Functional Analysis
Homework 3 Solutions

Due Tuesday, 6 February 2018

Problem 1. Show that $C[0,1]$ and $C[a,b]$ are isometric (with the usual supremum metric on each space).

Proof. Consider the map $T : C[a,b] \to C[0,1]$ defined by

$$(Tf)(x) = f\big((b-a)x + a\big).$$

First, notice that $x \in [0,1]$ if and only if $(b-a)x + a \in [a,b]$, and therefore Tf is a well-defined function on $[0,1]$. Moreover, Tf is continuous because it is a composition of continuous functions. Therefore, T is well-defined from $C[a,b]$ to $C[0,1]$.

We now claim that T is an isometry. Note that from the definition of T it follows that $T(cf + g) = c(Tf) + (Tg)$ for any $f, g \in C[a,b]$ and $c \in \mathbb{F}$, i.e., T is *linear*. Because of this it suffices to show that T is *norm-preserving*, i.e., $\|Tf\|_\infty = \|f\|_\infty$ for any $f \in C[a,b]$. Indeed, there is some $y \in [0,1]$ which attains the norm of Tf, and so

$$\|Tf\|_\infty = \max_{x\in[0,1]} \big|f\big((b-a)x + a\big)\big| = \big|f\big((b-a)y + a\big)\big| \le \max_{x\in[a,b]} |f(x)| = \|f\|_\infty .$$

Note that T^{-1} is a map of the same form as T, so the inequality is reversed as well. In fact, this shows that y attains the norm of Tf if and only if $(b-a)y + a$ attains the norm of f.

Since T is norm-preserving and linear, it is an isometry because

$$d(f,g) = \|f - g\|_\infty = \|T(f-g)\|_\infty = \|Tf - Tg\|_\infty .$$

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Problem 2. A map $f : X \to Y$ between metric spaces (X,d) and $(Y,\tilde{d})$ is said to be *uniformly continuous* if and only if for every $\varepsilon > 0$ there is some $\delta > 0$ such that for all $x_1, x_2 \in X$, if $d(x_1,x_2) < \delta$ then $\tilde{d}\big(f(x_1), f(x_2)\big) < \varepsilon$.

Suppose that Y is a complete metric space and $A \subseteq X$. Let $f : A \to Y$ be uniformly continuous (with the induced metric on A).

(a) Prove that if (a_n) is Cauchy in A, then $\big(f(a_n)\big)$ is Cauchy in Y.

(b) Prove that if $a \in \overline{A}$, then for any sequence (a_n) in A converging to a, the image sequence $\big(f(a_n)\big)$ converges and the limit is independent of the choice of the sequence (a_n).

(c) Prove that f extends uniquely to a continuous function $\bar{f} : \overline{A} \to Y$.

Proof. Suppose that Y is a complete metric space and $A \subseteq X$. Let $f : A \to Y$ be uniformly continuous (with the induced metric on A).

(a) Suppose that (a_n) is Cauchy in A. Let $\varepsilon > 0$. By the uniform continuity of f, there is some $\delta > 0$ such that for any $x_1, x_2 \in A$, if $d(x_1,x_2) < \delta$, then $\tilde{d}\big(f(x_1), f(x_2)\big) < \varepsilon$. Since (a_n) is Cauchy, there is some $N \in \mathbb{N}$ such that for all $n, m \ge N$, $d(a_n, a_m) < \delta$, and hence $\tilde{d}\big(f(a_n), f(a_m)\big)$ is Cauchy in Y.

(b) Let $a \in \overline{A}$. Then there is a sequence (a_n) in A converging to a. Since convergent sequences are Cauchy, (a_n) is Cauchy. By the previous part, $\big(f(a_n)\big)$ is Cauchy in Y Since Y is complete, this sequence converges to some $y \in Y$.

Now suppose that (a'_n) is some other sequence in A converging to a. Then we claim that $d(a_n, a'_n) \to 0$. Indeed, notice that

$$d(a_n, a'_n) \leq d(a_n, a) + d(a, a'_n),$$

and these latter two sequences each converge to zero. By the uniform continuity of f, we can again conclude that $\tilde{d}\big(f(a_n), f(a'_n)\big) \to 0$. Indeed, once $d(a_n, a'_n) < \delta$, we automatically have $\tilde{d}\big(f(a_n), f(a'_n)\big) < \varepsilon$. Finally,

$$\tilde{d}\big(f(a'_n), y\big) \leq \tilde{d}\big(f(a'_n), f(a_n)\big) + \tilde{d}\big(f(a_n), y\big) \to 0,$$

and therefore $\big(f(a'_n)\big)$ converges to y as well. Thus the limit is independent of the choice of the sequence from A converging to a.

