Functional Analysis
Homework 1 Solutions

Due Tuesday, 22 January 2018

Problem 1. If (X, d) is a metric space, prove that

$$|d(x,z) - d(y,z)| \leq d(x,y). \tag{1}$$

Proof. Let $x, y, z \in X$. By the triangle inequality, we have $d(x,z) \leq d(x,y) + d(y,z)$, and therefore

$$d(x,z) - d(y,z) \leq d(x,y). \tag{2}$$

Again, by the triangle inequality, $d(y,z) \leq d(y,x) + d(x,z)$, and therefore by symmetry

$$-\big(d(x,z) - d(y,z)\big) \leq d(y,x) = d(x,y). \tag{3}$$

Combining (2) and (3) yields

$$|d(x,z) - d(y,z)| \leq d(x,y). \qquad \blacksquare$$

Problem 2. If X is the subspace of ℓ^∞ consisting of all sequences of zeros and ones, what is the induced metric on X?

Proof. Let $x = (x_n), y = (y_n) \in X$ be $\{0,1\}$-valued sequences. As such, for all $n \in \mathbb{N}$,

$$|x_n - y_n| = \begin{cases} 0 & x_n = y_n \\ 1 & x_n \neq y_n. \end{cases}$$

Therefore, the metric on X induced by the ℓ^∞ metric is

$$d(x,y) := \sup_{n \in \mathbb{N}} |x_n - y_n| = \begin{cases} 0 & \forall n \in \mathbb{N}, x_n = y_n \\ 1 & \exists n \in \mathbb{N}, x_n \neq y_n \end{cases} = \begin{cases} 0 & x = y \\ 1 & x \neq y, \end{cases}$$

which is the *discrete metric* on X. $\blacksquare$

Problem 3. Let (X, d) be a metric space and let $0 < \epsilon < 1$. Prove that the function d^ϵ is a metric on X. d^ϵ is said to be a *snowflake* of the metric d. Hint: use calculus to prove that $(1+x)^\epsilon \leq 1 + x^\epsilon$ for any $x > 0$, then use the fact that $t \mapsto t^\epsilon$ is an increasing function.

Lemma. For $t \geq 0$, and $0 < \epsilon < 1$,

$$(1+t)^\epsilon \leq 1 + t^\epsilon \tag{4}$$

Proof. Let $f(t) = (1+t)^\epsilon$ and $g(t) = 1 + t^\epsilon$. Since $\epsilon < 1$, we find that $\epsilon - 1 < 0$, and therefore, $(\cdot)^{\epsilon-1}$ is a decreasing function. Because $1 + t > t$, this implies $(1+t)^{\epsilon-1} \leq t^{\epsilon-1}$. Moreover, because $\epsilon > 0$, we find $\epsilon(1+t)^{\epsilon-1} \leq t^{\epsilon-1}$. Finally, since $f(0) = 1 = g(0)$ and $f'(t)\epsilon(1+t)^{\epsilon-1} \leq t^{\epsilon-1} = g'(t)$, we conclude that (4) follows. ■

Proof of 3. It is easy to see that d^ϵ satisfies the first three properties of a metric because d does. Therefore, it suffices to prove the triangle inequality for d^ϵ. Let $x, y, z \in X$. If any of x, y, z are equal, the inequality is trivial, so we may assume they are distinct and therefore the distances between each pair are strictly positive. By the triangle inequality for d, we have $d(x, z) = d(x, y) + d(y, z)$. Since the function $t \mapsto t^\epsilon$ is increasing, we find

$$\begin{aligned} d^\epsilon(x, z) &= \big(d(x, y) + d(y, z)\big)^\epsilon \\ &= d^\epsilon(x, y)\left(1 + \frac{d(y, z)}{d(x, y)}\right)^\epsilon \\ &\leq d^\epsilon(x, y)\left(1 + \left(\frac{d(y, z)}{d(x, y)}\right)^\epsilon\right) \\ &= d^\epsilon(x, y) + d^\epsilon(y, z). \end{aligned}$$

■

Problem 4. The *distance* $\operatorname{dist}(A, B)$ between two nonempty subsets A, B of a metric space (X, d) is defined to be

$$\operatorname{dist}(A, B) = \inf_{\substack{a \in A \\ b \in B}} d(a, b). \tag{5}$$

Show that dist is *not* a metric on the power set of X.

Proof. Although dist is nonnegative and symmetric, it doesn't satisfy the other two properties of a metric. Indeed, as an example consider any metric space (X, d) with four or more points. Then if $A, B \subseteq X$ are sets such that $A \cap B \neq \emptyset$ and $A \neq B$, then there is some point $a \in A \cap B$, and so $\operatorname{dist}(A, B) \leq d(a, a) = 0$.

