Functional Analysis
Take-home Exam 1
Solutions

Due Thursday, 22 February 2018

Instructions: ***Do 5 out of the 7 problems.*** *You are allowed to use your notes, previous homework (including my solutions), or the textbook to aid you in completing this exam. You are free to cite any result which we have previously proven. However, you are not allowed to consult other students or outside resources (e.g., online Q&A sites). If you have any questions about the wording, meaning, or what is expected of you, please email me for clarification.*

Problem 1. Let (X, d) be a metric space and $\mathcal{B}$ the collection of nonempty bounded subsets of X. For $A, B \in \mathcal{B}$, define

$$d_H(A, B) := \max\left\{\sup_{a\in A}\inf_{b\in B} d(a, b)\,,\ \sup_{b\in B}\inf_{a\in A} d(a, b)\right\}.$$

The function d_H is called the *Hausdorff distance.*

(a) Prove that d_H is a pseudometric on $\mathcal{B}$.

(b) Let $\mathcal{C} \subseteq \mathcal{B}$ be the subcollection of nonempty compact subsets of X. Prove that d_H is a metric on $\mathcal{C}$.

(c) Prove that if (X, d) is not complete, then $(\mathcal{C}, d_H)$ is not complete.

Proof. (a) Let A, B, C be nonempty bounded subsets of X.

Then $d_H(A, B)$ is a combination of maximums, suprema and infima of quantities of the form $d(a, b)$ which are nonnegative since d is a metric on X. Therefore $d_H(A, B) \geq 0$ as well, so d_H satisfies $(M1)$.

Moreover,

$$\begin{aligned}
d_H(A, B) = 0 &\iff \sup_{a\in A}\inf_{b\in B} d(a, b) = 0 \text{ and } \sup_{b\in B}\inf_{a\in A} d(a, b) = 0\\
&\iff \forall a \in A, \inf_{b\in B} d(a, b) = 0 \text{ and } \forall b \in B, \inf_{a\in A} d(a, b) = 0\\
&\iff \forall a \in A, a \in \overline{B} \text{ and } \forall b \in B, b \in \overline{A}\\
&\iff A \subseteq \overline{B} \text{ and } B \subseteq \overline{A}.
\end{aligned}$$

In particular, if $A = B$, then the latter condition is satisfied, so $d_H(A, B) = 0$. Thus d_H satisfies $(M2')$.

To see that d_H satisfies $(M3)$, simply note that

$$\begin{aligned}
d_H(A, B) &= \max\left\{\sup_{a\in A}\inf_{b\in B} d(a, b)\,,\ \sup_{b\in B}\inf_{a\in A} d(a, b)\right\}\\
&= \max\left\{\sup_{b\in B}\inf_{a\in A} d(a, b)\,,\ \sup_{a\in A}\inf_{b\in B} d(a, b)\right\}\\
&= \max\left\{\sup_{b\in B}\inf_{a\in A} d(b, a)\,,\ \sup_{a\in A}\inf_{b\in B} d(b, a)\right\}\\
&= d_H(B, A).
\end{aligned}$$

Finally, we must prove that d_H satisfies the triangle inequality. To this end, let $a \in A, b \in B$ and $c \in C$. Then by the triangle inequality for d, we have $d(a,b) \leq d(a,c) + d(c,b)$. Taking the infimum over $a \in A$ yields

$$\inf_{a\in A} d(a,b) \leq \left(\inf_{a\in A} d(a,c)\right) + d(c,b) \leq \left(\sup_{c'\in C}\inf_{a\in A} d(a,c')\right) + d(c,b) \leq d_H(A,C) + d(c,b).$$

Then, taking the infimum over $c \in C$ yields

$$\inf_{a\in A} d(a,b) \leq d_H(A,C) + \inf_{c\in C} d(c,b),$$

and then taking the supremum over $b \in B$ yields

$$\sup_{b\in B}\inf_{a\in A} d(a,b) \leq d_H(A,C) + \sup_{b\in B}\inf_{c\in C} d(c,b) \leq d_H(A,C) + d_H(C,B).$$

A symmetric argument shows that $\sup_{a\in A}\inf_{b\in B} d(a,b) \leq d_H(A,C) + d_H(C,B)$, thereby yielding the triangle inequality for d_H.