(c) Define a function $\overline{f} : \overline{A} \to Y$ by

$$\overline{f}(x) = \begin{cases} f(x) & x \in A \\ \lim_{n\to\infty} f(a_n) & x \in \overline{A} \setminus A, a_n \in A, a_n \to x. \end{cases} \tag{1}$$

This functions is well-defined by the previous part (the limit exists and it does not depend on the choice of the sequence). It remains to show that $\overline{f}$ is (uniformly) continuous. Let $\varepsilon > 0$. Since f is uniformly continuous on A, there is some δ associated to $\frac{\varepsilon}{3}$ coming from uniform continuity. Suppose that $x_1, x_2 \in \overline{A}$ and $d(x_1, x_2) < \frac{\delta}{3}$. By the definition of $\overline{f}$, there exists $a_1, a_2 \in A$ such that $d(a_i, x_i) < \frac{\delta}{3}$ and $\tilde{d}\big(f(a_i), \overline{f}(x_i)\big) < \frac{\varepsilon}{3}$ for $i = 1, 2$ (note: if $x_i \in A$, simply choose $a_i = x_i$). Then

$$d(a_1, a_2) \leq d(a_1, x_1) + d(x_1, x_2) + d(x_2, a_2) < \delta,$$

and therefore

$$\tilde{d}\big(\overline{f}(x_1), \overline{f}(x_2)\big) \leq \tilde{d}\big(\overline{f}(x_1), f(a_1)\big) + \tilde{d}\big(f(a_1), f(a_2)\big) + \tilde{d}\big(f(a_1), \overline{f}(x_2)\big) < \varepsilon.$$

Thus $\overline{f}$ is uniformly continuous.

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Problem 3. Suppose that X, Y are complete metric spaces, A is dense in X, and Y contains an isometric copy of A which is dense in Y. Prove that X and Y are isometric (hint: use the previous problem). This establishes that the completion of a metric space is unique.

Proof. Suppose that X, Y are complete metric spaces, A is dense in X, and Y contains an isometric copy of A which is dense in Y. That is, there is an isometry $i : A \to Y$ with $i(A)$ dense in Y. Note that an isometry is always uniformly continuous (just choose $\delta = \varepsilon$). Indeed, if $\varepsilon > 0$, then when $a_1, a_2 \in A$ with $d(a_1, a_2) < \varepsilon$, then

$$\tilde{d}\big(i(a_1), i(a_2)\big) = d(a_1, a_2) < \varepsilon.$$

By the previous problem, i extends to a (uniformly) continuous function $\bar{i}$ from $\overline{A}$ to Y, but since A is dense in X, $\overline{A} = X$. Thus $\bar{i} : X \to Y$.

We claim that $\bar{i}$ is an isometry. Indeed, let $x, x' \in X$. Then there are sequences (perhaps constant sequences in the case of $x, x' \in A$) $(a_n), (a'_n)$ in A converging to x, x', respectively. Thus $i(a_n) \to \bar{i}(x)$ and $i(a'_n) \to \bar{i}(x')$, respectively. Therefore

$$\tilde{d}\big(\bar{i}(x), \bar{i}(x')\big) = \lim_{n\to\infty} \tilde{d}\big(i(a_n), i(a'_n)\big) = \lim_{n\to\infty} d(a_n, a'_n) = d(x, x').$$

So, $\bar{i}$ is an isometry.

Finally, we claim that $\bar{i}$ is surjective. Notice that $\bar{i}(X)$ is a complete subset of Y. Thus $\bar{i}(X)$ is closed. Moreover, $\bar{i}(X)$ contains $i(A)$ which is dense in Y. Putting this together we find that $\bar{i}(X) \supseteq \overline{i(A)} = Y$, so $\bar{i}$ is surjective, and hence X, Y are isometric. ■

Problem 4. A function $f : X \to Y$ between metric spaces (X, d) and $(Y, \tilde{d})$ is said to be *Lipschitz* (or *Lipschitz continuous*) if there exists an $K > 0$ such that $\tilde{d}\big(f(x_1), f(x_2)\big) \le K d(x_1, x_2)$ for all $x_1, x_2 \in X$.

(a) Show that Lipschitz functions are uniformly continuous.

(b) Give an example to show that not all uniformly continuous functions are Lipschitz.

(c) Prove that the composition of Lipschitz functions is Lipschitz.

Proof. (a) Suppose that the map $f : X \to Y$ between metric spaces (X, d) and $(Y, \tilde{d})$ is Lipschitz with Lipschitz constant $K > 0$. Let $\varepsilon > 0$ and notice that whenever $x_1, x_2 \in X$ with $d(x_1, x_2) < \frac{\varepsilon}{K}$,

$$\tilde{d}\big(f(x_1), f(x_2)\big) \le K d(x_1, x_2) < \varepsilon.$$

Therefore f is uniformly continuous.