To see that the triangle inequality fails in general, consider distinct points $w, x, y, z \in X$ and the sets $A = \{w, x\}$, $B = \{x, y\}$ and $C = \{y, z\}$. By the previous paragraph, we see that $\operatorname{dist}(A, B) = 0 = \operatorname{dist}(B, C)$, but $\operatorname{dist}(A, C)$ is just the minimum of the values $d(w, y), d(w, z), d(x, y), d(x, z)$, all of which are strictly positive. Thus $\operatorname{dist}(A, C) > 0 = \operatorname{dist}(A, B) + \operatorname{dist}(B, C)$ so the triangle inequality fails. ■

Problem 5. For $p > 1$, give an example of a sequence in ℓ^p but not in ℓ^1.

Proof. Let $p > 1$. Then notice that $x = \left(\frac{1}{n}\right)$ is in ℓ^p but not in ℓ^1. Indeed, x is not in ℓ^1 because the hqarmonic series is divergent. On the other hand, since $\frac{1}{n^p}$

is a decreasing sequence, we can apply the integral test to the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$. In particular, since $p > 1$, $-p \neq -1$, and $-p+1 < 0$,

$$\int_1^{\infty} x^{-p}\,dx = \lim_{b\to\infty} \frac{x^{-p+1}}{-p+1}\Big|_1^b = \lim_{b\to\infty} \frac{b^{-p+1}}{-p+1} + \frac{1}{-p+1} = \frac{1}{-p+1}.$$

Because this integral is convergent, the associated series converges. Therefore $x \in \ell^p$. ■

Problem 6. Give an example of a sequence in c_0 but not in ℓ^p for *any* $1 \leq p < \infty$.

Proof. Consider the sequence $x = \left(\frac{1}{\log n}\right)$ and any $p \geq 1$. Notice that $x \in c_0$ since $\log n \to \infty$ (and therefore x converges to zero).

By L'Hôpital's rule

$$\lim_{x\to\infty} \frac{(\log x)^p}{x} = \lim_{x\to\infty} \frac{p(\log x)^{p-1}}{x}.$$

Moreover, this argument can be repeatedly applied until the exponent of the logarithm is nonpositive, at which point is it clear that the limit is zero.

A consequence of this is that for any $p \geq 1$, there is some sufficiently large $N_p > 1$ such that for all $n > N_p$, $\frac{1}{(\log n)^p} > \frac{1}{n}$. Therefore, by the comparison test, since the harmonic series diverges, so also does the series $\sum_{n=N_p}^{\infty} \frac{1}{(\log n)^p}$. Therefore, $x = \left(\frac{1}{\log n}\right) \notin \ell^p$. ■

Problem 7. Let A be nonempty set in a metric space (X, d). The *diameter* of A is

$$\operatorname{diam}(A) = \sup_{x,y\in A} d(x, y). \tag{6}$$

A set is *bounded* if it has finite diameter. The metric space (X, d) is said to be bounded if X is bounded.

(a) Give an example to show that in general $\operatorname{diam}(B(x; r)) \neq 2r$.

(b) Consider any metric space (X, d) and the function

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}. \tag{7}$$

Prove that $\tilde{d}$ is a metric on X (I suggest you avoid looking at the proof given in Kreyszig), and that $(X, \tilde{d})$ is bounded.

Proof.

(a) Consider a metric space (X, d) where d is the discrete metric. Notice that for any $x \in X$ and $0 < r < 1$, $B(x; r) = \{x\}$. Therefore, $\operatorname{diam}(B(x; 1)) = d(x, x) = 0$, and this is not $2r$.

(b) First note that the function $f(t) = \frac{t}{1+t}$ is increasing everywhere since $f'(t) = frac1(1+t)^2 > 0$. This, along with the triangle inequality for d, namely, $d(x,z) \le d(x,y) +_d (y,z)$ guarantees

$$\begin{aligned}
\tilde{d}(x,z) &= \frac{d(x,z)}{1+d(x,z)} \\
&\le \frac{d(x,y)+d(x,z)}{1+d(x,y)+d(y,z)} \\
&= \frac{d(x,y)}{1+d(x,y)+d(y,z)} + \frac{d(x,z)}{1+d(x,y)+d(y,z)} \\
&\le \frac{d(x,y)}{1+d(x,y)} + \frac{d(x,z)}{1+d(y,z)} \\
&= \tilde{d}(x,y) + \tilde{d}(y,z),
\end{aligned}$$

where the second inequality holds because all the terms involved are non-negative. ■