(b) Let $\mathcal{C}$ be the collection of nonempty compact subsets of X. By a theorem, compact sets in a metric space are closed and bounded, so d_H is a pseudometric on $\mathcal{C}$ by the previous argument. Moreover, by the previous argument, $d_H(A,B) = 0$ if and only if $A \subseteq \overline{B} = B$ and $B \subseteq \overline{A} = A$, i.e., $A = B$.

(c) Suppose $(\mathcal{C}, d_H)$ is complete. Let (x_n) be a Cauchy sequence in X. Let $C_n = \{x_n\}$, which is compact (note: any open cover of C_n has a subcover consisting of a single element). Moreover, $d_H(C_n, C_m) = d(x_n, x_m)$ because the suprema and infima are over singleton sets. Thus (C_n) is Cauchy in $\mathcal{C}$ and so by completeness converges to some nonempty compact set C. Let $c \in C$. Then since C_n is a singleton, we have the inequality $d(x_n, c) \leq d_H(C_n, C)$, and therefore $x_n \to c$ since $C_n \to C$. Therefore X is complete.

■

Problem 2. Prove the following corollaries of Hölder's inequality.

(a) If $1 \leq r \leq p, q \leq \infty$ satisfy $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and $x = (x_n) \in \ell^p$ and $y = (y_n) \in \ell^q$, then $z = (x_n y_n) \in \ell^r$ and

$$\|z\|_r \leq \|x\|_p \|y\|_q .$$

(b) If $1 \leq r \leq p_1, \dots, p_k \leq \infty$ satisfies $\frac{1}{r} = \frac{1}{p_1} + \cdots + \frac{1}{p_k}$, and if $x^{(i)} = (x_n^{(i)}) \in \ell^{p_i}$ for $1 \leq i \leq k$, then $x = (x_n^{(1)} \cdots x_n^{(k)}) \in \ell^r$ and

$$\|x\|_r \leq \prod_{i=1}^{k} \|x^{(i)}\|_{p_i}$$

Proof. (a) Suppose $1 \leq r \leq p, q \leq \infty$ satisfy $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and $x = (x_n) \in \ell^p$ and $y = (y_n) \in \ell^q$. Then $x^r = (x_n^r) \in \ell^{p/r}$ and $y^r = (y_n^r) \in \ell^{q/r}$. Indeed,

$$\|x^r\|_{p/r} = \left(\sum_{n=1}^{\infty} |x_n^r|^{p/r}\right)^{r/p} = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{r/p} = \|x\|_p^r .$$

Since $\frac{1}{p/r} + \frac{1}{q/r} = \frac{r}{p} + \frac{r}{q} = \frac{r}{r} = 1$, we can apply Hölder's inequality to conclude

$$\|z\|_1^r = \|z^r\|_1 \leq \|x^r\|_{p/r} \|y^r\|_{q/r} = \|x\|_p^r \|y\|_p^r .$$

Taking the r-th root of both sides, we obtain

$$\|z\|_1 \leq \|x\|_p \|y\|_p ,$$

as desired.

(b) The proof is by induction on k, with the base case $k = 2$ following from the previous item. Now suppose $k \in \mathbb{N}$ and the statement holds for $p_1, \ldots, p_k$ as in the statement. Let $1 \le s \le p_1, \ldots, p_{k+1} \le \infty$ satisfy $\frac{1}{r} = \frac{1}{p_1} + \cdots + \frac{1}{p_k} + \frac{1}{p_{k+1}} = \frac{1}{s} + \frac{1}{p_{k+1}}$ and $x^{(i)} = (x_n^{(i)}) \in \ell^{p_i}$ for $1 \le i \le k+1$. Then set $w = (x_n^{(1)} \cdots x_n^{(k)}$ and $z = (w_n x_n^{(k+1)})$. By the induction hypothesis, we know that $w \in \ell^s$ and

$$\|w\|_s \le \prod_{i=1}^{k} \|x^{(i)}\|_{p_i}.$$

By the previous item, we know that $z \in \ell^r$ and

$$\|z\|_r \le \|w\|_s \left\|x^{(k+1}\right\|_{p_{k+1}} \le \prod_{i=1}^{k+1} \|x^{(i)}\|_{p_i}.$$

By induction, this proves the result. ■

Problem 3. Suppose $1 \le p < q \le \infty$.

