Appendix 4: the Ehrenfest Theorem

Here is a modern version of the Ehrenfest theorem. The rate of change of the expected value of the observable O is

$$\frac{\partial}{\partial t} < O >= \frac{\partial}{\partial t}\left\langle \Psi \middle| \hat{O}\Psi \right\rangle = \frac{\partial}{\partial t}\left(\left| \Psi^* \right\rangle \hat{O} \left| \Psi \right\rangle \right). \tag{A4.1}$$

By applying the product rule to the time derivative, one gets

$$\frac{\partial}{\partial t} < O >= \frac{\partial \left| \Psi \right\rangle^*}{\partial t} \hat{O} \left| \Psi \right\rangle + \left| \Psi \right\rangle^* \frac{\partial \hat{O}}{\partial t} \left| \Psi \right\rangle + \Psi^* \hat{O} \frac{\partial \left| \Psi \right\rangle}{\partial t}. \tag{A4.2}$$

However, from TDSE and its conjugate, one gets

$$\frac{\partial \Psi}{\partial t} = -\frac{i}{\hbar} \hat{H} \Psi \tag{A4.3}$$

and

$$\frac{\partial \Psi^*}{\partial t} = \frac{i}{\hbar} \hat{H}^* \Psi^*. \tag{A4.4}$$ [1]

Plugging these into (A4.2), we get

$$\begin{aligned} \frac{\partial}{\partial t} < O >&= \frac{i}{\hbar} H^* \left| \Psi \right\rangle^* \hat{O} \left| \Psi \right\rangle + \left| \Psi \right\rangle^* \frac{\partial \hat{O}}{\partial t} \left| \Psi \right\rangle - \frac{i}{\hbar} \Psi^* \hat{O} H \left| \Psi \right\rangle = \\ &= \frac{i}{\hbar} \left\{ H^* \left| \Psi \right\rangle^* \hat{O} \left| \Psi \right\rangle - \Psi^* \hat{O} H \left| \Psi \right\rangle \right\} + \left\langle \frac{\partial \hat{O}}{\partial t} \right\rangle \end{aligned}, \tag{A4.5}$$

or

$$\frac{\partial}{\partial t} < O >= \frac{i}{\hbar} \left\{ \left\langle H\Psi \middle| \hat{O}\Psi \right\rangle - \left\langle \Psi\hat{O} \middle| H\Psi \right\rangle \right\} + \left\langle \frac{\partial \hat{O}}{\partial t} \right\rangle, \tag{A4.6}$$

where the last term is the expectation value of $\partial \hat{O} / \partial t$.

But $\hat{H}$ is Hermitian, and therefore

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[1] Remember that TDSE is $i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi$.

$$\left\langle \hat{H}\Psi \mid \hat{O}\Psi \right\rangle = \left\langle \Psi \mid \hat{H}\hat{O}\Psi \right\rangle . \qquad \text{(A4.6)}$$

Plugging this back into (A4.6), we obtain

$$\frac{\partial}{\partial t} < O > = \frac{i}{\hbar} \left\{ \left\langle \Psi \middle| H\hat{O}\Psi \right\rangle - \left\langle \Psi \middle| \hat{O}H\Psi \right\rangle \right\} + \left\langle \frac{\partial \hat{O}}{\partial t} \right\rangle . \qquad \text{(A4.7)}$$

But

$$\left\langle \Psi \middle| H\hat{O}\Psi \right\rangle = < H\hat{O} > \qquad \text{(A4.8)}$$

and

$$\left\langle \Psi \middle| \hat{O}H\Psi \right\rangle = < \hat{O}H > . \qquad \text{(A4.9)}$$

Hence,

$$\frac{\partial}{\partial t} < O > = \frac{i}{\hbar} \left\{ < \hat{H}\hat{O} > - < \hat{O}\hat{H} > \right\} + \left\langle \frac{\partial \hat{O}}{\partial t} \right\rangle . \qquad \text{(A4.10)}$$

Given that the difference of the averages is equal to the average of the differences,

$$\frac{\partial}{\partial t} < O > = \frac{i}{\hbar} \left\langle \hat{H}\hat{O} - \hat{O}\hat{H} \right\rangle + \left\langle \frac{\partial \hat{O}}{\partial t} \right\rangle , \qquad \text{(A4.11)}$$

or

$$\frac{\partial}{\partial t} < O > = \frac{i}{\hbar} \left\langle \left[\hat{H}, \hat{O} \right] \right\rangle + \left\langle \frac{\partial \hat{O}}{\partial t} \right\rangle , \qquad \text{(A4.12)}$$

which is the Ehrenfest theorem.