Appendix 6: Time-Energy Uncertainty (so called)

The Generalized Uncertainty Principle applied to the Hamiltonian operator $\hat{H}$ and the generic operator $\hat{O}$ says that

$$\sigma_H{}^2 \sigma_O{}^2 \geq \left(\frac{i}{2i} \left\langle \left[\hat{H}, \hat{O} \right] \right\rangle \right)^2 . \qquad \text{(A6.1)}$$

If $\hat{O}$ does not depend explicitly on time, we can write the general form of the Ehrenfest Theorem as

$$\frac{\hbar}{i} \frac{\partial}{\partial t} < O > = \left\langle \left[\hat{H}, \hat{O} \right] \right\rangle . \qquad \text{(A6.2)}$$

By plugging it into the previous equation, we obtain

$$\sigma_H{}^2 \sigma_O{}^2 \geq \left(\frac{i}{2i} \frac{\hbar}{i} \frac{\partial}{\partial t} < O > \right)^2 = \left(\frac{\hbar}{2} \right)^2 \left(\frac{\partial}{\partial t} < O > \right)^2 . \qquad \text{(A6.3)}$$

Since standard deviation is always positive, we have

$$\sigma_H \sigma_O \geq \left(\frac{\hbar}{2} \right) \left| \frac{\partial}{\partial t} < O > \right| . \qquad \text{(A6.4)}$$

Now let us set

$$\sigma_O = \left| \frac{d}{dt} < O > \right| \Delta t , \qquad \text{(A6.5)}$$

where Δt is a time interval. Hence,

$$\Delta t = \frac{\sigma_O}{\left| \frac{d}{dt} < O > \right|} .^{[1]} \qquad \text{(A6.6)}$$

[1] To see what this formula means, consider this. When $\Delta t = 1$, then $\sigma_O = \left| \frac{d}{dt} < O > \right|$, that is, the rate of change of the expectation value is one standard deviation; when $\Delta t = 2$,

By plugging this into the previous equation, we obtain

$$\sigma_H \Delta t \geq \frac{\hbar}{2}, \qquad \text{(A6.7)}$$

the Time-Energy Uncertainty Principle (TEUP). The name is misleading because Δt is not the standard deviation of an ensemble of time measurements (there is no operator for time because time is *not* an observable), but the amount of time $< O >$ takes to change by O's standard deviation. TEUP tells us that in an ensemble the smaller σ_H (the standard deviation of the Hamiltonian of the system, that is, its energy), the larger Δt must be. Hence, when the standard deviation of the energy is small, the system changes slowly and gradually. As an extreme case, in a stationary state $\sigma_H = 0$ (the returns of energy measurements are always the same), and therefore $\Delta t = \infty$, that is, the expectation values are constant. Conversely, if Δt is small (the system changes rapidly), then σ_H must be large.

Occasionally, TEUP is coupled with the claim that if Δt is the time during which the energy of a system is measured and ΔE is the experimental error in the measurement of energy, then

$$\Delta E \Delta t \geq \frac{\hbar}{2}, \qquad \text{(A6.8)}$$

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then $\sigma_O = \frac{1}{2}\left|\frac{d}{dt} < O >\right|$, and so on. So, Δt is the amount of time $< O >$ takes to change by O's standard deviation. Since σ_O represent O's uncertainty, we can be sure that when $< O >$ has changed by at least σ_O, the system has undergone change. By contrast, if $< O >$ has not changed by at least σ_O, the measurement discrepancy may be simply due to the probabilistic nature of quantum mechanics.

which entails that the exact measurement of energy would have to take forever.

This association is understandable, since (A6.8) was obtained by Heisenberg via an analysis of a Gedanken experiment with a Stern-Gerlach device. Furthermore, Einstein's criticisms of (A6.8) and Bohr's successful defense of it have become part of the lore of quantum mechanics. However, at best such connection is controversial. For, some have argued that not only quantum mechanics allows the measurement of the energy of a system with arbitrary precision and speed, but (A6.8) is also refuted by several counterexamples (Greenstein, G., and Zajonc, A., (1997): 89-91; Home, D., (1997): 16).

TEUP has some interesting consequences affecting measurement. To a good approximation, when a hydrogen atom decays from an excited stationary state *j* to, say the ground state, it emits a photon whose energy is:

$$hf = E_j - E_0. \quad \text{(A6.9)}$$

However, in spite of the fact that energies in stationary states should be sharp, it turns out that upon measurement, *not* all photons emitted by the atoms in an ensemble undergoing the same decay have the same energy (the same frequency). Instead, the energies of the emitted photons fluctuate about the predicted value of (A6.9) by ΔE_j (called the "width" of the *j* state). The reason for the discrepancy is explained by TEUP. Let τ_j (called the "natural lifetime" of the state) be the minimum amount of time that must pass before we can say that a Hydrogen atom has in fact gone from the *j* state to the ground state. Then, TEUP becomes

$$\Delta E_j \tau_j \geq \frac{\hbar}{2}. \quad \text{(A6.10)}$$

It turns out that τ_j can be experimentally obtained by studying the emitted photons: it hovers about 10^{-8} seconds and grows as j does. Consequently, the higher the energy involved, the smaller the width of the state need be.