(a) Prove that $\ell^p \subsetneq \ell^q$.

By the previous item, ℓ^p can be equipped with both the norms $\|\cdot\|_p$ (ℓ^p norm) and $\|\cdot\|_q$ (ℓ^q norm).

(b) Prove that the identity map $\mathrm{id} : (\ell^p, \|\cdot\|_q) \to (\ell^p, \|\cdot\|_p)$ is not Lipschitz.

(c) Prove that ℓ^p is not closed (in $\|\cdot\|_q$) in ℓ^q, and is therefore not complete in this norm.

Proof. Suppose $1 \le p < q \le \infty$.

(a) Let $x \in \ell^p$. If $q = \infty$, then we are done because $x \in \ell^p \subseteq c_0$ and convergent sequences are necessarily bounded. So suppose $q < \infty$. Set $\mathbb{N}_1 = \{n \in \mathbb{N} \mid x_n > 1\}$, which is a finite set since $x \in \ell^p \subseteq c_0$. Then for each $n \in \mathbb{N} \setminus \mathbb{N}_1$, we have $|x_n|^q \le |x_n|^p$ since $p < q$. Therefore,

$$\sum_{n=1}^{\infty} |x_n|^q = \sum_{n\in\mathbb{N}_1} |x_n|^q + \sum_{k\in\mathbb{N}\setminus\mathbb{N}_1} |x_k|^q \le \sum_{n\in\mathbb{N}_1} |x_n|^q + \sum_{k\in\mathbb{N}\setminus\mathbb{N}_1} |x_k|^p \le \sum_{n\in\mathbb{N}_1} |x_n|^q + \|x\|_p^p < \infty.$$

Thus $x \in \ell^q$.

(b) This is equivalent to showing that $\frac{\|x\|_p}{\|x\|_q}$ is not bounded. Consider the sequence $x_n = (1, \ldots, 1, 0, 0, \ldots)$ with n ones. Then $\|x\|_p = n^{1/p}$ and $\|x\|_q = n^{1/q}$, so $\frac{\|x\|_p}{\|x\|_q} = n^{1/p - 1/q}$. Since $p < q$, $\frac{1}{p} - \frac{1}{q} > 0$, so the ratio of these norms is unbounded as $n \to \infty$. Thus the identity map is not Lipschitz.

(c) Since $p < q$ we can find some $0 < \varepsilon < 1$ so that $p < \varepsilon q$. Then define

$$x_n = \left(\frac{1}{1^{1/\varepsilon q}}, \frac{1}{2^{1/\varepsilon q}}, \ldots, \frac{1}{n^{1/\varepsilon q}}, 0, 0, \ldots\right).$$

Then x_n lies in both ℓ^p and ℓ^q because it has only finitely many nonzero terms. Let $x \in \ell^q$ be defined by

$$x = \left(\frac{1}{1^{1/\varepsilon q}}, \frac{1}{2^{1/\varepsilon q}}, \ldots, \frac{1}{n^{1/\varepsilon q}}, \ldots\right).$$

The sequence x does indeed lie in ℓ^q since

$$\sum_{k=1}^{\infty} \left|\frac{1}{k^{1/\varepsilon q}}\right|^q = \sum_{k=1}^{\infty} \frac{1}{k^{1/\varepsilon}},$$

which converges since $\frac{1}{\varepsilon} > 1$. Moreover,

$$\|x_n - x\|_q^q = \sum_{k=n+1}^{\infty} \frac{1}{k^{1/\varepsilon}},$$

which goes to zero as $n \to \infty$ because the series converges. Therefore $x_n \to x$ in ℓ^q norm.

However, the element x is not in ℓ^p since

$$\sum_{k=1}^{\infty} \left| \frac{1}{k^{1/\varepsilon q}} \right|^p = \sum_{k=1}^{\infty} \frac{1}{k^{p/\varepsilon q}},$$

which diverges since $\frac{p}{\varepsilon q} < 1$. Therefore ℓ^p is not closed in ℓ^q (with the ℓ^q norm). ∎

Problem 4. This problem concerns Lipschitz functions.

