ and $C[a,b]$ are isometric (with the usual supremum metric on each space).

Problem 2. A map $f : X \to Y$ between metric spaces (X, d) and $(Y, \tilde{d})$ is said to be *uniformly continuous* if and only if for every $\epsilon > 0$ there is some $\delta > 0$ such that for all $x_1, x_2 \in X$, if $d(x_1, x_2) < \delta$ then $\tilde{d}\big(f(x_1), f(x_2)\big) < \epsilon$.

Suppose that Y is a complete metric space and $A \subseteq X$. Let $f : A \to Y$ be uniformly continuous (with the induced metric on A).

(a) Prove that if (a_n) is Cauchy in A, then $\big(f(a_n)\big)$ is Cauchy in Y.

(b) Prove that if $a \in \overline{A}$, then for any sequence (a_n) in A converging to a, the image sequence $\big(f(a_n)\big)$ converges and the limit is independent of the choice of the sequence (a_n).

(c) Prove that f extends uniquely to a continuous function $\bar{f} : \overline{A} \to Y$.

Problem 3. Suppose that X, Y are complete metric spaces, A is dense in X, and Y contains an isometric copy of A which is dense in Y. Prove that X and Y are isometric (hint: use the previous problem). This establishes that the completion of a metric space is unique.

Problem 4. A function $f : X \to Y$ between metric spaces (X, d) and $(Y, \tilde{d})$ is said to be *Lipschitz* (or *Lipschitz continuous*) if there exists an $K > 0$ such that $\tilde{d}\big(f(x_1), f(x_2)\big) \le K d(x_1, x_2)$ for all $x_1, x_2 \in X$.

(a) Show that Lipschitz functions are uniformly continuous.

(b) Give an example to show that not all uniformly continuous functions are Lipschitz.

(c) Prove that the composition of Lipschitz functions is Lipschitz.

Problem 5. A function $f : X \to Y$ between metric spaces (X, d) and $(Y, \tilde{d})$ is said to be *bilipschitz* if f is Lipschitz, injective and its inverse is Lipschitz. When f is bilipschitz and surjective, we say that X, Y are *bilipschitz equivalent*, which by Problem 4(c) is transitive (it is obviously reflexive and symmetric) and therefore and an equivalence relation on metric spaces.

(a) Prove that $f : X \to Y$ is bilipschitz if and only if there is some $K > 0$ with

$$\frac{1}{K} d(x_1, x_2) \le \tilde{d}\big(f(x_1), f(x_2)\big) \le K d(x_1, x_2).$$

Suppose that X, Y are bilipschitz equivalent[1].

(b) Prove that U is open in X if and only if $f(U)$ is open in Y[2].

(c) Conclude that if X is complete, then Y is complete.

(d) Prove that if X is bounded, then Y is bounded.

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[1] Isometric spaces are completely indistinguishable as metric spaces, but in many important ways, so are metric spaces which are only bilipschitz equivalent.

[2] A bijective function with this property is called a *homeomorphism*; in other words, a bijective continuous function whose inverse is continuous. A homeomorphism between X, Y indicates that X and Y are indistinguishable *topologically*, since topologies are completely specified by their open sets.