(b) Consider the function $\sqrt{\cdot} : [0, 1] \to [0, 1]$. This function is continuous (because it is the inverse of the continuous function $x \mapsto x^2$ defined on the *interval* $[0, 1]$), and since $[0, 1]$ is a closed and bounded set (more importantly, it is *compact*), $\sqrt{\cdot}$ is uniformly continuous.

However, it is not Lipschitz. Indeed, notice that if $x > 0$, then

$$\frac{\big|\sqrt{x} - \sqrt{0}\big|}{|x - 0|} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}.$$

As $x \to 0^+$, this quotient is unbounded, and therefore $\sqrt{\cdot}$ is not Lipschitz.

(c) Suppose that $f : X_1 \to X_2$ and $g : X_2 \to X_3$ are Lipschitz functions with associated constants K_f, K_g. Then, for any $a, b \in X_1$, we find

$$d_3\big((g \circ f)(a), (g \circ f)(b)\big) = d_3\Big(g\big(f(a)\big), g\big(f(b)\big)\Big) \le K_g d_2\big(f(a), f(b)\big) \le K_g K_f d_1(a, b).$$

So, $g \circ f$ is Lipschitz with constant $K_g K_f$.

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Problem 5. A function $f : X \to Y$ between metric spaces (X, d) and $(Y, \tilde{d})$ is said to be *bilipschitz* if f is Lipschitz, injective and its inverse is Lipschitz. When f is bilipschitz and surjective, we say that X, Y are *bilipschitz equivalent*, which by Problem 4(c) is transitive (it is obviously reflexive and symmetric) and therefore and an equivalence relation on metric spaces.

(a) Prove that $f : X \to Y$ is bilipschitz if and only if there is some $K > 0$ with

$$\frac{1}{K} d(x_1, x_2) \le \tilde{d}\big(f(x_1), f(x_2)\big) \le K d(x_1, x_2).$$

Suppose that X, Y are bilipschitz equivalent[1].

(b) Prove that U is open in X if and only if $f(U)$ is open in Y[2].

(c) Conclude that if X is complete, then Y is complete.

(d) Prove that if X is bounded, then Y is bounded.

[1] Isometric spaces are completely indistinguishable as metric spaces, but in many important ways, so are metric spaces which are only bilipschitz equivalent.

[2] A bijective function with this property is called a *homeomorphism*; in other words, a bijective continuous function whose inverse is continuous. A homeomorphism between X, Y indicates that X and Y are indistinguishable *topologically*, since topologies are completely specified by their open sets.

Proof. (a) Suppose that $f : X \to Y$ is bilipschitz. Thus f is Lipschitz with constant K_1 f^{-1} is Lipschitz with constant K_2. Let $K = \max\{K_1, K_2\}$. Then, for any $x_1, x_2 \in X$, we have

$$\tilde{d}\big(f(x_1), f(x_2)\big) \le K_1 d(x_1, x_2) \le K d(x_1, x_2).$$

Similarly,

$$d(x_1, x_2) = d\big(f^{-1}(f(x_1)), f^{-1}(f(x_2))\big) \le K_2 \tilde{d}\big(f(x_1), f(x_2)\big) \le K \tilde{d}\big(f(x_1), f(x_2)\big).$$

Dividing the this latter display by K and combining the two displays, we obtain

$$\frac{1}{K} d(x_1, x_2) \le \tilde{d}\big(f(x_1), f(x_2)\big) \le K d(x_1, x_2).$$

For the remainder of the problem, we assume that X, Y are bilipschitz equivalent (i.e., f is surjective).

(a) Since f and f^{-1} are both Lipschitz, they are both (uniformly) continuous by the previous problem. Therefore, since f^{-1} is continuous, if U is open in X, then $(f^{-1})^{-1}(U) = f(U)$ is open in Y. Similarly, if $f(U)$ is open in Y, then since f is continuous, $f^{-1}(f(U)) = U$ is open in X.

(b) Suppose that X is complete and let (y_n) be a Cauchy sequence in Y. Given $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that if $n, m \ge N$, then $\tilde{d}(y_n, y_m) < \frac{\varepsilon}{K}$. Thus

$$d\big(f^{-1}(y_n), f^{-1}(y_m)\big) \le K \tilde{d}(y_n, y_m) < \varepsilon.$$

Therefore $(x_n) := \big(f^{-1}(y_n)\big)$ is Cauchy in X, and since X is complete, this converges to some $x \in X$. Since f is continuous $f(x_n) = f(f^{-1}(y_n)) = y_n$ converges to $f(x) \in Y$. Therefore Y is complete.

(c) Suppose that X is bounded, so that $\sup_{x,x' \in X} d(x, x') = M < \infty$. Since f is surjective, $f(X) = Y$. Therefore,

$$\sup_{y,y' \in Y} \tilde{d}(y, y') = \sup_{x,x' \in X} \tilde{d}\big(f(x), f(x')\big) \le \sup_{x,x' \in X} K d(x, x') = MK < \infty.$$

Therefore Y is bounded.

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