(a) Prove that a differentiable function $f : (a, b) \to \mathbb{R}$ is Lipschitz if and only if its derivative is bounded.

(b) Are all Lipschitz functions $f : \mathbb{R} \to \mathbb{R}$ differentiable? If so, prove it. If not, construct a counterexample.

(c) Let X be a metric space and Y a normed space. Show that the set $\mathrm{Lip}(X, Y)$ of all Lipschitz functions $f : X \to Y$ forms a vector space (with addition and scalar multiplication defined pointwise).

(d) Define a function $\|\cdot\|_{\mathrm{Lip}} : \mathrm{Lip}(X, Y) \to \mathbb{R}$ by

$$\|f\|_{\mathrm{Lip}} := \inf\{K > 0 \mid \forall x_1, x_2 \in X, \|f(x_1) - f(x_2)\|_Y \le K d(x_1, x_2)\}.$$

In other words, $\|f\|_{\mathrm{Lip}}$ is the infimum of all Lipschitz constants for f. Prove that $\|\cdot\|_{\mathrm{Lip}}$ is a pseudonorm, but not a norm.

Proof. (a) Suppose that $f : (a, b) \to \mathbb{R}$ is differentiable. If f is Lipschitz with constant K, then for each $x \in (a, b)$, we know $|f(x + h) - f(x)| \le K |(x + h) - x| = K |h|$. Therefore,

$$|f'(x)| = \left| \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \right| = \lim_{h \to 0} \left| \frac{f(x + h) - f(x)}{h} \right| \le \lim_{h \to 0} \frac{K |h|}{|h|} = K.$$

Thus f' is bounded by the Lipschitz constant of f.

Conversely if f' is bounded by some M, then for each $x, y \in (a, b)$, by the Mean Value Theorem there is some $c \in (a, b)$ such that

$$\frac{f(x) - f(y)}{x - y} = f'(c),$$

and therefore

$$|f(x) - f(y)| = |f'(c)(x - y)| \le M |x - y|.$$

Therefore the bound on the derivative f' is a Lipschitz constant for f.

(b) Not all Lipschitz functions $f : \mathbb{R} \to \mathbb{R}$ are differentiable. Consider $f(x) = |x|$. This is not differentiable at zero because

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1 \ne -1 = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} \frac{f(h) - f(0)}{h}.$$

However, f is Lipschitz with constant 1 because

$$|f(x) - f(y)| = ||x| - |y|| \le |x - y|,$$

which can be proven by cases.

(c) Notice that $\|\cdot\|_{\text{Lip}}$ is nonnegative because it is an infimum a (nonempty) set of positive numbers. To see that $\|\cdot\|_{\text{Lip}}$ is homogeneous, simply take $c \in \mathbb{F}$ and f Lipschitz with constant K to obtain

$$\|(cf)(x_1) - (cf)(x_2)\|_Y = \|c(f(x_1) - f(x_2)\|_Y = |c| \, \|f(x_1) - f(x_2)\|_Y \leq |c| \, K d(x_1, x_2),$$

so that $|c| \, K$ is a Lipschitz constant for (cf). In particular, this guarantees that $\|cf\|_{\text{Lip}} \leq |c| \, \|f\|_{\text{Lip}}$. If $c \neq 0$, then this can also be reversed since $\|f\|_{\text{Lip}} = \left\|\frac{1}{c} cf\right\|_{\text{Lip}} \leq \left|\frac{1}{c}\right| \|cf\|_{\text{Lip}}$, Moreover, if $c = 0$, this implies that $\|cf\|_{\text{Lip}} = 0 = |0| \, \|f\|_{\text{Lip}}$, so $\|\cdot\|_{\text{Lip}}$ is homogeneous. Finally, we show that $\|\cdot\|_{\text{Lip}}$ is subadditive. Let f, g be Lipschitz with constants K_f, K_g. Then

$$\begin{aligned}
\|(f+g)(x_1) - (f+g)(x_2)\|_Y &= \|f(x_1) + g(x_1) - f(x_2) - g(x_2)\|_Y \\
&\leq \|f(x_1) - f(x_2)\|_Y + \|g(x_1) - g(x_2)\|_Y \\
&\leq K_f d(x_1, x_2) + K_g d(x_1, x_2) \\
&= (K_f + K_g) d(x_1, x_2).
\end{aligned}$$

Taking the infimum over all such K_f, K_g, we find that $\|f\|_{\text{Lip}} + \|g\|_{\text{Lip}}$ is a Lipschitz constant for $f + g$, and therefore $\|f + g\|_{\text{Lip}} \leq \|f\|_{\text{Lip}} + \|g\|_{\text{Lip}}$.

