Functional Analysis
Homework 2

Due Tuesday, 29 January 2018

Problem 1. Let A be a nonempty subset of a metric space (X, d). Let $D(x, A) := \inf_{a \in A} d(x, a)$ (This D is not a metric). Prove that $D(x, A) = 0$ if and only if $x \in \overline{A}$.

Problem 2. Prove that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ and that equality does not hold in general. You should ask yourself (and figure out) what happens for unions, but you don't need to include it in your solutions.

Problem 3. Let (X, d) be a metric space and $A \subseteq X$. Prove that $X \setminus \mathrm{int}(A) = \overline{X \setminus A}$

Problem 4. A point $x \in X$ is a *boundary point* of a set A in a metric space (X, d) if every neighborhood of x (equivalently, every open ball centered at x) intersects both A and $X \setminus A$. The set of boundary points of A is called the *boundary* of A and is denoted ∂A.

(a) Write ∂A as an intersection of two sets.

(b) Describe (with proof) $\partial \mathbb{Q}$ in $\mathbb{R}$ with the standard Euclidean metric.

Problem 5. A point x in a metric space (X, d) is said to be *isolated* if $\{x\}$ is a neighborhood of x. A metric space is *discrete*[1] if every point is isolated. Let X be a discrete metric space.

(a) Prove that the singletons are open in X.

(b) Conclude that every subset of X is clopen.

(c) Explain in one sentence why if Y is any metric space and $f : X \to Y$ is any function, then f is continuous.

Problem 6. Let $C[a, b]$ have the usual supremum metric $d(f, g) = \max x \in [a, b]\, |f(x) - g(x)|$. Prove that a sequence of functions $f_n \in C[a, b]$ converges to $f \in C[a, b]$ in the metric d if and only if f_n converges to f uniformly. For this reason, the supremum metric is sometimes called the *uniform metric*.

Problem 7. Prove that if a subsequence of a Cauchy sequence converges, then the entire sequence converges.

Problem 8. Give an example to show that the image of an open (respectively, closed, bounded) set under a continuous map is not necessarily open (resp., closed, bounded). Unless your examples are highly nontrivial, you do not need to prove that the functions you provide are continuous or that the sets you describe are open (resp., closed, bounded).

Problem 9. Consider $C[0, 1]$ with the L^1 metric $d(f, g) = \int_0^1 |f - g|$. If $f_n, f \in C[0, 1]$ and f_n converges to f in the metric d, does that imply that f_n converges pointwise to f? If it does, prove it; if not, find a counterexample. (Hint: because of the homogeneity in the metric d, it suffices to consider the case where f is the zero function.)

[1] This is different than saying that d is the discrete metric. For example, $\mathbb{Z}$ with the metric induced by the standard Euclidean metric is a discrete metric space, but this is not the discrete metric on $\mathbb{Z}$.