Functional Analysis
Homework 2 Solutions

Due Tuesday, 29 January 2018

Problem 1. Let A be a nonempty subset of a metric space (X, d). Let $D(x, A) := \inf_{a \in A} d(x, a)$ (This D is not a metric). Prove that $D(x, A) = 0$ if and only if $x \in \overline{A}$.

Proof. Suppose that $D(x, A) = 0$. Let U be any neighborhood of x. So there is some $r > 0$ with $B(x; r) \subseteq U$. Then since $D(x, A) = 0$, there is some $a \in A$ such that $d(x, a) < r$. Therefore $a \in A \cap B(x; r)$, and so this set is nonempty. Hence $x \in \overline{A}$.

Conversely, suppose that $x \in \overline{A}$. Then for any $n \in \mathbb{N}$, there is some $x_n \in A \cap B(x; \frac{1}{n})$. Thus $D(x, A) = \inf_{a \in A} d(x, a) \leq d(x, x_n) < \frac{1}{n}$ for each $n \in \mathbb{N}$. Hence $D(x, A) = 0$. ■

Problem 2. Prove that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ and that equality does not hold in general. You should ask yourself (and figure out) what happens for unions, but you don't need to include it in your solutions.

Problem 3. Suppose $x \in \overline{A \cap B}$. Then any neighborhood U of x intersects $A \cap B$, and therefore it intersects both A and B. Thus $x \in \overline{A}$ and $x \in \overline{B}$ since U was arbitrary. Therefore $x \in \overline{A} \cap \overline{B}$.

Equality does not hold in general. Consider $A = (-\infty, 0)$ and $B = (0, \infty)$ in $\mathbb{R}$ equipped with the Euclidean metric. Then it is clear that $\overline{A} = (-\infty, 0]$ and similarly, $\overline{B} = [0, \infty)$. Hence $\overline{A} \cap \overline{B} = \{0\}$. On the other hand, $\overline{A \cap B} = \overline{\emptyset} = \emptyset$.

Problem 4. Let (X, d) be a metric space and $A \subseteq X$. Prove that $X \setminus \operatorname{int}(A) = \overline{X \setminus A}$

Proof. Since $\operatorname{int}(A)$ is an open set contained in A, $X \setminus \operatorname{int}(A)$ is a closed set containing $X \setminus A$. Therefore $X \setminus \operatorname{int}(A) \supseteq \overline{X \setminus A}$.

Since $\operatorname{int}(A)$ is the *largest* open set contained in A, we know that for any $x \in X \setminus \operatorname{int}(A)$ and any $r > 0$, $B(x; r) \cap (X \setminus A) \neq \emptyset$ (indeed, if this were empty, then $B(x; r) \subseteq A$ and so $\operatorname{int}(A) \cup B(x; r)$ would be a larger open set contained in A). Therefore, since $r > 0$ was arbitrary, $x \in \overline{X \setminus A}$, and hence $X \setminus \operatorname{int}(A) = \overline{X \setminus A}$. ■

Problem 5. A point $x \in X$ is a *boundary point* of a set A in a metric space (X, d) if every neighborhood of x (equivalently, every open ball centered at x) intersects both A and $X \setminus A$. The set of boundary points of A is called the *boundary* of A and is denoted ∂A.

(a) Write ∂A as an intersection of two sets.

(b) Describe (with proof) $\partial \mathbb{Q}$ in $\mathbb{R}$ with the standard Euclidean metric.

Proof. (a) Claim: $\partial A = \overline{A} \cap \overline{X \setminus A}$, and this is essentially just definition pushing. Take any element $x \in \partial A$. Then any neighborhood U of x intersects both A and $X \setminus A$. Since U was arbitrary, $x \in \overline{A}$ and $x \in \overline{X \setminus A}$.

Conversely, if $x \in \overline{A} \cap \overline{X \setminus A}$, then for any neighborhood U of x, U intersects both A and $X \setminus A$. Hence $x \in \partial A$, proving the claim.

Notice that from an earlier question, we can also write $\partial A = X \setminus \big(\operatorname{int}(A) \cup \operatorname{int}(X \setminus A)\big)$.

(b) Notice that $\mathbb{Q}$ does not contain any open intervals in $\mathbb{R}$ (because between any two rationals there is an irrational). Therefore no point of $\mathbb{R}$ is an interior point of $\mathbb{Q}$, and thus $\text{int}(\mathbb{Q}) = \emptyset$.

Similarly, $\mathbb{R} \setminus \mathbb{Q}$ does not contain any open intervals in $\mathbb{R}$ (because between any two irrational numbers there is a rational number). Therefore no point of $\mathbb{R}$ is an interior point of $\mathbb{R} \setminus \mathbb{Q}$, and therefore $\text{int}(\mathbb{R} \setminus \mathbb{Q}) = \emptyset$. By the previous problem, we find that $\partial\mathbb{Q} = \mathbb{R} \setminus \big(\text{int}(\mathbb{Q}) \cup \text{int}(\mathbb{R} \setminus \mathbb{Q})\big) = \mathbb{R}$.

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Problem 6. A point x in a metric space (X, d) is said to be *isolated* if $\{x\}$ is a neighborhood of x. A metric space is *discrete*[1] if every point is isolated. Let X be a discrete metric space.

(a) Prove that the singletons are open in X.

(b) Conclude that every subset of X is clopen.

(c) Explain in one sentence why if Y is any metric space and $f : X \to Y$ is any function, then f is continuous.