The last step is to see that this is not a norm precisely because any constant function $f : X \to Y$ has $\|f\|_{\text{Lip}} = 0$. In fact, one can also show that these are the *only* functions whose Lipschitz pseudonorm is zero. ■

Problem 5. Let $(X, d_X), (X, d_Y)$ be metric spaces.

(a) Show that the natural map

$$d\big((x_1, y_1), (x_2, y_2)\big) := \sqrt{\big(d_X(x_1, x_2)\big)^2 + \big(d_Y(y_1, y_2)\big)^2}$$

is a metric on $X \times Y$ using Minkowski's inequality.

(b) Show that $X \times Y$ is complete if and only if X, Y are complete.

(c) Show that $X \times Y$ is bounded if and only if X, Y are bounded.

Proof. (a) Consider $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$. Notice that d is nonnegative because it is square root of a sum of squares. Moreover,

$$\begin{aligned}
d\big((x_1, y_1), (x_2, y_2)\big) = 0 &\iff \sqrt{\big(d_X(x_1, x_2)\big)^2 + \big(d_Y(y_1, y_2)\big)^2} = 0 \\
&\iff d_X(x_1, x_2) = 0 = d_Y(y_1, y_2) \\
&\iff x_1 = x_2 \text{ and } y_1 = y_2 \\
&\iff (x_1, y_1) = (x_2, y_2).
\end{aligned}$$

d is symmetric because d_X and d_Y are symmetric. Finally, for the triangle inequality, notice that

$$\begin{aligned}
d\big((x_1, y_1), (x_3, y_3)\big) &= \sqrt{\big(d_X(x_1, x_3)\big)^2 + \big(d_Y(y_1, y_3)\big)^2} \\
&\leq \sqrt{\big(d_X(x_1, x_2) + d_X(x_2, x_3)\big)^2 + \big(d_Y(y_1, y_2) + d_Y(y_2, y_3)\big)^2} \\
&\leq \sqrt{\big(d_X(x_1, x_2)\big)^2 + \big(d_Y(y_1, y_2)\big)^2} + \sqrt{\big(d_X(x_2, x_3)\big)^2 + \big(d_Y(y_2, y_3)\big)^2} \\
&= d\big((x_1, y_1), (x_2, y_2)\big) + d\big((x_2, y_2), (x_3, y_3)\big),
\end{aligned}$$

where the first inequality follows from the triangle inequality and the monotonicity of the $\cdot^2$ and $\sqrt{\cdot}$ functions, and the second inequality is due to Minkowski's inequality.

(b) By a homework problem (about $\|\cdot\|_2$ and $\|\cdot\|_1$) we have the relationship

$$\frac{1}{\sqrt{2}}\big(d_X(x_1,x_2)+d_Y(y_1,y_2)\big) \le d\big((x_1,y_1),(x_2,y_2)\big) \le d_X(x_1,x_2)+d_Y(y_1,y_2). \tag{1}$$

Therefore, if X, Y are complete, then for any Cauchy sequence (x_n, y_n) in $X \times Y$, the first inequality above guarantees that both $(x_n), (y_n)$ are Cauchy in X, Y respectively. Since these spaces are complete, there exist x, y in X, Y respectively such that $x_n \to x$ and $y_n \to y$. Then, the first inequality guarantees that $(x_n, y_n) \to (x, y)$.

Conversely, suppose that $X \times Y$ is complete and that (x_n) is a Cauchy sequence in X. Let $y \in Y$ and notice that $d\big((x_n, y), (x_m, y)\big) = d_X(x_n, x_m)$, so that (x_n, y) is Cauchy in $X \times Y$. Therefore it converges to some point (x, y) (technically, we don't know *a priori* that the second coordinate is y, but upon examination of the formula for d, this is only option so we know it *a fortiori*). Thus $x_n \to x$ (since $d\big((\cdot, y), (\cdot, y)\big) = d_X$). Therefore X is complete. A symmetric argument shows Y is complete.

