Functional Analysis
Take-home Exam 1

Due Thursday, 22 February 2018

Instructions: ***Do 5 out of the 7 problems.*** *You are allowed to use your notes, previous homework (including my solutions), or the textbook to aid you in completing this exam. You are free to cite any result which we have previously proven. However, you are not allowed to consult other students or outside resources (e.g., online Q&A sites). If you have any questions about the wording, meaning, or what is expected of you, please email me for clarification.*

Problem 1. Let (X, d) be a metric space and $\mathcal{B}$ the collection of nonempty bounded subsets of X. For $A, B \in \mathcal{B}$, define

$$d_H(A, B) := \max \left\{ \sup_{a \in A} \inf_{b \in B} d(a, b)\,,\ \sup_{b \in B} \inf_{a \in A} d(a, b) \right\}.$$

The function d_H is called the *Hausdorff distance.*

(a) Prove that d_H is a pseudometric on $\mathcal{B}$.

(b) Let $\mathcal{C} \subseteq \mathcal{B}$ be the subcollection of nonempty compact subsets of X. Prove that d_H is a metric on $\mathcal{C}$.

(c) Prove that if (X, d) is not complete, then $(\mathcal{C}, d_H)$ is not complete.

Problem 2. Prove the following corollaries of Hölder's inequality.

(a) If $1 \le r \le p, q \le \infty$ satisfy $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and $x = (x_n) \in \ell^p$ and $y = (y_n) \in \ell^q$, then $z = (x_n y_n) \in \ell^r$ and

$$\|z\|_r \le \|x\|_p \, \|y\|_q \, .$$

(b) If $1 \le r \le p_1, \ldots, p_k \le \infty$ satisfies $\frac{1}{r} = \frac{1}{p_1} + \cdots + \frac{1}{p_k}$, and if $x^{(i)} = (x_n^{(i)}) \in \ell^{p_i}$ for $1 \le i \le k$, then $x = (x_n^{(1)} \cdots x_n^{(k)}) \in \ell^r$ and

$$\|x\|_r \le \prod_{i=1}^{k} \|x^{(i)}\|_{p_i}$$

Problem 3. Suppose $1 \le p < q \le \infty$.

(a) Prove that $\ell^p \subsetneq \ell^q$.

By the previous item, ℓ^p can be equipped with both the norms $\|\cdot\|_p$ (ℓ^p norm) and $\|\cdot\|_q$ (ℓ^q norm).

(b) Prove that the identity map $\mathrm{id} : (\ell^p, \|\cdot\|_q) \to (\ell^p, \|\cdot\|_p)$ is not Lipschitz.

(c) Prove that ℓ^p is not closed (in $\|\cdot\|_q$) in ℓ^q, and is therefore not complete in this norm.

Problem 4. This problem concerns Lipschitz functions.

(a) Prove that a differentiable function $f : (a, b) \to \mathbb{R}$ is Lipschitz if and only if its derivative is bounded.

(b) Are all Lipschitz functions $f : \mathbb{R} \to \mathbb{R}$ differentiable? If so, prove it. If not, construct a counterexample.

(c) Let X be a metric space and Y a normed space. Show that the set $\mathrm{Lip}(X,Y)$ of all Lipschitz functions $f : X \to Y$ forms a vector space (with addition and scalar multiplication defined pointwise).

(d) Define a function $\|\cdot\|_{\mathrm{Lip}} : \mathrm{Lip}(X,Y) \to \mathbb{R}$ by

$$\|f\|_{\mathrm{Lip}} := \inf\{K > 0 \mid \forall x_1, x_2 \in X, \|f(x_1) - f(x_2)\|_Y \leq K d(x_1, x_2)\}.$$

In other words, $\|f\|_{\mathrm{Lip}}$ is the infimum of all Lipschitz constants for f. Prove that $\|\cdot\|_{\mathrm{Lip}}$ is a pseudonorm, but not a norm.

Problem 5. Let $(X, d_X), (X, d_Y)$ be metric spaces.

(a) Show that the natural map

$$d\big((x_1, y_1), (x_2, y_2)\big) := \sqrt{\big(d_X(x_1, x_2)\big)^2 + \big(d_Y(y_1, y_2)\big)^2}$$

is a metric on $X \times Y$ using Minkowski's inequality.

(b) Show that $X \times Y$ is complete if and only if X, Y are complete.

(c) Show that $X \times Y$ is bounded if and only if X, Y are bounded.

Problem 6. Recall that an *inner product* on a vector space X over a field $\mathbb{F}$ is a function $\langle\cdot,\cdot\rangle : X^2 \to \mathbb{F}$ which is linear in the first coordinate, conjugate linear in the second coordinate (simply linear if $\mathbb{F} = \mathbb{R}$), and satisfies

$$\begin{aligned} &\langle x, x\rangle \geq 0, && \text{nonegativity/positivity} \\ &\langle x, x\rangle = 0 \iff x = 0, && \text{definiteness} \\ &\langle x, y\rangle = \overline{\langle y, x\rangle}. && \text{hermitian/symmetry} \end{aligned}$$

(a) Prove that the map $\langle\cdot,\cdot\rangle : (\ell^2)^2 \to \mathbb{F}$ defined by

$$\langle x, y\rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}$$

is well-defined and yields an inner product on ℓ^2. Moreover, $\langle x, x\rangle = \|x\|_2^2$.

(b) Similarly, prove that the map $\langle\cdot,\cdot\rangle : (C[0,1])^2 \to \mathbb{F}$ defined by

$$\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\, dx$$

is well-defined and yields an inner product on $C[0,1]$. Moreover, $\langle f, f\rangle = \|f\|_2^2$.

(Note: $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$; also, $z \in \mathbb{R} \iff z = \overline{z}$. Moreover, some standard properties of the conjugate are $z\overline{z} = |z|^2$, $\overline{z + w} = \overline{z} + \overline{w}$, $\overline{zw} = \overline{z}\,\overline{w}$, and $\overline{\overline{z}} = z$.)

Problem 7. A metric space X is said to be *totally bounded* if for every $\varepsilon > 0$, there is a cover of X by finitely many balls of radius ε.

(a) Prove that a totally bounded metric space is bounded.

(b) Give an example of a bounded metric space which is not totally bounded.

(c) Prove that a metric space is sequentially compact if and only if it is complete and totally bounded. (Hint: for some of these implications, it might be helpful to recall the idea of the proofs of the Bolzano–Weierstrass and Heine–Borel theorems.)