Proof. (a) Take any $x \in X$. Since X is discrete, x is isolated and so $\{x\}$ is a neighborhood of x. Therefore, there is some $r > 0$ such that $B(x; r) \subseteq \{x\}$ and we necessarily get equality here. Therefore, $\{x\}$ is open.

(b) Note that any set $A = \bigcup_{a \in A}\{a\}$ is a union of open sets and is therefore open. Since any set is open, the complement of any set is open, and therefore any set is closed.

(c) For any open set U in Y, $f^{-1}(U)$ is open by the previous item, and therefore f is continuous.

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Problem 7. Let $C[a, b]$ have the usual supremum metric $d(f, g) = \max x \in [a, b]\, |f(x) - g(x)|$. Prove that a sequence of functions $f_n \in C[a, b]$ converges to $f \in C[a, b]$ in the metric d if and only if f_n converges to f uniformly. For this reason, the supremum metric is sometimes called the *uniform metric*.

Proof. Suppose that $f_n \to f$ in the metric d. Then, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $d(f_n, f) < \epsilon$. Therefore, for any $x \in [a, b]$,

$$|f_n(x) - f(x)| \leq \max_{y \in [a,b]} |f_n(y) - f(y)| = d(f_n, f) < \epsilon.$$

Since this N is independent of x, f_n converges uniformly to f.

Conversely, suppose that f_n converges to f uniformly. Then for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ and any $x \in [a, b]$, $|f_n(x) - f(x)| < \epsilon$. Taking the supremum over all $x \in [a, b]$ (which is a maximum by the Extreme Value Theorem since $f_n - f$ is continuous), we find

$$d(f_n, f) = \max_{x \in [a,b]} |f_n(x) - f(x)| \leq \epsilon.$$

(By the way, if you are worried that we got $\leq \epsilon$ instead of $< \epsilon$ in the last line, just start the second paragraph with $\frac{\epsilon}{2}$ instead.) ■

Problem 8. Prove that if a subsequence of a Cauchy sequence converges, then the entire sequence converges.

Proof. Suppose that x_n is a Cauchy sequence in (X, d) and there is a subsequence x_{n_k} which converges to some element $x \in X$. Let $\epsilon > 0$. By the Cauchy property, there is some $N_1 \in \mathbb{N}$ such that for all $j, k \geq N_1$, $d(x_j, x_k) < \frac{\epsilon}{2}$. By the convergence of the subsequence, there is some $N_2 \in \mathbb{N}$ such that for every $k \geq N_2$,

[1] This is different than saying that d is the discrete metric. For example, $\mathbb{Z}$ with the metric induced by the standard Euclidean metric is a discrete metric space, but this is not the discrete metric on $\mathbb{Z}$.

$d(x_{n_k}, x) < \frac{\epsilon}{2}$. Now, n_k is a strictly increasing sequence of positive integers, and so there is some N_3 such that for all $k \geq N_3$, $n_k \geq N_1$. Let $N = \max\{N_1, N_2, N_3\}$. Then for any $j \geq N$,

$$d(x_j, x) \leq d(x_j, x_{n_j}) + d(x_{n_j}, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

where the first factor of $\frac{\epsilon}{2}$ follows from the Cauchy property because $j \geq N_1$ and $j \geq N_3$ which entails $n_j \geq N_1$. The second factor of $\frac{\epsilon}{2}$ follows from the convergence of the subsequence since $n_j \geq j \geq N_2$. ■

Problem 9. Give an example to show that the image of an open (respectively, closed, bounded) set under a continuous map is not necessarily open (resp., closed, bounded). Unless your examples are highly nontrivial, you do not need to prove that the functions you provide are continuous or that the sets you describe are open (resp., closed, bounded).

Proof. **open** Consider the continuous function $\sin : \mathbb{R} \to \mathbb{R}$. Then $\sin(\mathbb{R}) = [0, 1]$ which is not open in $\mathbb{R}$.

closed Consider the continuous function $\exp : \mathbb{R} \to \mathbb{R}$. Then $\exp\big((-\infty, 0]\big) = (0, 1]$ which is not closed in $\mathbb{R}$.

bounded Consider the continuous function $x \mapsto \frac{1}{x}$ on the bounded interval $(0, 1)$. Then the image of $(0, 1)$ is $(1, \infty)$ which is not bounded.

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Problem 10. Consider $C[0, 1]$ with the L^1 metric $d(f, g) = \int_0^1 |f - g|$. If $f_n, f \in C[0, 1]$ and f_n converges to f in the metric d, does that imply that f_n converges pointwise to f? If it does, prove it; if not, find a counterexample. (Hint: because of the homogeneity in the metric d, it suffices to consider the case where f is the zero function.)

Proof. The function does not necessarily converge pointwise. Consider the functions $f_n \in C[0, 1]$ defined by $f_n(x) = x^n$. It is a standard example that this sequence converges pointwise to the discontinuous function which is 0 on $[0, 1)$ and 1 at 1. However, f_n converges to the zero function f in the metric d. Indeed,

$$d(f_n, f) = \int_0^1 |f_n - f| = \int_0^1 f_n = \frac{x^{n+1}}{n+1}\bigg|_0^1 = \frac{1}{n+1},$$

which clearly goes to zero as $n \to \infty$.

Note: this example fails to converge pointwise to the same function at only a single point, but it still does converge pointwise to *something*. However, this is not necessarily the case. It is possible to construct sequences of functions f_n which converge to the zero function in the metric d, but which don't converge pointwise *anywhere*. ■