(c) Suppose that X, Y are bounded with constants M, N respectively. Then the second inequality in (1) guarantees that $X \times Y$ is bounded by $M + N$ (actually, a more careful analysis would yield that $X \times Y$ is bounded with constant $\sqrt{M^2 + N^2}$, but I was feeling lazy). Now suppose that $X \times Y$ is bounded with some constant K. Then the first inequality in (1) guarantees that X, Y are each bounded with constant $\sqrt{2}K$ (actually, by being more careful, we can show that X, Y are bounded with constant K, but I was feeling lazy). ■

Problem 6. Recall that an *inner product* on a vector space X over a field $\mathbb{F}$ is a function $\langle\cdot,\cdot\rangle : X^2 \to \mathbb{F}$ which is linear in the first coordinate, conjugate linear in the second coordinate (simply linear if $\mathbb{F} = \mathbb{R}$), and satisfies

$$\begin{aligned}
&\langle x, x\rangle \ge 0, && \text{nonegativity/positivity}\\
&\langle x, x\rangle = 0 \iff x = 0, && \text{definiteness}\\
&\langle x, y\rangle = \overline{\langle y, x\rangle}. && \text{hermitian/symmetry}
\end{aligned}$$

(a) Prove that the map $\langle\cdot,\cdot\rangle : (\ell^2)^2 \to \mathbb{F}$ defined by

$$\langle x, y\rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}$$

is well-defined and yields an inner product on ℓ^2. Moreover, $\langle x, x\rangle = \|x\|_2^2$.

(b) Similarly, prove that the map $\langle\cdot,\cdot\rangle : (C[0,1])^2 \to \mathbb{F}$ defined by

$$\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx$$

is well-defined and yields an inner product on $C[0,1]$. Moreover, $\langle f, f\rangle = \|f\|_2^2$.

(Note: $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$; also, $z \in \mathbb{R} \iff z = \overline{z}$. Moreover, some standard properties of the conjugate are $z\overline{z} = |z|^2$, $\overline{z + w} = \overline{z} + \overline{w}$, $\overline{zw} = \overline{z}\,\overline{w}$, and $\overline{\overline{z}} = z$.)

Proof. (a) We first show that the map $\langle\cdot,\cdot\rangle : (\ell^2)^2 \to \mathbb{F}$ defined in the problem is actually well-defined. To this end, let $x, y \in \ell^2$. Then since $\frac{1}{2} + \frac{1}{2} = 1$, by Hölder's inequality we know that $(x_n y_n) \in \ell^1$ and $\|(x_n y_n)\|_1 \le \|x\|_2 \|y\|_2$. Therefore, the series

$$\langle x, y\rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}$$

converges absolutely (and therefore also converges since $\mathbb{F}$ is complete). Therefore the map is well-defined.

Moreover, let $z \in \ell^2$ and $c \in \mathbb{F}$. Notice that

$$\begin{aligned}
\langle cx + z, y \rangle &= \sum_{n=1}^{\infty} (cx_n + z_n)\overline{y_n} \\
&= c \sum_{n=1}^{\infty} x_n \overline{y_n} + \sum_{n=1}^{\infty} z_n \overline{y_n} \\
&= c \langle x, y \rangle + \langle z, y \rangle .
\end{aligned}$$

Thus $\langle \cdot, \cdot \rangle$ is linear in the first coordinate.

Similarly,

$$\begin{aligned}
\langle x, cy + z \rangle &= \sum_{n=1}^{\infty} x_n \overline{(cy_n + z_n)} \\
&= \sum_{n=1}^{\infty} x_n \left(\overline{c}\, \overline{y_n} + \overline{z_n} \right) \\
&= \overline{c} \sum_{n=1}^{\infty} x_n \overline{y_n} + \sum_{n=1}^{\infty} x_n \overline{z_n} \\
&= \overline{c} \langle x, y \rangle + \langle x, z \rangle .
\end{aligned}$$

Thus $\langle \cdot, \cdot \rangle$ is conjugate linear in the second coordinate.

