Functional Analysis
Homework 1

Due Tuesday, 22 January 2018

Problem 1. If (X, d) is a metric space, prove that

$$|d(x,z) - d(y,z)| \leq d(x,y). \tag{1}$$

Problem 2. If X is the subspace of ℓ^∞ consisting of all sequences of zeros and ones, what is the induced metric on X?

Problem 3. Let (X, d) be a metric space and let $0 < \epsilon < 1$. Prove that the function d^ϵ is a metric on X. d^ϵ is said to be a *snowflake* of the metric d. Hint: use calculus to prove that $(1 + x)^\epsilon \leq 1 + x^\epsilon$ for any $x > 0$, then use the fact that $t \mapsto t^\epsilon$ is an increasing function.

Problem 4. The *distance* $\operatorname{dist}(A, B)$ between two nonempty subsets A, B of a metric space (X, d) is defined to be

$$\operatorname{dist}(A,B) = \inf_{\substack{a \in A \\ b \in B}} d(a,b). \tag{2}$$

Show that dist is *not* a metric on the power set of X.

Problem 5. For $p > 1$, give an example of a sequence in ℓ^p but not in ℓ^1.

Problem 6. Give an example of a sequence in c_0 but not in ℓ^p for *any* $1 \leq p < \infty$.

Problem 7. Let A be nonempty set in a metric space (X, d). The *diameter* of A is

$$\operatorname{diam}(A) = \sup_{x,y \in A} d(x,y). \tag{3}$$

A set is *bounded* if it has finite diameter. The metric space (X, d) is said to be bounded if X is bounded.

(a) Give an example to show that in general $\operatorname{diam}(B(x; r)) \neq 2r$.

(b) Consider any metric space (X, d) and the function

$$\tilde{d}(x,y) = \frac{d(x,y)}{1 + d(x,y)}. \tag{4}$$

Prove that $\tilde{d}$ is a metric on X (I suggest you avoid looking at the proof given in Kreyszig), and that $(X, \tilde{d})$ is bounded.