For nonnegativity,

$$\langle x, x \rangle = \sum_{n=1}^{\infty} x_n \overline{x_n} = \sum_{n=1}^{\infty} |x_n|^2 = \|x\|_2^2 ,$$

which is nonnegative and definite because $\|\cdot\|_2$ is as well.

Finally, for the hermitian property,

$$\begin{aligned}
\langle x, y \rangle &= \sum_{n=1}^{\infty} x_n \overline{y_n} \\
&= \sum_{n=1}^{\infty} \overline{\overline{x_n}}\, \overline{y_n} \\
&= \sum_{n=1}^{\infty} \overline{\overline{x_n} y_n} \\
&= \overline{\sum_{n=1}^{\infty} \overline{x_n} y_n} \\
&= \langle y, x \rangle .
\end{aligned}$$

(b) We first show that the map $\langle \cdot, \cdot \rangle : (\ell^2)^2 \to \mathbb{F}$ defined in the problem is actually well-defined, but this is almost trivial. Simply notice that if $f, g \in C[0,1]$, then $f\overline{g} \in C[0,1]$ as well, and so the integral is well-defined. Moreover, by the Hölder inequality for L^p norms, we find that $|\langle f, g \rangle| \leq \int_0^1 \left| f(x)\overline{g(x)} \right| dx \leq \|f\|_2 \|g\|_2$.

Let $h \in \ell^2$ and $c \in \mathbb{F}$. Notice that

$$\begin{aligned}\langle cf + h, g\rangle &= \int_0^1 (cf(x) + h(x))\overline{g(x)} \\ &= c\int_0^1 f(x)\overline{g(x)} + \int_0^1 h(x)\overline{g(x)} \\ &= c\,\langle f, g\rangle + \langle h, g\rangle\,.\end{aligned}$$

Thus $\langle \cdot, \cdot\rangle$ is linear in the first coordinate.

Similarly,

$$\begin{aligned}\langle f, cg + h\rangle &= \int_0^1 f(x)\overline{(cg(x) + h(x))} \\ &= \int_0^1 f(x)\big(\overline{c}\,\overline{g(x)} + \overline{h(x)}\big) \\ &= \overline{c}\int_0^1 f(x)\overline{g(x)} + \int_0^1 f(x)\overline{h(x)} \\ &= \overline{c}\,\langle f, g\rangle + \langle f, h\rangle\,.\end{aligned}$$

Thus $\langle \cdot, \cdot\rangle$ is conjugate linear in the second coordinate.

For nonnegativity,

$$\langle f, f\rangle = \int_0^1 f(x)\overline{f(x)} = \int_0^1 |f(x)|^2 = \|f\|_2^2\,,$$

which is nonnegative and definite because $\|\cdot\|_2$ is as well.

Finally, for the hermitian property,

$$\begin{aligned}\langle f, g\rangle &= \int_0^1 f(x)\overline{g(x)} \\ &= \int_0^1 \overline{\overline{f(x)}}\,\overline{g(x)} \\ &= \int_0^1 \overline{\overline{f(x)}g(x)} \\ &= \overline{\int_0^1 \overline{f(x)}g(x)} \\ &= \langle g, f\rangle\,.\end{aligned}$$

We now make a few more remarks about this function. Consider the metric on $(C[0,1])^2$ defined in Problem 5 where the metric used on each copy of $C[0,1]$ is the L^2 metric. We remark that this new metric actually corresponds to a norm on $(C[0,1])^2$ (you can check homogeneity and subadditivity just follows from Minkowski's inequality and subadditivity of the L^2 norm). Moreover, one can show that the inner product is uniformly continuous on bounded subsets of $(C[0,1])^2$ and so it extends uniquely to a continuous function on the completion of this space, which one can show is just $(L^2[0,1])^2$. Moreover, all the properties (linearity in the first coordinate, conjugate linearity in the second coordinate, nonnegativity, definiteness, the hermitian property) are also inherited by this extension. So, really, there is an inner product on $L^2[0,1]$ which, for pairs of continuous functions, is just the integral of the product (with the conjugate of the second function). ■

Problem 7. A metric space X is said to be *totally bounded* if for every $\varepsilon > 0$, there is a cover of X by finitely many balls of radius ε.

(a) Prove that a totally bounded metric space is bounded.

(b) Give an example of a bounded metric space which is not totally bounded.

(c) Prove that a metric space is sequentially compact if and only if it is complete and totally bounded. (Hint: for some of these implications, it might be helpful to recall the idea of the proofs of the Bolzano–Weierstrass and Heine–Borel theorems.)

Proof. (a) Suppose that X is a totally bounded metric space. Then the is a cover of X by a finite number of balls $B(x_i; 1)$ $(1 \leq i \leq n)$ of radius 1. Set $M = \max_{1\leq i,j\leq n} d(x_i, x_j)$. Then notice that for any $x, y \in X$, there are some $1 \leq r, s \leq n$ such that $x \in B(x_r; 1)$ and $y \in B(x_s; 1)$, and hence

$$d(x, y) \leq d(x, x_r) + d(x_r, x_s) + d(x_s, y) < 1 + M + 1.$$

Taking the supremum over $x, y \in X$, we obtain $\sup_{x,y\in X} d(x, y) \leq M + 2$.

(b) Let X be an infinite set equipped with the discrete metric and let $0 < \varepsilon < 1$. We claim that there is no finite cover of X by balls of radius ε. Indeed, for any $x \in X$, $B(x; \varepsilon) = \{x\}$. Therefore, the union of a finite number of balls of radius ε is a finite set and hence cannot cover X. Thus X is not totally bounded. At the same time, X is bounded because $\sup_{x,y\in X} d(x, y) = 1$.

(c) $(\Rightarrow)$ Suppose that X is sequentially compact. Then any Cauchy sequence has a convergent subsequence, and therefore converges by a homework problem. Hence X is complete.

Now suppose that X is not totally bounded. Then there is some $\varepsilon > 0$ such that X cannot be covered by a finite number of balls of radius ε. Take any $x_1 \in X$. Then create a sequence (x_n) inductively by letting x_n be any element of $X \setminus \bigcup_{j=1}^{n-1} B(x_j; \varepsilon)$, which is necessarily nonempty. Finally, we notice that (x_n) has no convergent subsequence because for any $m > n \in \mathbb{N}$, we know $x_m \notin B(x_n; \varepsilon)$ and thus $d(x_n, x_m) \geq \varepsilon$. Therefore, no subsequence of (x_n) can be Cauchy, and therefore no subsequence can converge (because convergent sequences are Cauchy). Therefore X is not sequentially compact.

$(\Leftarrow)$ (This proof mimics the proof of the Bolzano–Weierstrass theorem). Suppose that X is complete and totally bounded and let (x_n) be any sequence in X. Since X is totally bounded, there is a finite cover of X by balls of radius 1. By the Pigeonhole Principle, one of these balls, which we will call X_1, must contain infinitely many terms of the sequence (x_n). Let $(x_n^{(1)})$ be the subsequence of (x_n) which lies in X_1.

We then construct X_{k+1} and $(x_n^{(k+1)})$ from X_k and $(x_n^{(k)})$ as follows. Cover X, and as a consequence X_k, by a finite collection of balls of radius $\frac{1}{k+1}$. Then by the Pigeonhole Principle one of these balls, say $B(y, \frac{1}{k+1})$ must contain infinitely many elements of the sequence $(x_n^{(k)})$. Let $X_{k+1} := X_k \cap B(y, \frac{1}{k+1})$ and let $(x_n^{(k+1)})$ be the subsequence of $(x_n^{(k)})$ which lies in X_{k+1}.

Therefore, by induction we have a nested sequence of sets $X_{k+1} \subseteq X_k$ with $\operatorname{diam}(X_k) \leq \frac{2}{k}$, each X_{k+1} contains a sequence $(x_n^{(k+1)})$ which is a subsequence of $(x_n^{(k)})$ (and $(x_n^{(1)})$ is a subsequence of (x_n)). Consider the sequence $(y_n) = (x_n^{(n)})$, which is also a subsequence of the original (x_n). It is clear that if $n, m \geq N$, then $y_n, y_m \in X_N$ and hence $d(y_n, y_m) \leq \operatorname{diam}(X_N) = \frac{2}{N}$. Therefore (y_n) is a Cauchy sequence in X, and since X is complete, it converges. Hence X is sequentially compact. ■